

**INTRODUCTION TO**  
**CONTINUUM**  
**MECHANICS**

**Third Edition**

**W. Michael Lai**  
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**Erhard Krempf**

# **Introduction to Continuum Mechanics**

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# Introduction to Continuum Mechanics

Third Edition

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## Preface to the Third Edition

The first edition of this book was published in 1974, nearly twenty years ago. It was written as a text book for an introductory course in continuum mechanics and aimed specifically at the junior and senior level of undergraduate engineering curricula which choose to introduce to the students at the undergraduate level the general approach to the subject matter of continuum mechanics. We are pleased that many instructors of continuum mechanics have found this little book serves that purpose well. However, we have also understood that many instructors have used this book as one of the texts for a beginning graduate course in continuum mechanics. It is this latter knowledge that has motivated us to write this new edition. In this present edition, we have included materials which we feel are suitable for a beginning graduate course in continuum mechanics. The following are examples of the additions:

1. Anisotropic elastic solid which includes the concept of material symmetry and the constitutive equations for monoclinic, orthotropic, transversely isotropic and isotropic materials.
2. Finite deformation theory which includes derivations of the various finite deformation tensors, the Piola-Kirchhoff stress tensors, the constitutive equations for an incompressible nonlinear elastic solid together with some boundary value problems.
3. Some solutions of classical elasticity problems such as thick-walled pressure vessels (cylinders and spheres), stress concentrations and bending of curved bars.
4. Objective tensors and objective time derivatives of tensors including corotational derivative and convected derivatives.
5. Differential type, rate type and integral type linear and nonlinear constitutive equations for viscoelastic fluids and some solutions for the simple fluid in viscometric flows.
6. Equations in cylindrical and spherical coordinates are provided including the use of different coordinates for the deformed and the undeformed states.

We wish to state that notwithstanding the additions, the present edition is still intended to be "introductory" in nature, so that the coverage is not extensive. We hope that this new edition can serve a dual purpose: for an introductory course at the undergraduate level by omitting some of the "intermediate level" material in the book and for a beginning graduate course in continuum mechanics at the graduate level.

W. Michael Lai  
David Rubin  
Erhard Krempf

July, 1993

## Preface to the First Edition

This text is prepared for the purpose of introducing the concept of continuum mechanics to beginners in the field. Special attention and care have been given to the presentation of the subject matter so that it is within the grasp of those readers who have had a good background in calculus, some differential equations, and some rigid body mechanics. For pedagogical reasons the coverage of the subject matter is far from being extensive, only enough to provide for a better understanding of later courses in the various branches of continuum mechanics and related fields. The major portion of the material has been successfully class-tested at Rensselaer Polytechnic Institute for undergraduate students. However, the authors believe the text may also be suitable for a beginning graduate course in continuum mechanics.

We take the liberty to say a few words about the second chapter. This chapter introduces second-order tensors as linear transformations of vectors in a three dimensional space. From our teaching experience, the concept of linear transformation is the most effective way of introducing the subject. It is a self-contained chapter so that prior knowledge of linear transformations, though helpful, is not required of the students. The third-and higher-order tensors are introduced through the generalization of the transformation laws for the second-order tensor. Indicical notation is employed whenever it economizes the writing of equations. Matrices are also used in order to facilitate computations. An appendix on matrices is included at the end of the text for those who are not familiar with matrices.

Also, let us say a few words about the presentation of the basic principles of continuum physics. Both the differential and integral formulation of the principles are presented. The differential formulations are given in Chapters 3,4, and 6, at places where quantities needed in the formulation are defined while the integral formulations are given later in Chapter 7. This is done for a pedagogical reason: the integral formulations as presented required slightly more mathematical sophistication on the part of a beginner and may be either postponed or omitted without affecting the main part of the text.

This text would never have been completed without the constant encouragement and advice from Professor F. F. Ling, Chairman of Mechanics Division at RPI, to whom the authors wish to express their heartfelt thanks. Gratefully acknowledged is the financial support of the Ford Foundation under a grant which is directed by Dr. S. W. Yerazunis, Associate Dean of Engineering. The authors also wish to thank Drs. V. C. Mow and W. B. Browner, Jr. for their many useful suggestions. Special thanks are given to Dr. H. A. Scarton for painstakingly compiling a list of errata and suggestions on the preliminary edition. Finally, they are indebted to Mrs. Geri Frank who typed the entire manuscript.

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## Introduction

### 1.1 CONTINUUM THEORY

Matter is formed of molecules which in turn consist of atoms and sub-atomic particles. Thus matter is not continuous. However, there are many aspects of everyday experience regarding the behaviors of materials, such as the deflection of a structure under loads, the rate of discharge of water in a pipe under a pressure gradient or the drag force experienced by a body moving in the air etc., which can be described and predicted with theories that pay no attention to the molecular structure of materials. The theory which aims at describing relationships between gross phenomena, neglecting the structure of material on a smaller scale, is known as continuum theory. The continuum theory regards matter as indefinitely divisible. Thus, within this theory, one accepts the idea of an infinitesimal volume of materials referred to as a particle in the continuum, and in every neighborhood of a particle there are always neighbor particles. Whether the continuum theory is justified or not depends on the given situation; for example, while the continuum approach adequately describes the behavior of real materials in many circumstances, it does not yield results that are in accord with experimental observations in the propagation of waves of extremely small wavelength. On the other hand, a rarefied gas may be adequately described by a continuum in certain circumstances. At any case, it is misleading to justify the continuum approach on the basis of the number of molecules in a given volume. After all, an infinitesimal volume in the limit contains no molecules at all. Neither is it necessary to infer that quantities occurring in continuum theory must be interpreted as certain particular statistical averages. In fact, it has been known that the same continuum equation can be arrived at by different hypothesis about the molecular structure and definitions of gross variables. While molecular-statistical theory, whenever available, does enhance the understanding of the continuum theory, the point to be made is simply that whether the continuum theory is justified in a given situation is a matter of experimental test, not of philosophy. Suffice it to say that more than a hundred years of experience have justified such a theory in a wide variety of situations.

### 1.2 Contents of Continuum Mechanics

Continuum mechanics studies the response of materials to different loading conditions. Its subject matter can be divided into two main parts: (1) general principles common to all media,

## 2 Introduction

and (2) constitutive equations defining idealized materials. The general principles are axioms considered to be self-evident from our experience with the physical world, such as conservation of mass, balance of linear momentum, of moment of momentum, of energy, and the entropy inequality law. Mathematically, there are two equivalent forms of the general principles: (1) the integral form, formulated for a finite volume of material in the continuum, and (2) the field equations for differential volume of material (particle) at every point of the field of interest. Field equations are often derived from the integral form. They can also be derived directly from the free body of a differential volume. The latter approach seems to suit beginners. In this text both approaches are presented, with the integral form given toward the end of the text. Field equations are important wherever the variations of the variables in the field are either of interest by itself or are needed to get the desired information. On the other hand, the integral forms of conservation laws lend themselves readily to certain approximate solutions.

The second major part of the theory of continuum mechanics concerns the “constitutive equations” which are used to define idealized material. Idealized materials represent certain aspects of the mechanical behavior of the natural materials. For example, for many materials under restricted conditions, the deformation caused by the application of loads disappears with the removal of the loads. This aspect of the material behavior is represented by the constitutive equation of an elastic body. Under even more restricted conditions, the state of stress at a point depends linearly on the changes of lengths and mutual angle suffered by elements at the point measured from the state where the external and internal forces vanish. The above expression defines the linearly elastic solid. Another example is supplied by the classical definition of viscosity which is based on the assumption that the state of stress depends linearly on the instantaneous rates of change of length and mutual angle. Such a constitutive equation defines the linearly viscous fluid. The mechanical behavior of real materials varies not only from material to material but also with different loading conditions for a given material. This leads to the formulation of many constitutive equations defining the many different aspects of material behavior. In this text, we shall present four idealized models and study the behavior they represent by means of some solutions of simple boundary-value problems. The idealized materials chosen are (1) the linear isotropic and anisotropic elastic solid (2) the incompressible nonlinear isotropic elastic solid (3) the linearly viscous fluid including the inviscid fluid, and (4) the Non-Newtonian incompressible fluid.

One important requirement which must be satisfied by all quantities used in the formulation of a physical law is that they be coordinate-invariant. In the following chapter, we discuss such quantities.

## 2

# Tensors

As mentioned in the introduction, all laws of continuum mechanics must be formulated in terms of quantities that are independent of coordinates. It is the purpose of this chapter to introduce such mathematical entities. We shall begin by introducing a short-hand notation - the indicial notation - in Part A of this chapter, which will be followed by the concept of tensors introduced as a linear transformation in Part B. The basic field operations needed for continuum formulations are presented in Part C and their representations in curvilinear coordinates in Part D.

### Part A The Indicial Notation

#### 2A1 Summation Convention, Dummy Indices

Consider the sum

$$s = a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n \quad (2A1.1)$$

We can write the above equation in a compact form by using the summation sign:

$$s = \sum_{i=1}^n a_i x_i \quad (2A1.2)$$

It is obvious that the following equations have exactly the same meaning as Eq. (2A1.2)

$$s = \sum_{j=1}^n a_j x_j \quad (2A1.3)$$

$$s = \sum_{m=1}^n a_m x_m \quad (2A1.4)$$

etc.

#### 4 Indicical Notation

The index  $i$  in Eq. (2A1.2), or  $j$  in Eq. (2A1.3), or  $m$  in Eq. (2A1.4) is a dummy index in the sense that the sum is independent of the letter used.

We can further simplify the writing of Eq.(2A1.1) if we adopt the following convention: Whenever an index is repeated once, it is a dummy index indicating a summation with the index running through the integers  $1, 2, \dots, n$ .

This convention is known as Einstein's summation convention. Using the convention, Eq. (2A1.1) shortens to

$$s = a_i x_i \quad (2A1.5)$$

We also note that

$$a_i x_i = a_m x_m = a_j x_j = \dots \quad (2A1.6)$$

It is emphasized that expressions such as  $a_i b_i x_i$  are not defined within this convention. That is, an index should *never* be repeated more than once when the summation convention is used. Therefore, an expression of the form

$$\sum_{i=1}^n a_i b_i x_i$$

must retain its summation sign.

In the following we shall always take  $n$  to be 3 so that, for example,

$$a_i x_i = a_m x_m = a_1 x_1 + a_2 x_2 + a_3 x_3$$

$$a_{ii} = a_{mm} = a_{11} + a_{22} + a_{33}$$

$$a_i e_i = a_1 e_1 + a_2 e_2 + a_3 e_3$$

The summation convention obviously can be used to express a double sum, a triple sum, etc. For example, we can write

$$\sum_{i=1}^3 \sum_{j=1}^3 a_{ij} x_i x_j \quad (2A1.7)$$

simply as

$$a_{ij} x_i x_j \quad (2A1.8)$$

Expanding in full, the expression (2A1.8) gives a sum of nine terms, i.e.,

$$\begin{aligned} a_{ij} x_i x_j = & a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3 + a_{21}x_2x_1 + a_{22}x_2x_2 \\ & + a_{23}x_2x_3 + a_{31}x_3x_1 + a_{32}x_3x_2 + a_{33}x_3x_3 \end{aligned} \quad (2A1.9)$$

For beginners, it is probably better to perform the above expansion in two steps, first, sum over  $i$  and then sum over  $j$  (or vice versa), i.e.,

$$a_{ij} x_i x_j = a_{1j}x_1x_j + a_{2j}x_2x_j + a_{3j}x_3x_j$$

where

$$a_{1j}x_1x_j = a_{11}x_1x_1 + a_{12}x_1x_2 + a_{13}x_1x_3$$

etc.

Similarly, the triple sum

$$\sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 a_{ijk}x_ix_jx_k \quad (2A1.10)$$

will simply be written as

$$a_{ijk}x_ix_jx_k \quad (2A1.11)$$

The expression (2A1.11) represents the sum of 27 terms.

We emphasize again that expressions such as  $a_{ii}x_ix_jx_j$  or  $a_{ijk}x_ix_ix_jx_k$  are not defined in the summation convention, they do not represent

$$\sum_{i=1}^3 \sum_{j=1}^3 a_{ii}x_ix_jx_j \quad \text{or} \quad \sum_{i=1}^3 \sum_{j=1}^3 \sum_{k=1}^3 a_{ijk}x_ix_ix_jx_k$$

## 2A2 Free Indices

Consider the following system of three equations

$$\begin{aligned} x'_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ x'_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ x'_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \quad (2A2.1)$$

Using the summation convention, Eqs. (2A2.1) can be written as

$$\begin{aligned} x'_1 &= a_{1m}x_m \\ x'_2 &= a_{2m}x_m \\ x'_3 &= a_{3m}x_m \end{aligned} \quad (2A2.2)$$

which can be shortened into

$$x'_i = a_{im}x_m, \quad i = 1,2,3 \quad (2A2.3)$$

An index which appears only once in each term of an equation such as the index  $i$  in Eq. (2A2.3) is called a “free index.” A free index takes on the integral number 1, 2, or 3 *one at a time*. Thus Eq. (2A2.3) is short-hand for three equations each having a sum of three terms on its right-hand side [i.e., Eqs. (2A2.1)].

A further example is given by

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$$e'_i = Q_{mi}e_m, \quad i = 1,2,3 \tag{2A2.4}$$

representing

$$\begin{aligned} e'_1 &= Q_{11}e_1 + Q_{21}e_2 + Q_{31}e_3 \\ e'_2 &= Q_{12}e_1 + Q_{22}e_2 + Q_{32}e_3 \\ e'_3 &= Q_{13}e_1 + Q_{23}e_2 + Q_{33}e_3 \end{aligned} \tag{2A2.5}$$

We note that  $x'_j = a_{jm}x_m, j=1,2,3$ , is the same as Eq. (2A2.3) and  $e'_j = Q_{mj}e_m, j=1,2,3$  is the same as Eq. (2A2.4). However,

$$a_i = b_j$$

is a meaningless equation. *The free index appearing in every term of an equation must be the same.* Thus the following equations are meaningful

$$\begin{aligned} a_i + k_i &= c_i \\ a_i + b_i c_i d_j &= 0 \end{aligned}$$

If there are two free indices appearing in an equation such as

$$T_{ij} = A_{im} A_{jm} \quad i = 1,2,3; j = 1,2,3 \tag{2A2.6}$$

then the equation is a short-hand writing of 9 equations; each has a sum of 3 terms on the right-hand side. In fact,

$$\begin{aligned} T_{11} &= A_{1m}A_{1m} = A_{11}A_{11} + A_{12}A_{12} + A_{13}A_{13} \\ T_{12} &= A_{1m}A_{2m} = A_{11}A_{21} + A_{12}A_{22} + A_{13}A_{23} \\ T_{13} &= A_{1m}A_{3m} = A_{11}A_{31} + A_{12}A_{32} + A_{13}A_{33} \\ &\dots\dots\dots \\ &\dots\dots\dots \\ T_{33} &= A_{3m}A_{3m} = A_{31}A_{31} + A_{32}A_{32} + A_{33}A_{33} \end{aligned}$$

Again, equations such as

$$T_{ij} = T_{ik}$$

have no meaning.

**2A3 Kronecker Delta**

The Kronecker delta, denoted by  $\delta_{ij}$ , is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \tag{2A3.1}$$

That is,

$$\begin{aligned}\delta_{11} &= \delta_{22} = \delta_{33} = 1 \\ \delta_{12} &= \delta_{13} = \delta_{21} = \delta_{23} = \delta_{31} = \delta_{32} = 0\end{aligned}$$

In other words, the matrix of the Kronecker delta is the identity matrix, i.e.,

$$[\delta_{ij}] = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2A3.2)$$

We note the following:

$$(a) \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3 \quad (2A3.3)$$

$$\begin{aligned}(b) \delta_{1m}a_m &= \delta_{11}a_1 + \delta_{12}a_2 + \delta_{13}a_3 = a_1 \\ \delta_{2m}a_m &= \delta_{21}a_1 + \delta_{22}a_2 + \delta_{23}a_3 = a_2 \\ \delta_{3m}a_m &= \delta_{31}a_1 + \delta_{32}a_2 + \delta_{33}a_3 = a_3\end{aligned}$$

Or, in general

$$\delta_{im}a_m = a_i \quad (2A3.4)$$

$$\begin{aligned}(c) \delta_{1m}T_{mj} &= \delta_{11}T_{1j} + \delta_{12}T_{2j} + \delta_{13}T_{3j} = T_{1j} \\ \delta_{2m}T_{mj} &= T_{2j} \\ \delta_{3m}T_{mj} &= T_{3j}\end{aligned}$$

or, in general

$$\delta_{im}T_{mj} = T_{ij} \quad (2A3.5)$$

In particular,

$$\delta_{im}\delta_{mj} = \delta_{ij} \quad (2A3.6)$$

$$\delta_{im}\delta_{mn}\delta_{nj} = \delta_{ij}$$

etc.

(d) If  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are unit vectors perpendicular to each other, then

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad (2A3.7)$$

## 2A4 Permutation Symbol

The permutation symbol, denoted by  $\varepsilon_{ijk}$  is defined by

$$\varepsilon_{ijk} = \begin{cases} +1 \\ -1 \\ 0 \end{cases} \equiv \text{according to whether } i, j, k \left\{ \begin{array}{l} \text{form an even} \\ \text{form an odd} \\ \text{do not form a} \end{array} \right\} \text{ permutation of } 1, 2, 3 \quad (2A4.1)$$

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i.e.,

$$\begin{aligned}\epsilon_{123} &= \epsilon_{231} = \epsilon_{312} = +1 \\ \epsilon_{132} &= \epsilon_{321} = \epsilon_{213} = -1 \\ \epsilon_{111} &= \epsilon_{112} = \dots = 0\end{aligned}$$

We note that

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji} \quad (2A4.2)$$

If  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  form a right-handed triad, then

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1, \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2, \quad \mathbf{e}_1 \times \mathbf{e}_1 = \mathbf{0}, \dots$$

which can be written for short as

$$\mathbf{e}_i \times \mathbf{e}_j = \epsilon_{ijk} \mathbf{e}_k = \epsilon_{jki} \mathbf{e}_k = \epsilon_{kij} \mathbf{e}_k \quad (2A4.3)$$

Now, if  $\mathbf{a} = a_i \mathbf{e}_i$ , and  $\mathbf{b} = b_j \mathbf{e}_j$ , then

$$\mathbf{a} \times \mathbf{b} = (a_i \mathbf{e}_i) \times (b_j \mathbf{e}_j) = a_i b_j (\mathbf{e}_i \times \mathbf{e}_j) = a_i b_j \epsilon_{ijk} \mathbf{e}_k$$

i.e.,

$$\mathbf{a} \times \mathbf{b} = a_i b_j \epsilon_{ijk} \mathbf{e}_k \quad (2A4.4)$$

The following useful identity can be proven (see Prob. 2A7)

$$\epsilon_{ijm} \epsilon_{klm} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} \quad (2A4.5)$$

### 2A5 Manipulations with the Indicial Notation

(a) *Substitution*

If

$$a_i = U_{im} b_m \quad (i)$$

and

$$b_i = V_{im} c_m \quad (ii)$$

then, in order to substitute the  $b_i$ 's in (i) into (i) we first change the free index in (ii) from  $i$  to  $m$  and the dummy index  $m$  to some other letter, say  $n$  so that

$$b_m = V_{mn} c_n \quad (iii)$$

Now, (i) and (iii) give

$$a_i = U_{im} V_{mn} c_n \quad (iv)$$

Note (iv) represents three equations each having the sum of nine terms on its right-hand side.

(b) *Multiplication*

If

$$p = a_m b_m \quad (i)$$

and

$$q = c_m d_m \quad (ii)$$

then,

$$pq = a_m b_m c_m d_m \quad (iii)$$

It is important to note that  $pq \neq a_m b_m c_m d_m$ . In fact, the right hand side of this expression is not even defined in the summation convention and further it is obvious that

$$pq \neq \sum_{m=1}^3 a_m b_m c_m d_m.$$

Since the dot product of vectors is distributive, therefore, if  $\mathbf{a} = a_i \mathbf{e}_i$  and  $\mathbf{b} = b_j \mathbf{e}_j$ , then

$$\mathbf{a} \cdot \mathbf{b} = (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) = a_i b_j (\mathbf{e}_i \cdot \mathbf{e}_j) \quad (iv)$$

In particular, if  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are unit vectors perpendicular to one another, then  $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$  so that

$$\mathbf{a} \cdot \mathbf{b} = a_i b_j \delta_{ij} = a_i b_i = a_j b_j = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (v)$$

(c) *Factoring*

If

$$T_{ij} n_j - \lambda n_i = 0 \quad (i)$$

then, using the Kronecker delta, we can write

$$n_i = \delta_{ij} n_j \quad (ii)$$

so that (i) becomes

$$T_{ij} n_j - \lambda \delta_{ij} n_j = 0 \quad (iii)$$

Thus,

$$(T_{ij} - \lambda \delta_{ij}) n_j = 0 \quad (iv)$$

(d) *Contraction*

The operation of identifying two indices and so summing on them is known as contraction. For example,  $T_{ii}$  is the contraction of  $T_{ij}$ ,

$$T_{ii} = T_{11} + T_{22} + T_{33} \quad (i)$$

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If

$$T_{ij} = \lambda\theta\delta_{ij} + 2\mu E_{ij} \quad (\text{ii})$$

then

$$T_{ii} = \lambda\theta\delta_{ii} + 2\mu E_{ii} = 3\lambda\theta + 2\mu E_{ii} \quad (\text{iii})$$

## Part B Tensors

### 2B1 Tensor – A Linear Transformation

Let  $T$  be a transformation, which transforms any vector into another vector. If  $T$  transforms  $\mathbf{a}$  into  $\mathbf{c}$  and  $\mathbf{b}$  into  $\mathbf{d}$ , we write  $T\mathbf{a} = \mathbf{c}$  and  $T\mathbf{b} = \mathbf{d}$ .

If  $T$  has the following linear properties:

$$T(\mathbf{a}+\mathbf{b}) = T\mathbf{a}+T\mathbf{b} \quad (2B1.1a)$$

$$T(\alpha\mathbf{a}) = \alpha T\mathbf{a} \quad (2B1.1b)$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are two arbitrary vectors and  $\alpha$  is an arbitrary scalar then  $T$  is called a **linear transformation**. It is also called a **second-order tensor** or simply a **tensor**.<sup>†</sup> An alternative and equivalent definition of a linear transformation is given by the single linear property:

$$T(\alpha\mathbf{a}+\beta\mathbf{b}) = \alpha T\mathbf{a}+\beta T\mathbf{b} \quad (2B1.2)$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are two arbitrary vectors and  $\alpha$  and  $\beta$  are arbitrary scalars.

If two tensors,  $T$  and  $S$ , transform any arbitrary vector  $\mathbf{a}$  in an identical way, then these tensors are equal to each other, i.e.,  $T\mathbf{a}=S\mathbf{a} \rightarrow T=S$ .

#### Example 2B1.1

Let  $T$  be a transformation which transforms every vector into a fixed vector  $\mathbf{n}$ . Is this transformation a tensor?

*Solution.* Let  $\mathbf{a}$  and  $\mathbf{b}$  be any two vectors, then by the definition of  $T$ ,

$$T\mathbf{a} = \mathbf{n}, T\mathbf{b} = \mathbf{n} \text{ and } T(\mathbf{a}+\mathbf{b}) = \mathbf{n}$$

Clearly,

$$T(\mathbf{a}+\mathbf{b}) \neq T\mathbf{a}+T\mathbf{b}$$

Thus,  $T$  is not a linear transformation. In other words, it is not a tensor.

<sup>†</sup> Scalars and vectors are sometimes called the zeroth and first order tensor, respectively. Even though they can also be defined algebraically, in terms of certain operational rules, we choose not to do so. The geometrical concept of scalars and vectors, which we assume that the students are familiar with, is quite sufficient for our purpose.

## Example 2B1.2

Let  $\mathbf{T}$  be a transformation which transforms every vector into a vector that is  $k$  times the original vector. Is this transformation a tensor?

*Solution.* Let  $\mathbf{a}$  and  $\mathbf{b}$  be arbitrary vectors and  $\alpha$  and  $\beta$  be arbitrary scalars, then by the definition of  $\mathbf{T}$ ,

$$\mathbf{T}\mathbf{a} = k\mathbf{a}, \mathbf{T}\mathbf{b} = k\mathbf{b}, \text{ and } \mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = k(\alpha\mathbf{a} + \beta\mathbf{b})$$

Clearly,

$$\mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha(k\mathbf{a}) + \beta(k\mathbf{b}) = \alpha\mathbf{T}\mathbf{a} + \beta\mathbf{T}\mathbf{b}$$

Thus, by Eq. (2B1.2),  $\mathbf{T}$  is a linear transformation. In other words, it is a tensor.

In the previous example, if  $k=0$  then the tensor  $\mathbf{T}$  transforms all vectors into zero. This tensor is the zero tensor and is symbolized by  $\mathbf{0}$ .

## Example 2B1.3

Consider a transformation  $\mathbf{T}$  that transforms every vector into its mirror image with respect to a fixed plane. Is  $\mathbf{T}$  a tensor?

*Solution.* Consider a parallelogram in space with its sides represented by vectors  $\mathbf{a}$  and  $\mathbf{b}$  and its diagonal represented the resultant  $\mathbf{a} + \mathbf{b}$ . Since the parallelogram remains a parallelogram after the reflection, the diagonal (the resultant vector) of the reflected parallelogram is clearly both  $\mathbf{T}(\mathbf{a} + \mathbf{b})$ , the reflected  $(\mathbf{a} + \mathbf{b})$ , and  $\mathbf{T}\mathbf{a} + \mathbf{T}\mathbf{b}$ , the sum of the reflected  $\mathbf{a}$  and the reflected  $\mathbf{b}$ . That is,  $\mathbf{T}(\mathbf{a} + \mathbf{b}) = \mathbf{T}\mathbf{a} + \mathbf{T}\mathbf{b}$ . Also, for an arbitrary scalar  $\alpha$ , the reflection of  $\alpha\mathbf{a}$  is obviously the same as  $\alpha$  times the reflection of  $\mathbf{a}$  (i.e.,  $\mathbf{T}(\alpha\mathbf{a}) = \alpha\mathbf{T}\mathbf{a}$ ) because both vectors have the same magnitude given by  $\alpha$  times the magnitude of  $\mathbf{a}$  and the same direction. Thus, by Eqs. (2B1.1),  $\mathbf{T}$  is a tensor.

## Example 2B1.4

When a rigid body undergoes a rotation about some axis, vectors drawn in the rigid body in general change their directions. That is, the rotation transforms vectors drawn in the rigid body into other vectors. Denote this transformation by  $\mathbf{R}$ . Is  $\mathbf{R}$  a tensor?

*Solution.* Consider a parallelogram embedded in the rigid body with its sides representing vectors  $\mathbf{a}$  and  $\mathbf{b}$  and its diagonal representing the resultant  $\mathbf{a} + \mathbf{b}$ . Since the parallelogram remains a parallelogram after a rotation about any axis, the diagonal (the resultant vector) of the rotated parallelogram is clearly both  $\mathbf{R}(\mathbf{a} + \mathbf{b})$ , the rotated  $(\mathbf{a} + \mathbf{b})$ , and  $\mathbf{R}\mathbf{a} + \mathbf{R}\mathbf{b}$ , the sum of the rotated  $\mathbf{a}$  and the rotated  $\mathbf{b}$ . That is  $\mathbf{R}(\mathbf{a} + \mathbf{b}) = \mathbf{R}\mathbf{a} + \mathbf{R}\mathbf{b}$ . A similar argument as that used in the previous example leads to  $\mathbf{R}(\alpha\mathbf{a}) = \alpha\mathbf{R}\mathbf{a}$ . Thus,  $\mathbf{R}$  is a tensor.

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Example 2B1.5

Let  $\mathbf{T}$  be a tensor that transforms the specific vectors  $\mathbf{a}$  and  $\mathbf{b}$  according to

$$\mathbf{T}\mathbf{a} = \mathbf{a} + 2\mathbf{b}, \quad \mathbf{T}\mathbf{b} = \mathbf{a} - \mathbf{b}$$

Given a vector  $\mathbf{c} = 2\mathbf{a} + \mathbf{b}$ , find  $\mathbf{T}\mathbf{c}$ .

*Solution.* Using the linearity property of tensors

$$\mathbf{T}\mathbf{c} = \mathbf{T}(2\mathbf{a} + \mathbf{b}) = 2\mathbf{T}\mathbf{a} + \mathbf{T}\mathbf{b} = 2(\mathbf{a} + 2\mathbf{b}) + (\mathbf{a} - \mathbf{b}) = 3\mathbf{a} + 3\mathbf{b}$$

## 2B2 Components of a Tensor

The components of a vector depend on the base vectors used to describe the components. This will also be true for tensors. Let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  be unit vectors in the direction of the  $x_1, x_2, x_3$ -axes respectively, of a rectangular Cartesian coordinate system. Under a transformation  $\mathbf{T}$ , these vectors,  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  become  $\mathbf{T}\mathbf{e}_1, \mathbf{T}\mathbf{e}_2,$  and  $\mathbf{T}\mathbf{e}_3$ . Each of these  $\mathbf{T}\mathbf{e}_i$  ( $i=1,2,3$ ), being a vector, can be written as:

$$\begin{aligned} \mathbf{T}\mathbf{e}_1 &= T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3 \\ \mathbf{T}\mathbf{e}_2 &= T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3 \\ \mathbf{T}\mathbf{e}_3 &= T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3 \end{aligned} \tag{2B2.1a}$$

or

$$\mathbf{T}\mathbf{e}_i = T_{ji}\mathbf{e}_j \tag{2B2.1b}$$

It is clear from Eqs. (2B2.1a) that

$$T_{11} = \mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_1, \quad T_{12} = \mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_2, \quad T_{21} = \mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_1, \quad \dots$$

or in general

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j \tag{2B2.2}$$

The components  $T_{ij}$  in the above equations are defined as the components of the tensor  $\mathbf{T}$ . These components can be put in a matrix as follows:

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

This matrix is called the matrix of the tensor  $\mathbf{T}$  with respect to the set of base vectors  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  or  $\{\mathbf{e}_i\}$  for short. We note that, because of the way we have chosen to denote the components of transformation of the base vectors, the elements of the first column are components of the vector  $\mathbf{T}\mathbf{e}_1$ , those in the second column are the components of the vector  $\mathbf{T}\mathbf{e}_2$ , and those in the third column are the components of  $\mathbf{T}\mathbf{e}_3$ .

## Example 2B2.1

Obtain the matrix for the tensor  $\mathbf{T}$  which transforms the base vectors as follows:

$$\mathbf{T}\mathbf{e}_1 = 4\mathbf{e}_1 + \mathbf{e}_2$$

$$\mathbf{T}\mathbf{e}_2 = 2\mathbf{e}_1 + 3\mathbf{e}_3$$

$$\mathbf{T}\mathbf{e}_3 = -\mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3$$

*Solution.* By Eq. (2B2.1a) it is clear that:

$$[\mathbf{T}] = \begin{bmatrix} 4 & 2 & -1 \\ 1 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$


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## Example 2B2.2

Let  $\mathbf{T}$  transform every vector into its mirror image with respect to a fixed plane. If  $\mathbf{e}_1$  is normal to the reflection plane ( $\mathbf{e}_2$  and  $\mathbf{e}_3$  are parallel to this plane), find a matrix of  $\mathbf{T}$ .

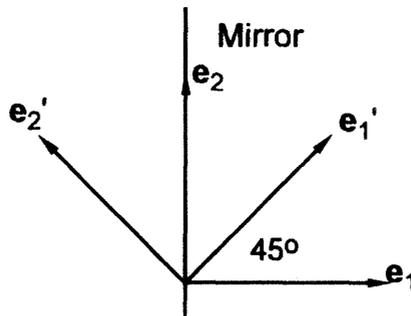


Fig. 2B.1

*Solution.* Since the normal to the reflection plane is transformed into its negative and vectors parallel to the plane are not altered:

$$\mathbf{T}\mathbf{e}_1 = -\mathbf{e}_1$$

$$\mathbf{T}\mathbf{e}_2 = \mathbf{e}_2$$

$$\mathbf{T}\mathbf{e}_3 = \mathbf{e}_3$$

Thus,

$$[\mathbf{T}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_i$$

We note that this is only one of the infinitely many matrices of the tensor  $\mathbf{T}$ , each depends on a particular choice of base vectors. In the above matrix, the choice of  $\mathbf{e}_i$  is indicated at the bottom right corner of the matrix. If we choose  $\mathbf{e}'_1$  and  $\mathbf{e}'_2$  to be on a plane perpendicular to the mirror with each making  $45^\circ$  with the mirror as shown in Fig. 2B.1 and  $\mathbf{e}'_3$  points straight out from the paper. Then we have

$$\mathbf{T}\mathbf{e}'_1 = \mathbf{e}'_2$$

$$\mathbf{T}\mathbf{e}'_2 = \mathbf{e}'_1$$

$$\mathbf{T}\mathbf{e}'_3 = \mathbf{e}'_3$$

Thus, with respect to  $\{\mathbf{e}'_i\}$ , the matrix of the tensor is

$$[\mathbf{T}]' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}'_i$$

Throughout this book, we shall denote the matrix of a tensor  $\mathbf{T}$  with respect to the basis  $\mathbf{e}_i$  by either  $[\mathbf{T}]$  or  $[T_{ij}]$  and with respect to the basis  $\mathbf{e}'_i$  by either  $[\mathbf{T}]'$  or  $[T'_{ij}]$ . The last two matrices should not be confused with  $[\mathbf{T}']$ , which represents the matrix of the tensor  $\mathbf{T}'$  with respect to the basis  $\mathbf{e}_i$ .

### Example 2B2.3

Let  $\mathbf{R}$  correspond to a right-hand rotation of a rigid body about the  $x_3$ -axis by an angle  $\theta$ . Find a matrix of  $\mathbf{R}$ .

*Solution.* From Fig. 2B.2 it is clear that

$$\mathbf{R}\mathbf{e}_1 = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$$

$$\mathbf{R}\mathbf{e}_2 = -\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2$$

$$\mathbf{R}\mathbf{e}_3 = \mathbf{e}_3$$

Thus,

$$[\mathbf{R}] = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{e}_i$$

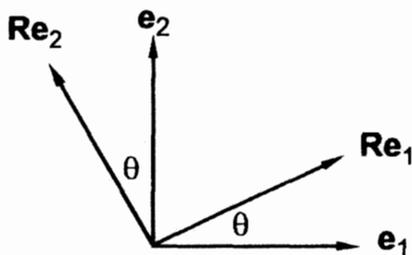


Fig. 2B.2

### 2B3 Components of a Transformed Vector

Given the vector  $\mathbf{a}$  and the tensor  $\mathbf{T}$ , we wish to compute the components of  $\mathbf{b} = \mathbf{T}\mathbf{a}$  from the components of  $\mathbf{a}$  and the components of  $\mathbf{T}$ . Let the components of  $\mathbf{a}$  with respect to  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$  be  $[a_1, a_2, a_3]$ , i.e.,

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \quad (\text{i})$$

then

$$\mathbf{b} = \mathbf{T}\mathbf{a} = \mathbf{T}(a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3) = a_1\mathbf{T}\mathbf{e}_1 + a_2\mathbf{T}\mathbf{e}_2 + a_3\mathbf{T}\mathbf{e}_3 \quad (\text{ii})$$

Thus,

$$\begin{aligned} b_1 &= \mathbf{e}_1 \cdot \mathbf{b} = a_1(\mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_1) + a_2(\mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_2) + a_3(\mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_3) \\ b_2 &= \mathbf{e}_2 \cdot \mathbf{b} = a_1(\mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_1) + a_2(\mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_2) + a_3(\mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_3) \\ b_3 &= \mathbf{e}_3 \cdot \mathbf{b} = a_1(\mathbf{e}_3 \cdot \mathbf{T}\mathbf{e}_1) + a_2(\mathbf{e}_3 \cdot \mathbf{T}\mathbf{e}_2) + a_3(\mathbf{e}_3 \cdot \mathbf{T}\mathbf{e}_3) \end{aligned} \quad (\text{iii})$$

By Eq. (2B2.2), we have,

$$\begin{aligned} b_1 &= T_{11}a_1 + T_{12}a_2 + T_{13}a_3 \\ b_2 &= T_{21}a_1 + T_{22}a_2 + T_{23}a_3 \\ b_3 &= T_{31}a_1 + T_{32}a_2 + T_{33}a_3 \end{aligned} \quad (2B3.1a)$$

We can write the above three equations in matrix form as:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad (2B3.1b)$$

or

$$[\mathbf{b}] = [\mathbf{T}][\mathbf{a}] \quad (2B3.1c)$$

We can concisely derive Eq. (2B3.1a) using indicial notation as follows: From  $\mathbf{a} = a_i \mathbf{e}_i$ , we get  $\mathbf{Ta} = \mathbf{T}a_i \mathbf{e}_i = a_i \mathbf{T}e_i$ . Since  $\mathbf{T}e_i = T_{ji} \mathbf{e}_j$ , (Eq. (2B2.1b)), therefore,

$$b_k = \mathbf{b} \cdot \mathbf{e}_k = \mathbf{Ta} \cdot \mathbf{e}_k = a_i T_{ji} \mathbf{e}_j \cdot \mathbf{e}_k = a_i T_{ji} \delta_{jk} = a_i T_{ki}$$

i.e.,

$$b_k = T_{ki} a_i \quad (2B3.1d)$$

Eq. (2B3.1d) is nothing but Eq. (2B3.1a) in indicial notation. We see that for the tensorial equation  $\mathbf{b} = \mathbf{Ta}$ , there corresponds a matrix equation of exactly the same form, i.e.,  $[\mathbf{b}] = [\mathbf{T}][\mathbf{a}]$ . This is the reason we adopted the convention that  $\mathbf{T}e_1 = T_{11}e_1 + T_{21}e_2 + T_{31}e_3$ , etc. If we had adopted the convention  $\mathbf{T}e_1 = T_{11}e_1 + T_{12}e_2 + T_{13}e_3$ , etc., then we would have obtained  $[\mathbf{b}] = [\mathbf{T}]^T [\mathbf{a}]$  for the tensorial equation  $\mathbf{b} = \mathbf{Ta}$ , which would not be as natural.

### Example 2B3.1

Given that a tensor  $\mathbf{T}$  which transforms the base vectors as follows:

$$\mathbf{T}e_1 = 2e_1 - 6e_2 + 4e_3$$

$$\mathbf{T}e_2 = 3e_1 + 4e_2 - e_3$$

$$\mathbf{T}e_3 = -2e_1 + e_2 + 2e_3$$

How does this tensor transform the vector  $\mathbf{a} = e_1 + 2e_2 + 3e_3$ ?

*Solution.* Using Eq. (2B3.1b)

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -2 \\ -6 & 4 & 1 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

or

$$\mathbf{b} = 2e_1 + 5e_2 + 8e_3$$

## 2B4 Sum of Tensors

Let  $\mathbf{T}$  and  $\mathbf{S}$  be two tensors and  $\mathbf{a}$  be an arbitrary vector. The sum of  $\mathbf{T}$  and  $\mathbf{S}$ , denoted by  $\mathbf{T} + \mathbf{S}$ , is defined by:

$$(\mathbf{T} + \mathbf{S})\mathbf{a} = \mathbf{T}\mathbf{a} + \mathbf{S}\mathbf{a} \quad (2B4.1)$$

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It is easily seen that by this definition  $\mathbf{T} + \mathbf{S}$  is indeed a tensor.

To find the components of  $\mathbf{T} + \mathbf{S}$ , let

$$\mathbf{W} = \mathbf{T} + \mathbf{S} \quad (2B4.2a)$$

Using Eqs. (2B2.2) and (2B4.1), the components of  $\mathbf{W}$  are obtained to be

$$W_{ij} = \mathbf{e}_i \cdot (\mathbf{T} + \mathbf{S})\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{S}\mathbf{e}_j$$

i.e.,

$$W_{ij} = T_{ij} + S_{ij} \quad (2B4.2b)$$

In matrix notation, we have

$$[\mathbf{W}] = [\mathbf{T}] + [\mathbf{S}] \quad (2B4.2c)$$

### 2B5 Product of Two Tensors

Let  $\mathbf{T}$  and  $\mathbf{S}$  be two tensors and  $\mathbf{a}$  be an arbitrary vector, then  $\mathbf{TS}$  and  $\mathbf{ST}$  are defined to be the transformations (easily seen to be tensors)

$$(\mathbf{TS})\mathbf{a} = \mathbf{T}(\mathbf{S}\mathbf{a}) \quad (2B5.1)$$

and

$$(\mathbf{ST})\mathbf{a} = \mathbf{S}(\mathbf{T}\mathbf{a}) \quad (2B5.2)$$

Thus the components of  $\mathbf{TS}$  are

$$(\mathbf{TS})_{ij} = \mathbf{e}_i \cdot (\mathbf{TS})\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{T}(\mathbf{S}\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{T}S_{mj}\mathbf{e}_m = S_{mj}\mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_m = T_{im}S_{mj}$$

i.e.,

$$(\mathbf{TS})_{ij} = T_{im}S_{mj} \quad (2B5.3)$$

Similarly,

$$(\mathbf{ST})_{ij} = S_{im}T_{mj} \quad (2B5.4)$$

In fact, Eq. (2B5.3) is equivalent to the matrix equation:

$$[\mathbf{TS}] = [\mathbf{T}][\mathbf{S}] \quad (2B5.5)$$

whereas, Eq. (2B5.4) is equivalent to the matrix equation:

$$[\mathbf{ST}] = [\mathbf{S}][\mathbf{T}] \quad (2B5.6)$$

The two matrix products are in general different. Thus, it is clear that in general, the tensor product is not commutative (i.e.,  $\mathbf{TS} \neq \mathbf{ST}$ ).

If  $\mathbf{T}$ ,  $\mathbf{S}$ , and  $\mathbf{V}$  are three tensors, then

$$(\mathbf{T}(\mathbf{SV}))\mathbf{a} = \mathbf{T}((\mathbf{SV})\mathbf{a}) = \mathbf{T}(\mathbf{S}(\mathbf{V}\mathbf{a}))$$

and

$$(\mathbf{TS})(\mathbf{Va}) = \mathbf{T}(\mathbf{S}(\mathbf{Va}))$$

i.e.,

$$\mathbf{T}(\mathbf{SV}) = (\mathbf{TS})\mathbf{V} \quad (2B5.7)$$

Thus, the tensor product is associative. It is, therefore, natural to define the integral positive powers of a transformation by these simple products, so that

$$\mathbf{T}^2 = \mathbf{TT}, \quad \mathbf{T}^3 = \mathbf{TTT}, \quad \dots \quad (2B5.8)$$

#### Example 2B5.1

(a) Let  $\mathbf{R}$  correspond to a  $90^\circ$  right-hand rigid body rotation about the  $x_3$ -axis. Find the matrix of  $\mathbf{R}$ .

(b) Let  $\mathbf{S}$  correspond to a  $90^\circ$  right-hand rigid body rotation about the  $x_1$ -axis. Find the matrix of  $\mathbf{S}$ .

(c) Find the matrix of the tensor that corresponds to the rotation (a) then (b).

(d) Find the matrix of the tensor that corresponds to the rotation (b) then (a).

(e) Consider a point  $P$  whose initial coordinates are  $(1,1,0)$ . Find the new position of this point after the rotations of part (c). Also find the new position of this point after the rotations of part (d).

*Solution.* (a) For this rotation the transformation of the base vectors is given by

$$\mathbf{R}\mathbf{e}_1 = \mathbf{e}_2$$

$$\mathbf{R}\mathbf{e}_2 = -\mathbf{e}_1$$

$$\mathbf{R}\mathbf{e}_3 = \mathbf{e}_3$$

so that,

$$[\mathbf{R}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) In a similar manner to (a) the transformation of the base vectors is given by

$$\mathbf{S}\mathbf{e}_1 = \mathbf{e}_1$$

$$\mathbf{S}\mathbf{e}_2 = \mathbf{e}_3$$

$$\mathbf{S}\mathbf{e}_3 = -\mathbf{e}_2$$

so that,

$$[\mathbf{S}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

(c) Since  $\mathbf{S}(\mathbf{R}\mathbf{a}) = (\mathbf{SR})\mathbf{a}$ , the resultant rotation is given by the single transformation  $\mathbf{SR}$  whose components are given by the matrix

$$[\mathbf{SR}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

(d) In a manner similar to (c) the resultant rotation is given by the single transformation  $\mathbf{RS}$  whose components are given by the matrix

$$[\mathbf{RS}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

(e) Let  $\mathbf{r}$  be the initial position of the point  $P$ . Let  $\mathbf{r}^*$  and  $\mathbf{r}^{**}$  be the rotated position of  $P$  after the rotations of part (c) and part (d) respectively. Then

$$[\mathbf{r}^*] = [\mathbf{SR}][\mathbf{r}] = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

i.e.,

$$\mathbf{r}^* = -\mathbf{e}_1 + \mathbf{e}_3$$

and

$$[\mathbf{r}^{**}] = [\mathbf{RS}][\mathbf{r}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

i.e.,

$$\mathbf{r}^{**} = \mathbf{e}_2 + \mathbf{e}_3$$

This example further illustrates that the order of rotations is important.

## 2B6 Transpose of a Tensor

The transpose of a tensor  $\mathbf{T}$ , denoted by  $\mathbf{T}^T$ , is defined to be the tensor which satisfies the following identity for all vectors  $\mathbf{a}$  and  $\mathbf{b}$ :

$$\mathbf{a} \cdot \mathbf{T}\mathbf{b} = \mathbf{b} \cdot \mathbf{T}^T\mathbf{a} \quad (2B6.1)$$

It can be easily seen that  $\mathbf{T}^T$  is a tensor. From the above definition, we have

$$\mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j = \mathbf{e}_j \cdot \mathbf{T}^T\mathbf{e}_i$$

Thus,

$$T_{ij} = T_{ji}^T \quad (2B6.2)$$

or

$$[\mathbf{T}^T] = [\mathbf{T}]^T$$

i.e., the matrix of  $\mathbf{T}^T$  is the transpose of the matrix of  $\mathbf{T}$ .

We also note that by Eq. (2B6.1), we have

$$\mathbf{a} \cdot \mathbf{T}^T \mathbf{b} = \mathbf{b} \cdot (\mathbf{T}^T)^T \mathbf{a}$$

Thus,  $\mathbf{b} \cdot \mathbf{T} \mathbf{a} = \mathbf{b} \cdot (\mathbf{T}^T)^T \mathbf{a}$  for any  $\mathbf{a}$  and  $\mathbf{b}$ , so that

$$\mathbf{T} = (\mathbf{T}^T)^T \quad (2B6.3)$$

It can also be established that (see Prob. 2B13)

$$(\mathbf{TS})^T = \mathbf{S}^T \mathbf{T}^T \quad (2B6.4)$$

That is, the transpose of a product of the tensors is equal to the product of transposed tensors in reverse order. More generally,

$$(\mathbf{ABC...D})^T = \mathbf{D}^T \dots \mathbf{C}^T \mathbf{B}^T \mathbf{A}^T \quad (2B6.5)$$

## 2B7 Dyadic Product of Two Vectors

The dyadic product of vectors  $\mathbf{a}$  and  $\mathbf{b}$ , denoted by  $\mathbf{ab}$ , is defined to be the transformation which transforms an arbitrary vector  $\mathbf{c}$  according to the rule:

$$(\mathbf{ab})\mathbf{c} = \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \quad (2B7.1)$$

Now, for any  $\mathbf{c}$ ,  $\mathbf{d}$ ,  $\alpha$  and  $\beta$ , we have, from the above definition:

$$(\mathbf{ab})(\alpha\mathbf{c} + \beta\mathbf{d}) = \mathbf{a}(\mathbf{b} \cdot (\alpha\mathbf{c} + \beta\mathbf{d})) = \mathbf{a}((\alpha\mathbf{b} \cdot \mathbf{c}) + (\beta\mathbf{b} \cdot \mathbf{d})) = \alpha(\mathbf{ab})\mathbf{c} + \beta(\mathbf{ab})\mathbf{d}$$

Thus,  $\mathbf{ab}$  is a tensor. Letting  $\mathbf{W} = \mathbf{ab}$ , then the components of  $\mathbf{W}$  are:

$$W_{ij} = \mathbf{e}_i \cdot \mathbf{W} \mathbf{e}_j = \mathbf{e}_i \cdot (\mathbf{ab}) \mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{a}(\mathbf{b} \cdot \mathbf{e}_j) = a_i b_j$$

i.e.,

$$W_{ij} = a_i b_j \quad (2B7.2a)$$

In matrix notation, Eq. (2B7.2a) is

$$[\mathbf{W}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} [b_1, b_2, b_3] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \quad (2B7.2b)$$

In particular, the components of the dyadic product of the base vectors  $\mathbf{e}_i$  are:

$$[\mathbf{e}_1 \mathbf{e}_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{e}_1 \mathbf{e}_2] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \dots$$

Thus, it is clear that any tensor  $\mathbf{T}$  can be expressed as:

$$\mathbf{T} = T_{11}\mathbf{e}_1\mathbf{e}_1 + T_{12}\mathbf{e}_1\mathbf{e}_2 + \dots + T_{33}\mathbf{e}_3\mathbf{e}_3 \quad (2B7.3a)$$

i.e.,

$$\mathbf{T} = T_{ij}\mathbf{e}_i\mathbf{e}_j \quad (2B7.3b)$$

We note that another commonly used notation for the dyadic product of  $\mathbf{a}$  and  $\mathbf{b}$  is  $\mathbf{a} \otimes \mathbf{b}$ .

## 2B8 Trace of a Tensor

The trace of any dyad  $\mathbf{ab}$  is defined to be a scalar given by  $\mathbf{a} \cdot \mathbf{b}$ . That is,

$$\text{tr } \mathbf{ab} = \mathbf{a} \cdot \mathbf{b} \quad (2B8.1)$$

Furthermore, the trace is defined to be a linear operator that satisfies the relation:

$$\text{tr}(\alpha\mathbf{ab} + \beta\mathbf{cd}) = \alpha \text{tr } \mathbf{ab} + \beta \text{tr } \mathbf{cd} \quad (2B8.2)$$

Using Eq. (2B7.3b), the trace of  $\mathbf{T}$  can, therefore, be obtained as

$$\text{tr } \mathbf{T} = \text{tr}(T_{ij}\mathbf{e}_i\mathbf{e}_j) = T_{ij}\text{tr}(\mathbf{e}_i\mathbf{e}_j) = T_{ij}\mathbf{e}_i \cdot \mathbf{e}_j = T_{ij}\delta_{ij} = T_{ii}$$

that is,

$$\text{tr } \mathbf{T} = T_{ii} = T_{11} + T_{22} + T_{33} = \text{sum of diagonal elements} \quad (2B8.3)$$

It is obvious that

$$\text{tr } \mathbf{T}^T = \text{tr } \mathbf{T} \quad (2B8.4)$$

### Example 2B8.1

Show that for any second-order tensor  $\mathbf{A}$  and  $\mathbf{B}$

$$\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA}) \quad (2B8.5)$$

*Solution.* Let  $\mathbf{C} = \mathbf{AB}$ , then  $C_{ij} = A_{im}B_{mj}$ . Thus,

$$\text{tr } \mathbf{AB} = \text{tr } \mathbf{C} = C_{ii} = A_{im}B_{mi} \quad (i)$$

Let  $\mathbf{D} = \mathbf{BA}$ , then  $D_{ij} = B_{im}A_{mj}$ , and

$$\text{tr } \mathbf{BA} = \text{tr } \mathbf{D} = D_{ii} = B_{im}A_{mi} \quad (ii)$$

But  $B_{im}A_{mi} = B_{mi}A_{im}$  (change of dummy indices), that is

$$\text{tr } \mathbf{BA} = \text{tr } \mathbf{AB} \quad (iii)$$

## 2B9 Identity Tensor and Tensor Inverse

The linear transformation which transforms every vector into itself is called an identity tensor. Denoting this special tensor by  $\mathbf{I}$ , we have, for any vector  $\mathbf{a}$ ,

$$\mathbf{I}\mathbf{a} = \mathbf{a} \quad (2B9.1)$$

and in particular,

$$\mathbf{I}\mathbf{e}_1 = \mathbf{e}_1$$

$$\mathbf{I}\mathbf{e}_2 = \mathbf{e}_2$$

$$\mathbf{I}\mathbf{e}_3 = \mathbf{e}_3$$

Thus, the components of the identity tensor are:

$$I_{ij} = \mathbf{e}_i \cdot \mathbf{I}\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \quad (2B9.2a)$$

i.e.,

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2B9.2b)$$

It is obvious that the identity matrix is the matrix of  $\mathbf{I}$  for *all* rectangular Cartesian coordinates and that  $\mathbf{T}\mathbf{I} = \mathbf{I}\mathbf{T} = \mathbf{T}$  for any tensor  $\mathbf{T}$ . We also note that if  $\mathbf{T}\mathbf{a} = \mathbf{a}$  for any arbitrary  $\mathbf{a}$ , then  $\mathbf{T} = \mathbf{I}$ .

### Example 2B9.1

Write the tensor  $\mathbf{T}$ , defined by the equation  $\mathbf{T}\mathbf{a} = k\mathbf{a}$ , where  $k$  is a constant and  $\mathbf{a}$  is arbitrary, in terms of the identity tensor and find its components.

*Solution.* Using Eq. (2B9.1) we can write  $k\mathbf{a}$  as  $k\mathbf{I}\mathbf{a}$  so that  $\mathbf{T}\mathbf{a} = k\mathbf{a}$  becomes

$$\mathbf{T}\mathbf{a} = k\mathbf{I}\mathbf{a}$$

and since  $\mathbf{a}$  is arbitrary

$$\mathbf{T} = k\mathbf{I}$$

The components of this tensor are clearly,

$$T_{ij} = k\delta_{ij}$$

Given a tensor  $\mathbf{T}$ , if a tensor  $\mathbf{S}$  exists such that  $\mathbf{S}\mathbf{T} = \mathbf{I}$  then we call  $\mathbf{S}$  the inverse of  $\mathbf{T}$  or  $\mathbf{S} = \mathbf{T}^{-1}$ . (Note: With  $\mathbf{T}^{-1}\mathbf{T} = \mathbf{T}^{-1+1} = \mathbf{T}^0 = \mathbf{I}$ , the zeroth power of a tensor is the identity tensor). To find the components of the inverse of a tensor  $\mathbf{T}$  is to find the inverse of the matrix of  $\mathbf{T}$ . From the study of matrices we know that the inverse exists as long as  $\det \mathbf{T} \neq 0$  (that is,  $\mathbf{T}$

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is non-singular) and in this case,  $[\mathbf{T}]^{-1}[\mathbf{T}] = [\mathbf{T}][\mathbf{T}]^{-1} = [\mathbf{I}]$ . Thus, the inverse of a tensor satisfies the following reciprocal relation:

$$\mathbf{T}^{-1}\mathbf{T} = \mathbf{T}\mathbf{T}^{-1} = \mathbf{I} \quad (2B9.3)$$

We can easily show (see Prob. 2B15) that for the tensor inverse the following relations are satisfied,

$$(\mathbf{T}^T)^{-1} = (\mathbf{T}^{-1})^T \quad (2B9.4)$$

and

$$(\mathbf{ST})^{-1} = \mathbf{T}^{-1}\mathbf{S}^{-1} \quad (2B9.5)$$

We note that if the inverse exists then we have the reciprocal relation that

$$\mathbf{T}\mathbf{a} = \mathbf{b} \quad \text{and} \quad \mathbf{a} = \mathbf{T}^{-1}\mathbf{b}$$

This indicates that when a tensor is invertible there is a one to one mapping of vectors  $\mathbf{a}$  and  $\mathbf{b}$ . On the other hand, if a tensor  $\mathbf{T}$  does not have an inverse, then, for a given  $\mathbf{b}$ , there are in general more than one  $\mathbf{a}$  which transforms into  $\mathbf{b}$ . For example, consider the singular tensor  $\mathbf{T} = \mathbf{cd}$  (the dyadic product of  $\mathbf{c}$  and  $\mathbf{d}$ , which does not have an inverse because its determinant is zero), we have

$$\mathbf{T}\mathbf{a} = \mathbf{c}(\mathbf{d} \cdot \mathbf{a}) \equiv \mathbf{b}$$

Now, let  $\mathbf{h}$  be any vector perpendicular to  $\mathbf{d}$  (i.e.,  $\mathbf{d} \cdot \mathbf{h} = 0$ ), then

$$\mathbf{T}(\mathbf{a} + \mathbf{h}) = \mathbf{c}(\mathbf{d} \cdot \mathbf{a}) = \mathbf{b}$$

That is, all vectors  $\mathbf{a} + \mathbf{h}$  transform under  $\mathbf{T}$  into the same vector  $\mathbf{b}$ .

### 2B10 Orthogonal Tensor

An orthogonal tensor is a linear transformation, under which the transformed vectors preserve their lengths and angles. Let  $\mathbf{Q}$  denote an orthogonal tensor, then by definition,  $|\mathbf{Q}\mathbf{a}| = |\mathbf{a}|$  and  $\cos(\mathbf{a}, \mathbf{b}) = \cos(\mathbf{Q}\mathbf{a}, \mathbf{Q}\mathbf{b})$  for any  $\mathbf{a}$  and  $\mathbf{b}$ . Thus,

$$\mathbf{Q}\mathbf{a} \cdot \mathbf{Q}\mathbf{b} = \mathbf{a} \cdot \mathbf{b} \quad (2B10.1)$$

for any  $\mathbf{a}$  and  $\mathbf{b}$ .

Using the definitions of the transpose and the product of tensors:

$$(\mathbf{Q}\mathbf{a}) \cdot (\mathbf{Q}\mathbf{b}) = \mathbf{b} \cdot \mathbf{Q}^T(\mathbf{Q}\mathbf{a}) = \mathbf{b} \cdot (\mathbf{Q}^T\mathbf{Q})\mathbf{a} \quad (i)$$

Therefore,

$$\mathbf{b} \cdot (\mathbf{Q}^T\mathbf{Q})\mathbf{a} = \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{I}\mathbf{a} \quad (ii)$$

Since  $\mathbf{a}$  and  $\mathbf{b}$  are arbitrary, it follows that

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \quad (iii)$$

This means that  $\mathbf{Q}^{-1} = \mathbf{Q}^T$  and from Eq. (2B9.3),

$$\mathbf{Q}^T \mathbf{Q} = \mathbf{Q} \mathbf{Q}^T = \mathbf{I} \quad (2B10.2a)$$

In matrix notation, Eqs. (2B10.2a) take the form:

$$[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{Q}]^T[\mathbf{Q}] = [\mathbf{I}] \quad (2B10.2b)$$

and in subscript notation, these equations take the form:

$$Q_{im}Q_{jm} = Q_{mi}Q_{mj} = \delta_{ij} \quad (2B10.2c)$$

#### Example 2B10.1

The tensor given in Example 2B2.2, being a reflection, is obviously an orthogonal tensor. Verify that  $[\mathbf{T}][\mathbf{T}]^T = [\mathbf{I}]$  for the  $[\mathbf{T}]$  in that example. Also, find the determinant of  $[\mathbf{T}]$ .

*Solution.* Using the matrix of Example 2B7.1:

$$[\mathbf{T}][\mathbf{T}]^T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The determinant of  $[\mathbf{T}]$  is

$$|\mathbf{T}| = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1$$


---

#### Example 2B10.2

The tensor given in Example 2B2.3, being a rigid body rotation, is obviously an orthogonal tensor. Verify that  $[\mathbf{R}][\mathbf{R}]^T = [\mathbf{I}]$  for the  $[\mathbf{R}]$  in that example. Also find the determinant of  $[\mathbf{R}]$ .

*Solution.* It is clear that

$$[\mathbf{R}][\mathbf{R}]^T = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det[\mathbf{R}] \equiv |\mathbf{R}| = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = +1$$


---

The determinant of the matrix of any orthogonal tensor  $\mathbf{Q}$  is easily shown to be equal to either +1 or -1. In fact,

$$[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{I}]$$

therefore,

$$|[Q][Q]^T| = |Q||Q^T| = |I|$$

Now,  $|Q| = |Q^T|$ , and  $|I| = 1$ , therefore,  $|Q|^2 = 1$ , thus

$$|Q| = \pm 1 \quad (2B10.3)$$

From the previous examples we can see that the value of  $+1$  corresponds to *rotation* and  $-1$  corresponds to *reflection*.

### 2B11 Transformation Matrix Between Two Rectangular Cartesian Coordinate Systems.

Suppose  $\{e_i\}$  and  $\{e'_i\}$  are unit vectors corresponding to two rectangular Cartesian coordinate systems (see Fig. 2B.3). It is clear that  $\{e_i\}$  can be made to coincide with  $\{e'_i\}$  through either a rigid body rotation (if both bases are same handed) or a rotation followed by a reflection (if different handed). That is  $\{e_i\}$  and  $\{e'_i\}$  can be related by an orthogonal tensor  $Q$  through the equations

$$e'_i = Qe_i = Q_{mi}e_m \quad (2B11.1a)$$

i.e.,

$$\begin{aligned} e'_1 &= Q_{11}e_1 + Q_{21}e_2 + Q_{31}e_3 \\ e'_2 &= Q_{12}e_1 + Q_{22}e_2 + Q_{32}e_3 \\ e'_3 &= Q_{13}e_1 + Q_{23}e_2 + Q_{33}e_3 \end{aligned} \quad (2B11.1b)$$

where

$$Q_{im}Q_{jm} = Q_{mi}Q_{mj} = \delta_{ij}$$

or

$$QQ^T = Q^TQ = I$$

We note that  $Q_{11} = e_1 \cdot Qe_1 = e_1 \cdot e'_1 = \cosine$  of the angle between  $e_1$  and  $e'_1$ ,  $Q_{12} = e_1 \cdot Qe_2 = e_1 \cdot e'_2 = \cosine$  of the angle between  $e_1$  and  $e'_2$ , etc. In general,  $Q_{ij} = \cosine$  of the angle between  $e_i$  and  $e'_j$  which may be written:

$$Q_{ij} = \cos(e_i, e'_j) = e_i \cdot e'_j \quad (2B11.2)$$

The matrix of these directional cosines, i.e., the matrix

$$[Q] = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix}$$

is called the transformation matrix between  $\{e_i\}$  and  $\{e'_i\}$ . Using this matrix, we shall obtain, in the following sections, the relationship between the two sets of components, with respect to these two sets of base vectors, of either a vector or a tensor.

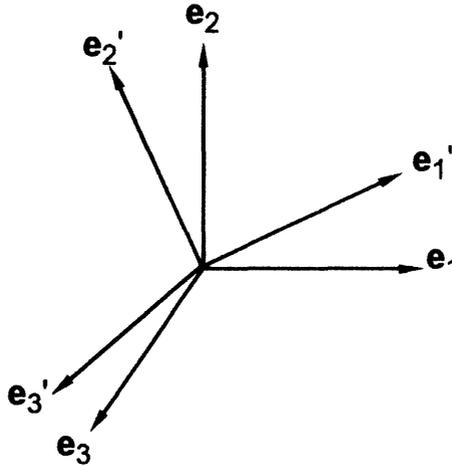


Fig. 2B.3

Example 2B11.1

Let  $\{e'_i\}$  be obtained by rotating the basis  $\{e_i\}$  about the  $e_3$  axis through  $30^\circ$  as shown in Fig. 2B.4. We note that in this figure,  $e_3$  and  $e'_3$  coincide.

*Solution.* We can obtain the transformation matrix in two ways.

(i) Using Eq. (2B11.2), we have

$$Q_{11} = \cos(e_1, e'_1) = \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad Q_{12} = \cos(e_1, e'_2) = \cos 120^\circ = -\frac{1}{2}, \quad Q_{13} = \cos(e_1, e'_3) = \cos 90^\circ = 0$$

$$Q_{21} = \cos(e_2, e'_1) = \cos 60^\circ = \frac{1}{2}, \quad Q_{22} = \cos(e_2, e'_2) = \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad Q_{23} = \cos(e_2, e'_3) = \cos 90^\circ = 0$$

$$Q_{31} = \cos(e_3, e'_1) = \cos 90^\circ = 0, \quad Q_{32} = \cos(e_3, e'_2) = \cos 90^\circ = 0, \quad Q_{33} = \cos(e_3, e'_3) = \cos 0^\circ = 1$$

(ii) It is easier to simply look at Fig. 2B.4 and decompose each of the  $e'_i$ 's into its components in the  $\{e_1, e_2, e_3\}$  directions, i.e.,

$$e'_1 = \frac{\sqrt{3}}{2}e_1 + \frac{1}{2}e_2$$

$$e'_2 = -\frac{1}{2}e_1 + \frac{\sqrt{3}}{2}e_2$$

$$\mathbf{e}'_3 = \mathbf{e}_3$$

Thus, by either method, the transformation matrix is

$$[\mathbf{Q}] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

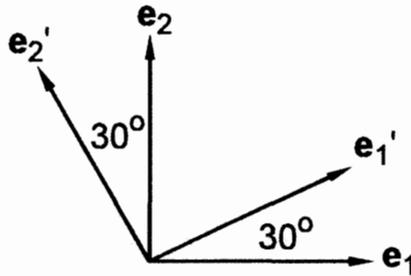


Fig. 2B.4

### 2B12 Transformation Laws for Cartesian Components of Vectors

Consider any vector  $\mathbf{a}$ , then the components of  $\mathbf{a}$  with respect to  $\{\mathbf{e}_i\}$  are

$$a_i = \mathbf{a} \cdot \mathbf{e}_i$$

and its components with respect to  $\{\mathbf{e}'_i\}$  are

$$a'_i = \mathbf{a} \cdot \mathbf{e}'_i$$

Now,  $\mathbf{e}'_i = Q_{mi}\mathbf{e}_m$ , [Eq. (2B11.1a)], therefore,

$$a'_i = \mathbf{a} \cdot Q_{mi}\mathbf{e}_m = Q_{mi}(\mathbf{a} \cdot \mathbf{e}_m)$$

i.e.,

$$a'_i = Q_{mi}a_m \quad (2B12.1a)$$

In matrix notation, Eq. (2B12.1a) is

$$\begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad (2B12.1b)$$

or

$$[\mathbf{a}]' = [\mathbf{Q}]^T[\mathbf{a}] \quad (2B12.1c)$$

Equation (2B12.1) is the transformation law relating components of the same vector with respect to different rectangular Cartesian unit bases. It is very important to note that in Eq. (2B12.1c),  $[\mathbf{a}]'$  denote the matrix of the vector  $\mathbf{a}$  with respect to the primed basis  $\mathbf{e}'_i$  and  $[\mathbf{a}]$  denote that with respect to the unprimed basis  $\mathbf{e}_i$ . Eq. (2B12.1) is not the same as  $\mathbf{a}' = \mathbf{Q}^T \mathbf{a}$ . The distinction is that  $[\mathbf{a}]$  and  $[\mathbf{a}]'$  are matrices of the same vector, where  $\mathbf{a}$  and  $\mathbf{a}'$  are two different vectors;  $\mathbf{a}'$  being the transformed vector of  $\mathbf{a}$  (through the transformation  $\mathbf{Q}^T$ ).

If we premultiply Eq. (2B12.1c) with  $[\mathbf{Q}]$ , we get

$$[\mathbf{a}] = [\mathbf{Q}][\mathbf{a}]' \quad (2B12.2a)$$

The indicial notation equation for Eq.(2B12.2a) is

$$a_i = Q_{im} a'_m \quad (2B12.2b)$$

#### Example 2B12.1

Given that the components of a vector  $\mathbf{a}$  with respect to  $\{\mathbf{e}_i\}$  are given by  $(2,0,0)$ , (i.e.,  $\mathbf{a} = 2\mathbf{e}_1$ ), find its components with respect to  $\{\mathbf{e}'_i\}$ , where the  $\mathbf{e}'_i$  axes are obtained by a  $90^\circ$  counter-clockwise rotation of the  $\mathbf{e}_i$  axes about the  $\mathbf{e}_3$ -axis.

*Solution.* The answer to the question is obvious from Fig. 2B.5, that is

$$\mathbf{a} = 2\mathbf{e}_1 = -2\mathbf{e}'_2$$

We can also obtain the answer by using Eq. (2B12.2a). First we find the transformation matrix. With  $\mathbf{e}'_1 = \mathbf{e}_2$ ,  $\mathbf{e}'_2 = -\mathbf{e}_1$  and  $\mathbf{e}'_3 = \mathbf{e}_3$ , by Eq. (2B11.1b), we have

$$[\mathbf{Q}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus,

$$[\mathbf{a}]' = [\mathbf{Q}]^T[\mathbf{a}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

i.e.,

$$\mathbf{a} = -2\mathbf{e}'_2$$

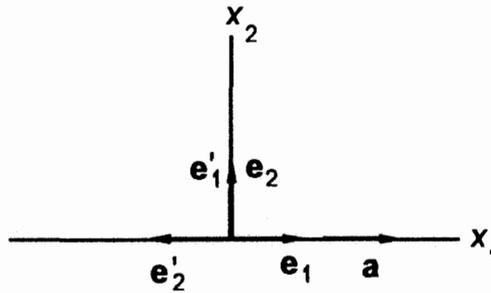


Fig. 2B.5

**2B13 Transformation Law for Cartesian Components of a Tensor**

Consider any tensor  $T$ , then the components of  $T$  with respect to the basis  $\{e_j\}$  are:

$$T_{ij} = e_i \cdot T e_j$$

Its components with respect to  $\{e'_i\}$  are:

$$T'_{ij} = e'_i \cdot T e'_j$$

With  $e'_i = Q_{mi} e_m$ ,

$$T'_{ij} = Q_{mi} e_m \cdot T Q_{nj} e_n = Q_{mi} Q_{nj} (e_m \cdot T e_n)$$

i.e.,

$$T'_{ij} = Q_{mi} Q_{nj} T_{mn} \tag{2B13.1a}$$

In matrix notation, Eq. (2B13.1a) reads

$$\begin{bmatrix} T'_{11} & T'_{12} & T'_{13} \\ T'_{21} & T'_{22} & T'_{23} \\ T'_{31} & T'_{32} & T'_{33} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{21} & Q_{31} \\ Q_{12} & Q_{22} & Q_{32} \\ Q_{13} & Q_{23} & Q_{33} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{21} & Q_{22} & Q_{23} \\ Q_{31} & Q_{32} & Q_{33} \end{bmatrix} \tag{2B13.1b}$$

or

$$[T'] = [Q]^T [T] [Q] \tag{2B13.1c}$$

We can also express the unprimed components in terms of the primed components. Indeed, premultiply Eq. (2B13.1c) with  $[\mathbf{Q}]$  and postmultiply it with  $[\mathbf{Q}]^T$ , we obtain, since  $[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{Q}]^T[\mathbf{Q}] = [\mathbf{I}]$ ,

$$[\mathbf{T}] = [\mathbf{Q}][\mathbf{T}'][\mathbf{Q}]^T \tag{2B13.2a}$$

Using indicial notation, Eq. (2B13.2a) reads

$$T_{ij} = Q_{im}Q_{jn}T'_{mn} \tag{2B13.2b}$$

Equations (2B13.1 & 2B13.2) are the transformation laws relating the components of the same tensor with respect to different Cartesian unit bases. It is important to note that in these equations,  $[\mathbf{T}]$  and  $[\mathbf{T}]'$  are different matrices of the same tensor  $\mathbf{T}$ . We note that the equation  $[\mathbf{T}]' = [\mathbf{Q}]^T[\mathbf{T}][\mathbf{Q}]$  differs from the equation  $\mathbf{T}' = \mathbf{Q}^T\mathbf{T}\mathbf{Q}$  in that the former relates the components of the same tensor  $\mathbf{T}$  whereas the latter relates the two different tensors  $\mathbf{T}$  and  $\mathbf{T}'$ .

### Example 2B13.1

Given the matrix of a tensor  $\mathbf{T}$  in respect to the basis  $\{\mathbf{e}_i\}$ :

$$[\mathbf{T}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find  $[\mathbf{T}]_{\mathbf{e}'_i}$ , i.e., find the matrix of  $\mathbf{T}$  with respect to the  $\{\mathbf{e}'_i\}$  basis, where  $\{\mathbf{e}'_i\}$  is obtained by rotating  $\{\mathbf{e}_i\}$  about  $\mathbf{e}_3$  through  $90^\circ$ . (see Fig. 2B.5).

*Solution.* Since  $\mathbf{e}'_1 = \mathbf{e}_2$ ,  $\mathbf{e}'_2 = -\mathbf{e}_1$  and  $\mathbf{e}'_3 = \mathbf{e}_3$ , by Eq. (2B11.1b), we have

$$[\mathbf{Q}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, Eq. (2B13.1c) gives

$$[\mathbf{T}]' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e.,  $T'_{11} = 2$ ,  $T'_{12} = -1$ ,  $T'_{13} = 0$ ,  $T'_{21} = -1$ , etc.

### Example 2B13.2

Given a tensor  $\mathbf{T}$  and its components  $T_{ij}$  and  $T'_{ij}$  with respect to two sets of bases  $\{\mathbf{e}_i\}$  and  $\{\mathbf{e}'_i\}$ . Show that  $T_{ii}$  is invariant with respect to this change of bases, i.e.,  $T_{ii} = T'_{ii}$ .

## 32 Tensors

*Solution.* The primed components are related to the unprimed components by Eq. (2B13.1a)

$$T'_{ij} = Q_{mi}Q_{nj}T_{mn}$$

Thus,

$$T'_{ii} = Q_{mi}Q_{ni}T_{mn}$$

But,  $Q_{mi}Q_{ni} = \delta_{mn}$  (Eq. (2B10.2c)), therefore,

$$T'_{ii} = \delta_{mn}T_{mn} = T_{mm}$$

i.e.,

$$T'_{11} + T'_{22} + T'_{33} = T_{11} + T_{22} + T_{33}$$


---

We see from Example 2B13.1, that we can calculate all nine components of a tensor  $\mathbf{T}$  with respect to  $\mathbf{e}'_i$  from the matrix  $[\mathbf{T}]_{\mathbf{e}'_i}$ , by using Eq. (2B13.1c). However, there are often times when we need only a few components. Then it is more convenient to use the Eq. (2B2.2) ( $T'_{ij} = \mathbf{e}'_i \cdot \mathbf{T} \mathbf{e}'_j$ ) which defines each of the specific components.

In matrix form this equation is written as:

$$T'_{ij} = [\mathbf{e}'_i]^T [\mathbf{T}] [\mathbf{e}'_j] \quad (2B13.4)$$

where  $[\mathbf{e}'_i]^T$  denotes a row matrix whose elements are the components of  $\mathbf{e}'_i$  with respect to the basis  $\{\mathbf{e}_i\}$ .

### Example 2B13.3

Obtain  $T'_{12}$  for the tensor  $\mathbf{T}$  and the bases  $\mathbf{e}_i$  and  $\mathbf{e}'_i$  given in Example 2B13.1

*Solution.* Since  $\mathbf{e}'_1 = \mathbf{e}_2$ , and  $\mathbf{e}'_2 = -\mathbf{e}_1$ , thus

$$T'_{12} = \mathbf{e}'_1 \cdot \mathbf{T} \mathbf{e}'_2 = \mathbf{e}_2 \cdot \mathbf{T}(-\mathbf{e}_1) = -\mathbf{e}_2 \cdot \mathbf{T} \mathbf{e}_1 = -T_{21} = -1$$

Alternatively, using Eq. (2B13.4)

$$T'_{12} = [\mathbf{e}'_1]^T [\mathbf{T}] [\mathbf{e}'_2] = [0, 1, 0] \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = [0, 1, 0] \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = -1$$

### 2B14 Defining Tensors by Transformation Laws

Equations (2B12.1) or (2B13.1) state that when the components of a vector or a tensor with respect to  $\{\mathbf{e}_i\}$  are known, then its components with respect to any  $\{\mathbf{e}'_i\}$  are uniquely determined from them. In other words, the components  $a_i$  or  $T_{ij}$  with respect to one set of  $\{\mathbf{e}_i\}$

completely characterizes a vector or a tensor. Thus, it is perfectly meaningful to use a statement such as “consider a tensor  $T_{ij}$ ” meaning consider the tensor  $\mathbf{T}$  whose components with respect to some set of  $\{\mathbf{e}_i\}$  are  $T_{ij}$ . In fact, an alternative way of defining a tensor is through the use of transformation laws relating the components of a tensor with respect to different bases. Confining ourselves to only rectangular Cartesian coordinate systems and using unit vectors along positive coordinate directions as base vectors, we now define Cartesian components of tensors of different orders in terms of their transformation laws in the following where the primed quantities are referred to basis  $\{\mathbf{e}'_i\}$  and unprimed quantities to basis  $\{\mathbf{e}_i\}$ , the  $\mathbf{e}'_i$  and  $\mathbf{e}_i$  are related by  $\mathbf{e}'_i = \mathbf{Q}\mathbf{e}_i$ ,  $\mathbf{Q}$  being an orthogonal transformation

$\alpha' = \alpha$	<b>zeroth-order tensor (or scalar)</b>
$a'_i = Q_{mi}a_m$	<b>first-order tensor (or vector)</b>
$T'_{ij} = Q_{mi}Q_{nj}T_{mn}$	<b>second-order tensor (or tensor)</b>
$T'_{ijk} = Q_{mi}Q_{nj}Q_{rk}T_{mnr}$	<b>third-order tensor</b>

etc.

Using the above transformation laws, one can easily establish the following three rules (a) the addition rule (b) the multiplication rule and (c) the quotient rule.

**(a) The addition rule:**

If  $T_{ij}$  and  $S_{ij}$  are components of any two tensors, then  $T_{ij} + S_{ij}$  are components of a tensor. Similarly if  $T_{ijk}$  and  $S_{ijk}$  are components of any two third order tensors, then  $T_{ijk} + S_{ijk}$  are components of a third order tensor.

To prove this rule, we note that since  $T'_{ijk} = Q_{mi}Q_{nj}Q_{rk}T_{mnr}$  and  $S'_{ijk} = Q_{mi}Q_{nj}Q_{rk}S_{mnr}$  we have,

$$T'_{ijk} + S'_{ijk} = Q_{mi}Q_{nj}Q_{rk}T_{mnr} + Q_{mi}Q_{nj}Q_{rk}S_{mnr} = Q_{mi}Q_{nj}Q_{rk}(T_{mnr} + S_{mnr})$$

Letting  $W'_{ijk} = T'_{ijk} + S'_{ijk}$  and  $W_{mnr} = T_{mnr} + S_{mnr}$  we have,

$$W'_{ijk} = Q_{mi}Q_{nj}Q_{rk}W_{mnr}$$

i.e,  $W'_{ijk}$  are components of a third order tensor.

**(b) The multiplication rule:**

Let  $a_i$  be components of any vector and  $T_{ij}$  be components of any tensor. We can form many kinds of products from these components. Examples are (a)  $a_i a_i$  (b)  $a_i a_j a_k$  (c)  $T_{ij} T_{kl}$ , etc. It can be proved that each of these products are components of a tensor, whose order is equal to the number of the free indices. For example,  $a_i a_i$  is a scalar (zeroth order tensor),  $a_i a_j a_k$  are components of a third order tensor,  $T_{ij} T_{kl}$  are components of a fourth order tensor.

To prove that  $T_{ij} T_{kl}$  are components of a fourth-order tensor, let  $M_{ijkl} = T_{ij} T_{kl}$ , then

$$M'_{ijkl} = T'_{ij}T'_{kl} = Q_{mi}Q_{nj}T_{mn}Q_{rk}Q_{sl}T_{rs} = Q_{mi}Q_{nj}Q_{rk}Q_{sl}T_{mn}T_{rs}$$

i.e.,

$$M'_{ijkl} = Q_{mi}Q_{nj}Q_{rk}Q_{sl}M_{mnr}$$

which is the transformation law for a fourth order tensor.

It is quite clear from the proof given above that the order of the tensor whose components are obtained from the multiplication of components of tensors is determined by the number of free indices; no free index corresponds to a scalar, one free index corresponds to a vector, two free indices correspond a second-order tensor, etc.

**(c) The quotient rule:**

If  $a_i$  are components of an arbitrary vector and  $T_{ij}$  are components of an arbitrary tensor and  $a_i = T_{ij}b_j$  for all coordinates, then  $b_i$  are components of a vector. To prove this, we note that since  $a_i$  are components of a vector, and  $T_{ij}$  are components of a second-order tensor, therefore,

$$a_i = Q_{im}a'_m \tag{i}$$

and

$$T_{ij} = Q_{im}Q_{jn}T'_{mn} \tag{ii}$$

Now, substituting Eqs. (i) and (ii) into the equation  $a_i = T_{ij}b_j$ , we have

$$Q_{im}a'_m = Q_{im}Q_{jn}T'_{mn}b'_j \tag{iii}$$

But, the equation  $a_i = T_{ij}b_j$  is true for all coordinates, thus, we also have

$$a'_m = T'_{mn}b'_n \tag{iv}$$

Thus, Eq. (iii) becomes

$$Q_{im}T'_{mn}b'_n = Q_{im}Q_{jn}T'_{mn}b'_j \tag{v}$$

Multiplying the above equation with  $Q_{ik}$  and noting that  $Q_{ik}Q_{im} = \delta_{km}$ , we get

$$T'_{kn}b'_n = Q_{jn}T'_{kn}b'_j \tag{vi}$$

i.e.,

$$T'_{kn}(b'_n - Q_{jn}b'_j) = 0 \tag{vii}$$

Since the above equation is to be true for any tensor  $\mathbf{T}$ , therefore, the parenthesis must be identically zero. Thus,

$$b'_n = Q_{jn}b'_j \tag{viii}$$

This is the transformation law for the components of a vector. Thus,  $b_i$  are components of a vector.

Another example which will be important later when we discuss the relationship between stress and strain for an elastic body is the following: If  $T_{ij}$  and  $E_{ij}$  are components of arbitrary second order tensors  $\mathbf{T}$  and  $\mathbf{E}$  then

$$T_{ij} = C_{ijkl}E_{kl}$$

for all coordinates, then  $C_{ijkl}$  are components of a fourth order tensor. The proof for this example follows that of the previous example.

### 2B15 Symmetric and Antisymmetric Tensors

A tensor is said to be symmetric if  $\mathbf{T} = \mathbf{T}^T$ . Thus, the components of a symmetric tensor have the property,

$$T_{ij} = T_{ij}^T = T_{ji} \tag{2B15.1}$$

i.e.,

$$T_{12} = T_{21}, \quad T_{13} = T_{31}, \quad T_{23} = T_{32}$$

A tensor is said to be antisymmetric if  $\mathbf{T} = -\mathbf{T}^T$ . Thus, the components of an antisymmetric tensor have the property

$$T_{ij} = -T_{ij}^T = -T_{ji} \tag{2B15.2}$$

i.e.,

$$T_{11} = T_{22} = T_{33} = 0$$

and

$$T_{12} = -T_{21}, \quad T_{13} = -T_{31}, \quad T_{23} = -T_{32}.$$

Any tensor  $\mathbf{T}$  can always be decomposed into the sum of a symmetric tensor and an antisymmetric tensor. In fact,

$$\mathbf{T} = \mathbf{T}^S + \mathbf{T}^A \tag{2B15.3}$$

where

$$\mathbf{T}^S = \frac{\mathbf{T} + \mathbf{T}^T}{2} \text{ is symmetric}$$

and

$$\mathbf{T}^A = \frac{\mathbf{T} - \mathbf{T}^T}{2} \text{ is antisymmetric}$$

It is not difficult to prove that the decomposition is unique (see Prob. 2B27)

Example 2B15.1

Show that if  $\mathbf{T}$  is symmetric and  $\mathbf{W}$  is antisymmetric, then  $\text{tr}(\mathbf{TW})=0$ .

*Solution.* We have, [see Example 2B8.4]

$$\text{tr}(\mathbf{TW})=\text{tr}(\mathbf{TW})^T=\text{tr}(\mathbf{W}^T\mathbf{T}^T) \tag{i}$$

Since  $\mathbf{T}$  is symmetric and  $\mathbf{W}$  is antisymmetric, therefore, by definition,  $\mathbf{T}=\mathbf{T}^T$ ,  $\mathbf{W}=-\mathbf{W}^T$ . Thus, (see Example 2B8.1)

$$\text{tr}(\mathbf{TW})=-\text{tr}(\mathbf{WT})=-\text{tr}(\mathbf{TW}) \tag{ii}$$

Consequently,  $2\text{tr}(\mathbf{TW})=0$ . That is,

$$\text{tr}(\mathbf{TW})=0 \tag{iii}$$

**2B16 The Dual Vector of an Antisymmetric Tensor**

The diagonal elements of an antisymmetric tensor are always zero, and, of the six non-diagonal elements, only three are independent, because  $T_{12} = -T_{21}$ ,  $T_{13} = -T_{31}$  and  $T_{23} = -T_{32}$ . Thus, an antisymmetric tensor has really only three components, just like a vector. Indeed, it does behavior like a vector. More specifically, for every antisymmetric tensor  $\mathbf{T}$ , there corresponds a vector  $\mathbf{t}^A$ , such that for every vector  $\mathbf{a}$  the transformed vector,  $\mathbf{Ta}$ , can be obtained from the cross product of  $\mathbf{t}^A$  with  $\mathbf{a}$ . That is,

$$\mathbf{Ta} = \mathbf{t}^A \times \mathbf{a} \tag{2B16.1}$$

This vector,  $\mathbf{t}^A$ , is called the **dual vector** (or **axial vector**) of the antisymmetric tensor. The form of the dual vector is given below:

From Eq.(2B16.1), we have, since  $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c} = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a}$ ,

$$T_{12} = \mathbf{e}_1 \cdot \mathbf{T}\mathbf{e}_2 = \mathbf{e}_1 \cdot \mathbf{t}^A \times \mathbf{e}_2 = \mathbf{t}^A \cdot \mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{t}^A \cdot \mathbf{e}_3 = -t_3^A$$

$$T_{31} = \mathbf{e}_3 \cdot \mathbf{T}\mathbf{e}_1 = \mathbf{e}_3 \cdot \mathbf{t}^A \times \mathbf{e}_1 = \mathbf{t}^A \cdot \mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{t}^A \cdot \mathbf{e}_2 = -t_2^A$$

$$T_{23} = \mathbf{e}_2 \cdot \mathbf{T}\mathbf{e}_3 = \mathbf{e}_2 \cdot \mathbf{t}^A \times \mathbf{e}_3 = \mathbf{t}^A \cdot \mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{t}^A \cdot \mathbf{e}_1 = -t_1^A$$

Similar derivations will give  $T_{21} = t_3^A$ ,  $T_{13} = t_2^A$ ,  $T_{32} = t_1^A$  and  $T_{11} = T_{22} = T_{33} = 0$ . Thus, *only* an antisymmetric tensor has a dual vector defined by Eq.(2B16.1). It is given by:

$$\mathbf{t}^A = -(T_{23}\mathbf{e}_1 + T_{31}\mathbf{e}_2 + T_{12}\mathbf{e}_3) = (T_{32}\mathbf{e}_1 + T_{13}\mathbf{e}_2 + T_{21}\mathbf{e}_3) \tag{2B16.2a}$$

or, in indicial notation

$$2\mathbf{t}^A = -\varepsilon_{ijk}T_{jk}\mathbf{e}_i \tag{2B16.2b}$$

## Example 2B16.1

Given

$$[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

(a) Decompose the tensor into a symmetric and an antisymmetric part.

(b) Find the dual vector for the antisymmetric part.

(c) Verify  $\mathbf{T}^A \mathbf{a} = \mathbf{t}^A \times \mathbf{a}$  for  $\mathbf{a} = \mathbf{e}_1 + \mathbf{e}_3$ .*Solution.* (a)  $[\mathbf{T}] = [\mathbf{T}^S] + [\mathbf{T}^A]$ , where

$$[\mathbf{T}^S] = \frac{[\mathbf{T}] + [\mathbf{T}]^T}{2} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$[\mathbf{T}^A] = \frac{[\mathbf{T}] - [\mathbf{T}]^T}{2} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

(b) The dual vector of  $\mathbf{T}^A$  is

$$\mathbf{t}^A = -(T_{23}^A \mathbf{e}_1 + T_{31}^A \mathbf{e}_2 + T_{12}^A \mathbf{e}_3) = -(0\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3) = \mathbf{e}_2 + \mathbf{e}_3.$$

(c) Let  $\mathbf{b} = \mathbf{T}^A \mathbf{a}$ , then

$$[\mathbf{b}] = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

i.e.,

$$\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$$

On the other hand,

$$\mathbf{t}^A \times \mathbf{a} = (\mathbf{e}_2 + \mathbf{e}_3) \times (\mathbf{e}_1 + \mathbf{e}_3) = -\mathbf{e}_3 + \mathbf{e}_1 + \mathbf{e}_2 = \mathbf{b}$$

## Example 2B16.2

Given that  $\mathbf{R}$  is a rotation tensor and that  $\mathbf{m}$  is a unit vector in the direction of the axis of rotation, prove that the dual vector  $\mathbf{q}$  of  $\mathbf{R}^A$  is parallel to  $\mathbf{m}$ .

*Solution.* Since  $\mathbf{m}$  is parallel to the axis of rotation, therefore,

$$\mathbf{R}\mathbf{m} = \mathbf{m} \tag{i}$$

Thus,  $(\mathbf{R}^T \mathbf{R})\mathbf{m} = \mathbf{R}^T \mathbf{m}$ . Since  $\mathbf{R}^T \mathbf{R} = \mathbf{I}$ , we have

$$\mathbf{R}^T \mathbf{m} = \mathbf{m} \tag{ii}$$

Thus, (i) and (ii) gives

$$(\mathbf{R} - \mathbf{R}^T)\mathbf{m} = \mathbf{0} \tag{iii}$$

But  $(\mathbf{R} - \mathbf{R}^T)\mathbf{m} = 2\mathbf{q} \times \mathbf{m}$ , where  $\mathbf{q}$  is the dual vector of  $\mathbf{R}^A$ . Thus,

$$\mathbf{q} \times \mathbf{m} = \mathbf{0} \tag{iv}$$

i.e.,  $\mathbf{q}$  is parallel to  $\mathbf{m}$ . We note that it can be shown (see Prob. 2B29 or Prob. 2B36) that if  $\theta$  denotes the right-hand rotation angle, then

$$\mathbf{q} = (\sin\theta)\mathbf{m} \tag{2B16.3}$$

### 2B17 Eigenvalues and Eigenvectors of a Tensor

Consider a tensor  $\mathbf{T}$ . If  $\mathbf{a}$  is a vector which transforms under  $\mathbf{T}$  into a vector parallel to itself, i.e.,

$$\mathbf{T}\mathbf{a} = \lambda\mathbf{a} \tag{2B17.1}$$

then  $\mathbf{a}$  is an **eigenvector** and  $\lambda$  is the corresponding **eigenvalue**.

If  $\mathbf{a}$  is an eigenvector with corresponding eigenvalue  $\lambda$  of the linear transformation  $\mathbf{T}$ , then any vector parallel to  $\mathbf{a}$  is also an eigenvector with the same eigenvalue  $\lambda$ . In fact, for any scalar  $\alpha$ ,

$$\mathbf{T}(\alpha\mathbf{a}) = \alpha\mathbf{T}\mathbf{a} = \alpha(\lambda\mathbf{a}) = \lambda(\alpha\mathbf{a}) \tag{i}$$

Thus, an eigenvector, as defined by Eq. (2B17.1), has an arbitrary length. For definiteness, we shall agree that all eigenvectors sought will be of unit length.

A tensor may have infinitely many eigenvectors. In fact, since  $\mathbf{I}\mathbf{a} = \mathbf{a}$ , any vector is an eigenvector for the identity tensor  $\mathbf{I}$ , with eigenvalues all equal to unity. For the tensor  $\beta\mathbf{I}$ , the same is true, except that the eigenvalues are all equal to  $\beta$ .

Some tensors have eigenvectors in only one direction. For example, for any rotation tensor, which effects a rigid body rotation about an axis through an angle not equal to integral multiples of  $\pi$ , only those vectors which are parallel to the axis of rotation will remain parallel to themselves.

Let  $\mathbf{n}$  be a unit eigenvector, then

$$\mathbf{T}\mathbf{n} = \lambda\mathbf{n} = \lambda\mathbf{I}\mathbf{n} \tag{2B17.2}$$

Thus,

$$(\mathbf{T} - \lambda\mathbf{I})\mathbf{n} = \mathbf{0} \tag{2B17.3a}$$

Let  $\mathbf{n} = \alpha_i \mathbf{e}_i$ , then in component form

$$(T_{ij} - \lambda \delta_{ij})\alpha_j = 0 \quad (2B17.3b)$$

In long form, we have

$$\begin{aligned} (T_{11} - \lambda)\alpha_1 + T_{12}\alpha_2 + T_{13}\alpha_3 &= 0 \\ T_{21}\alpha_1 + (T_{22} - \lambda)\alpha_2 + T_{23}\alpha_3 &= 0 \\ T_{31}\alpha_1 + T_{32}\alpha_2 + (T_{33} - \lambda)\alpha_3 &= 0 \end{aligned} \quad (2B17.3c)$$

Equations (2B17.3c) are a system of linear homogeneous equations in  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$ . Obviously, regardless of the values of  $\lambda$ , a solution for this system is  $\alpha_1 = \alpha_2 = \alpha_3 = 0$ . This is known as the **trivial solution**. This solution simply states the obvious fact that  $\mathbf{a} = \mathbf{0}$  satisfies the equation  $\mathbf{T}\mathbf{a} = \lambda\mathbf{a}$ , independent of the value of  $\lambda$ . To find the nontrivial eigenvectors for  $\mathbf{T}$ , we note that a homogeneous system of equations admits nontrivial solution only if the determinant of its coefficients vanishes. That is

$$|\mathbf{T} - \lambda \mathbf{I}| = 0 \quad (2B17.4a)$$

i.e.,

$$\begin{vmatrix} T_{11} - \lambda & T_{12} & T_{13} \\ T_{21} & T_{22} - \lambda & T_{23} \\ T_{31} & T_{32} & T_{33} - \lambda \end{vmatrix} = 0 \quad (2B17.4b)$$

For a given  $\mathbf{T}$ , the above equation is a cubic equation in  $\lambda$ . It is called the **characteristic equation** of  $\mathbf{T}$ . The roots of this **characteristic equation** are the **eigenvalues** of  $\mathbf{T}$ .

Equations (2B17.3), together with the equation

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \quad (2B17.5)$$

allow us to obtain eigenvectors of unit length. The following examples illustrate how eigenvectors and eigenvalues of a tensor can be obtained.

#### Example 2B17.1

If, with respect to some basis  $\{\mathbf{e}_i\}$ , the matrix of  $\mathbf{T}$  is

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

find the eigenvalues and eigenvectors for this tensor.

**Solution.** We note that this tensor is  $2\mathbf{I}$ , so that  $\mathbf{T}\mathbf{a} = 2\mathbf{I}\mathbf{a} = 2\mathbf{a}$ , for any vector  $\mathbf{a}$ . Therefore, by the definition of eigenvector, (see Eq. (2B17.1)), any direction is a direction for an eigenvector. The eigenvalues for all the directions are the same, which is 2. However, we can also

use Eq. (2B17.3) to find the eigenvalues and Eqs. (2B17.4) to find the eigenvectors. Indeed, Eq. (2B17.3) gives, for this tensor the following characteristic equation:

$$(2-\lambda)^3 = 0.$$

So we have a triple root  $\lambda=2$ . Substituting  $\lambda=2$  in Eqs. (2B17.3c), we obtain

$$(2-2)\alpha_1 = 0$$

$$(2-2)\alpha_2 = 0$$

$$(2-2)\alpha_3 = 0$$

Thus, all three equations are automatically satisfied for arbitrary values of  $\alpha_1$ ,  $\alpha_2$ , and  $\alpha_3$ , so that vectors in all directions are eigenvectors. We can choose any three directions as the three independent eigenvectors. In particular, we can choose the basis  $\{\mathbf{e}_i\}$  as a set of linearly independent eigenvectors.

#### Example 2B17.2

Show that if  $T_{21}=T_{31}=0$ , then  $\pm\mathbf{e}_1$  is an eigenvector of  $\mathbf{T}$  with eigenvalue  $T_{11}$ .

*Solution.* From  $\mathbf{T}\mathbf{e}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3$ , we have

$$\mathbf{T}\mathbf{e}_1 = T_{11}\mathbf{e}_1 \text{ and } \mathbf{T}(-\mathbf{e}_1) = T_{11}(-\mathbf{e}_1)$$

Thus, by definition, Eq. (2B17.1),  $\pm\mathbf{e}_1$  are eigenvectors with  $T_{11}$  as its eigenvalue. Similarly, if  $T_{12}=T_{32}=0$ , then  $\pm\mathbf{e}_2$  are eigenvectors with corresponding eigenvalue  $T_{22}$  and if  $T_{13}=T_{23}=0$ , then  $\pm\mathbf{e}_3$  are eigenvectors with corresponding eigenvalue  $T_{33}$ .

#### Example 2B17.3

Given that

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Find the eigenvalues and their corresponding eigenvectors.

*Solution.* The characteristic equation is

$$(2-\lambda)^2(3-\lambda) = 0$$

Thus,  $\lambda_1=3$ ,  $\lambda_2=\lambda_3=2$ . (note the ordering of the eigenvalues is arbitrary). These results are obvious in view of Example 2B17.2. In fact, that example also tells us that the eigenvector corresponding to  $\lambda_1=3$  is  $\mathbf{e}_3$  and eigenvectors corresponding to  $\lambda_2=\lambda_3=2$  are  $\mathbf{e}_1$  and  $\mathbf{e}_2$ . How-

ever, there are actually infinitely many eigenvectors corresponding to the double root. In fact, since  $\mathbf{T}\mathbf{e}_1=2\mathbf{e}_1$  and  $\mathbf{T}\mathbf{e}_2=2\mathbf{e}_2$ , therefore,

$$\mathbf{T}(\alpha\mathbf{e}_1+\beta\mathbf{e}_2) = \alpha\mathbf{T}\mathbf{e}_1+\beta\mathbf{T}\mathbf{e}_2 = 2\alpha\mathbf{e}_1+2\beta\mathbf{e}_2=2(\alpha\mathbf{e}_1+\beta\mathbf{e}_2)$$

i.e.,  $\alpha\mathbf{e}_1+\beta\mathbf{e}_2$  is an eigenvector with eigenvalue 2. This fact can also be obtained from Eqs.(2B17.3c). With  $\lambda=2$  these equations give

$$0\alpha_1=0$$

$$0\alpha_2=0$$

$$\alpha_3=0$$

Thus,  $\alpha_1$  and  $\alpha_2$  are arbitrary and  $\alpha_3=0$  so that any vector perpendicular to  $\mathbf{e}_3$ , i.e.,  $\mathbf{n}=\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2$  is an eigenvector.

#### Example 2B17.4

Find the eigenvalues and eigenvectors for the tensor

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

*Solution.* The characteristic equation gives

$$[\mathbf{T}-\lambda\mathbf{I}] = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 4 \\ 0 & 4 & -3-\lambda \end{bmatrix} = (2-\lambda)(\lambda^2-25) = 0$$

Thus, there are three distinct eigenvalues,  $\lambda_1=2$ ,  $\lambda_2=5$  and  $\lambda_3=-5$ .

Corresponding to  $\lambda_1=2$ , Eqs. (2B17.3c) give

$$0\alpha_1 = 0$$

$$\alpha_2+4\alpha_3 = 0$$

$$4\alpha_2-5\alpha_3 = 0$$

and Eq. (2B17.5) gives

$$\alpha_1^2+\alpha_2^2+\alpha_3^2=1$$

Thus,  $\alpha_2=\alpha_3=0$  and  $\alpha_1=\pm 1$ , so that the eigenvector corresponding to  $\lambda_1=2$  is  $\mathbf{n}_1=\pm\mathbf{e}_1$ . We note that from the Example 2B17.2, this eigenvalue 2 and the corresponding eigenvector  $\pm\mathbf{e}_1$  can be written down by inspection without computation.

Corresponding to  $\lambda_2=5$ , we have

$$3\alpha_1 = 0$$

$$-2\alpha_2 + 4\alpha_3 = 0$$

$$4\alpha_2 - 8\alpha_3 = 0$$

Thus (note the second and third equations are the same),

$$\alpha_1 = 0, \quad \alpha_2 = \pm 2/\sqrt{5}, \quad \alpha_3 = \pm 1/\sqrt{5}$$

and the eigenvector corresponding to  $\lambda_2 = 5$  is

$$\mathbf{n}_2 = \pm \frac{1}{\sqrt{5}}(2\mathbf{e}_2 + \mathbf{e}_3)$$

Corresponding to  $\lambda_3 = -5$ , similar computations give

$$\mathbf{n}_3 = \pm \frac{1}{\sqrt{5}}(-\mathbf{e}_2 + 2\mathbf{e}_3)$$

All the examples given above have three eigenvalues that are real. It can be shown that if a tensor is real (i.e., with real components) and symmetric, then all its eigenvalues are real. If a tensor is real but not symmetric, then two of the eigenvalues may be complex conjugates. The following example illustrates this possibility.

#### Example 2B17.5

Find the eigenvalues and eigenvectors for the rotation tensor  $\mathbf{R}$  corresponding to a  $90^\circ$  rotation about the  $\mathbf{e}_3$ -axis (see Example 2B5.1(a)).

*Solution.* The characteristic equation is

$$\begin{vmatrix} 0-\lambda & -1 & 0 \\ 1 & 0-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

i.e.,

$$\lambda^2(1-\lambda) + (1-\lambda) = (1-\lambda)(\lambda^2 + 1) = 0$$

Thus, only one eigenvalue is real, namely  $\lambda_1 = 1$ , the other two are imaginary,  $\lambda_{2,3} = \pm\sqrt{-1}$ . Correspondingly, there is only one real eigenvector. Only real eigenvectors are of interest to us, we shall therefore compute only the eigenvector corresponding to  $\lambda_1 = 1$ .

From

$$(0-1)\alpha_1 - \alpha_2 = 0$$

$$\alpha_1 - \alpha_2 = 0$$

$$(1-1)\alpha_3 = 0$$

and

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

We obtain  $\alpha_1=0, \alpha_2=0, \alpha_3=\pm 1$ , i.e.,  $\mathbf{n}=\pm \mathbf{e}_3$ , which, of course, is parallel to the axis of rotation.

**2B18 Principal Values and Principal Directions of Real Symmetric tensors**

In the following chapters, we shall encounter several tensors (stress tensor, strain tensor, rate of deformation tensor, etc.) which are symmetric, for which the following theorem, stated without proof, is important: “the eigenvalues of any real symmetric tensor are all real.” Thus, for a real symmetric tensor, there always exist at least three real eigenvectors which we shall also call the **principal directions**. The corresponding eigenvalues are called the **principal values**. We now prove that there always exist three principal directions which are mutually perpendicular.

Let  $\mathbf{n}_1$  and  $\mathbf{n}_2$  be two eigenvectors corresponding to the eigenvalues  $\lambda_1$  and  $\lambda_2$  respectively of a tensor  $\mathbf{T}$ . Then

$$\mathbf{T}\mathbf{n}_1 = \lambda_1\mathbf{n}_1 \tag{i}$$

and

$$\mathbf{T}\mathbf{n}_2 = \lambda_2\mathbf{n}_2 \tag{ii}$$

Thus,

$$\mathbf{n}_2 \cdot \mathbf{T}\mathbf{n}_1 = \lambda_1\mathbf{n}_1 \cdot \mathbf{n}_2 \tag{iii}$$

$$\mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_2 = \lambda_2\mathbf{n}_2 \cdot \mathbf{n}_1 \tag{iv}$$

The definition of the transpose of  $\mathbf{T}$  gives  $\mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_2 = \mathbf{n}_2 \cdot \mathbf{T}^T\mathbf{n}_1$ , thus for a symmetric tensor  $\mathbf{T}$ ,  $\mathbf{T}=\mathbf{T}^T$ , so that  $\mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_2 = \mathbf{n}_2 \cdot \mathbf{T}\mathbf{n}_1$ . Thus, from Eqs. (iii) and (iv), we have

$$(\lambda_1-\lambda_2)(\mathbf{n}_1 \cdot \mathbf{n}_2) = 0 \tag{v}$$

It follows that if  $\lambda_1$  is not equal to  $\lambda_2$ , then  $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$ , i.e.,  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are perpendicular to each other. We have thus proven that if the eigenvalues are all distinct, then the *three principal directions are mutually perpendicular*.

Next, let us suppose that  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are two eigenvectors corresponding to the same eigenvalue  $\lambda$ . Then, by definition,  $\mathbf{T}\mathbf{n}_1 = \lambda\mathbf{n}_1$  and  $\mathbf{T}\mathbf{n}_2 = \lambda\mathbf{n}_2$  so that for any  $\alpha$ , and  $\beta$ ,  $\mathbf{T}(\alpha\mathbf{n}_1+\beta\mathbf{n}_2)=\alpha\mathbf{T}\mathbf{n}_1+\beta\mathbf{T}\mathbf{n}_2=\lambda(\alpha\mathbf{n}_1+\beta\mathbf{n}_2)$ . That is  $\alpha\mathbf{n}_1+\beta\mathbf{n}_2$  is also an eigenvector with the same eigenvalue  $\lambda$ . In other words, if there are two distinct eigenvectors with the same eigenvalue, then, there are infinitely many eigenvectors (which forms a plane) with the same eigenvalue. This situation arises when the characteristic equation has a repeated root. Suppose the characteristic equation has roots  $\lambda_1$  and  $\lambda_2=\lambda_3=\lambda$  ( $\lambda_1$  distinct from  $\lambda$ ). Let  $\mathbf{n}_1$  be the eigenvector corresponding to  $\lambda_1$ , then  $\mathbf{n}_1$  is perpendicular to any eigenvector of  $\lambda$ . Now, corresponding to  $\lambda$ , the equations

$$(T_{11}-\lambda)\alpha_1+T_{12}\alpha_2+T_{13}\alpha_3 = 0 \tag{2B18.1a}$$

$$T_{21}\alpha_1 + (T_{22} - \lambda)\alpha_2 + T_{23}\alpha_3 = 0 \quad (2B18.1b)$$

$$T_{31}\alpha_1 + T_{32}\alpha_2 + (T_{33} - \lambda)\alpha_3 = 0 \quad (2B18.1c)$$

degenerate to one independent equation (see Example 2B17.3) so that there are infinitely many eigenvectors lying on the plane whose normal is  $\mathbf{n}_1$ . Therefore, though not unique, *there again exist three mutually perpendicular principal directions*.

In the case of a triple root, the above three equations will be automatically satisfied for whatever values of  $(\alpha_1, \alpha_2, \alpha_3)$  so that any vector is an eigenvector (see Example 2B17.1).

*Thus, for every real symmetric tensor, there always exists at least one triad of principal directions which are mutually perpendicular.*

### 2B19 Matrix of a Tensor with Respect to Principal Directions

We have shown that for a real symmetric tensor, there always exist three principal directions which are mutually perpendicular. Let  $\mathbf{n}_1, \mathbf{n}_2$  and  $\mathbf{n}_3$  be unit vectors in these directions. Then using  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  as base vectors, the components of the tensor are

$$T_{11} = \mathbf{n}_1 \cdot \mathbf{T} \mathbf{n}_1 = \mathbf{n}_1 \cdot \lambda_1 \mathbf{n}_1 = \lambda_1$$

$$T_{22} = \mathbf{n}_2 \cdot \mathbf{T} \mathbf{n}_2 = \mathbf{n}_2 \cdot \lambda_2 \mathbf{n}_2 = \lambda_2$$

$$T_{33} = \mathbf{n}_3 \cdot \mathbf{T} \mathbf{n}_3 = \mathbf{n}_3 \cdot \lambda_3 \mathbf{n}_3 = \lambda_3$$

$$T_{12} = \mathbf{n}_1 \cdot \mathbf{T} \mathbf{n}_2 = \mathbf{n}_1 \cdot \lambda_2 \mathbf{n}_2 = \lambda_2 (\mathbf{n}_1 \cdot \mathbf{n}_2) = 0 = T_{21}$$

$$T_{13} = \mathbf{n}_1 \cdot \mathbf{T} \mathbf{n}_3 = \mathbf{n}_1 \cdot \lambda_3 \mathbf{n}_3 = \lambda_3 (\mathbf{n}_1 \cdot \mathbf{n}_3) = 0 = T_{31}$$

$$T_{23} = \mathbf{n}_2 \cdot \mathbf{T} \mathbf{n}_3 = \mathbf{n}_2 \cdot \lambda_3 \mathbf{n}_3 = \lambda_3 (\mathbf{n}_2 \cdot \mathbf{n}_3) = 0 = T_{32}$$

That is

$$[\mathbf{T}]_{\mathbf{n}_i} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (2B19.1)$$

Thus, the matrix is diagonal and the diagonal elements are the eigenvalues of  $\mathbf{T}$ .

We now show that the principal values of a tensor  $\mathbf{T}$  include the maximum and minimum values that the diagonal elements of any matrix of  $\mathbf{T}$  can have.

First, for any unit vector  $\mathbf{e}'_1 = \alpha \mathbf{n}_1 + \beta \mathbf{n}_2 + \gamma \mathbf{n}_3$ ,

$$T'_{11} = \mathbf{e}'_1 \cdot \mathbf{T} \mathbf{e}'_1 = [\alpha \ \beta \ \gamma] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

i.e.,

$$T'_{11} = \lambda_1 \alpha^2 + \lambda_2 \beta^2 + \lambda_3 \gamma^2$$

Without loss of generality, let

$$\lambda_1 \geq \lambda_2 \geq \lambda_3$$

then noting that  $\alpha^2 + \beta^2 + \gamma^2 = 1$ , we have

$$\lambda_1 = \lambda_1(\alpha^2 + \beta^2 + \gamma^2) \geq \lambda_1\alpha^2 + \lambda_2\beta^2 + \lambda_3\gamma^2$$

i.e.,

$$\lambda_1 \geq T'_{11}$$

Also,

$$\lambda_1\alpha^2 + \lambda_2\beta^2 + \lambda_3\gamma^2 \geq \lambda_3(\alpha^2 + \beta^2 + \gamma^2) = \lambda_3$$

i.e.,

$$T'_{11} \geq \lambda_3$$

Thus, the  $\left\{ \begin{matrix} \text{maximum} \\ \text{minimum} \end{matrix} \right\}$  value of the principal values of  $\mathbf{T}$  is the  $\left\{ \begin{matrix} \text{maximum} \\ \text{minimum} \end{matrix} \right\}$  value of the diagonal elements of all  $[\mathbf{T}]$  of  $\mathbf{T}$ .

### 2B20 Principal Scalar Invariants of a Tensor

The characteristic equation of a tensor  $\mathbf{T}$ ,  $|T_{ij} - \lambda\delta_{ij}| = 0$  is a cubic equation in  $\lambda$ . It can be written as

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{2B20.1}$$

where

$$I_1 = T_{11} + T_{22} + T_{33} = T_{ii} = \text{tr } \mathbf{T}$$

$$I_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} = \frac{1}{2}(T_{ii}T_{jj} - T_{ij}T_{ji}) = \frac{1}{2}[(\text{tr } \mathbf{T})^2 - \text{tr } (\mathbf{T}^2)]$$

$$I_3 = \begin{vmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{vmatrix} = \det[\mathbf{T}]$$

Since by definition, the eigenvalues of  $\mathbf{T}$  do *not* depend on the choices of the base vectors, therefore the coefficients of Eq. (2B20.1) will not depend on any particular choice of basis. They are called the **principal scalar invariants** of  $\mathbf{T}$ .

We note that, in terms of the eigenvalues of  $\mathbf{T}$  which are the roots of Eq.(2B20.1), the  $I_i$ 's take the simpler form

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3$$

$$I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1$$

$$I_3 = \lambda_1 \lambda_2 \lambda_3 \quad (2B20.2)$$

Example 2B20.1

For the tensor of Example 2B17.4, first find the principal scalar invariants and then evaluate the eigenvalues using Eq. (2B20.1).

*Solution.* The matrix of  $\mathbf{T}$  is

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & -3 \end{bmatrix}$$

$$I_1 = 2 + 3 - 3 = 2$$

$$I_2 = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} + \begin{vmatrix} 2 & 0 \\ 0 & -3 \end{vmatrix} = -25$$

$$I_3 = |\mathbf{T}| = 2 \begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix} = -50$$

These values give the characteristic equation

$$\lambda^3 - 2\lambda^2 - 25\lambda + 50 = 0$$

or,

$$(\lambda - 2)(\lambda - 5)(\lambda + 5) = 0$$

Thus, the eigenvalues are  $\lambda = 2, 5, -5$  as previously determined.

## Part C Tensor Calculus

### 2C1 Tensor-valued functions of a Scalar

Let  $\mathbf{T}=\mathbf{T}(t)$  be a tensor-valued function of a scalar  $t$  (such as time). The derivative of  $\mathbf{T}$  with respect to  $t$  is defined to be a second-order tensor given by

$$\frac{d\mathbf{T}}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{T}(t+\Delta t) - \mathbf{T}(t)}{\Delta t} \quad (2C1.1)$$

The following identities can be easily established [only Eq. (2C1.2d) will be proven here]:

$$\frac{d}{dt}(\mathbf{T}+\mathbf{S}) = \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{S}}{dt} \quad (2C1.2a)$$

$$\frac{d}{dt}(\alpha(t)\mathbf{T}) = \frac{d\alpha}{dt}\mathbf{T} + \alpha \frac{d\mathbf{T}}{dt} \quad (2C1.2b)$$

$$\frac{d}{dt}(\mathbf{T}\mathbf{S}) = \frac{d\mathbf{T}}{dt}\mathbf{S} + \mathbf{T} \frac{d\mathbf{S}}{dt} \quad (2C1.2c)$$

$$\frac{d}{dt}(\mathbf{T}\mathbf{a}) = \frac{d\mathbf{T}}{dt}\mathbf{a} + \mathbf{T} \frac{d\mathbf{a}}{dt} \quad (2C1.2d)$$

$$\frac{d}{dt}(\mathbf{T}^T) = \left( \frac{d\mathbf{T}}{dt} \right)^T \quad (2C1.2e)$$

To prove Eq. (2C1.2d), we use the definition (2C1.1)

$$\begin{aligned} \frac{d}{dt}(\mathbf{T}\mathbf{a}) &= \lim_{\Delta t \rightarrow 0} \frac{\mathbf{T}(t+\Delta t)\mathbf{a}(t+\Delta t) - \mathbf{T}(t)\mathbf{a}(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\mathbf{T}(t+\Delta t)\mathbf{a}(t+\Delta t) - \mathbf{T}(t)\mathbf{a}(t) + \mathbf{T}(t)\mathbf{a}(t+\Delta t) - \mathbf{T}(t)\mathbf{a}(t+\Delta t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{[\mathbf{T}(t+\Delta t) - \mathbf{T}(t)]\mathbf{a}(t+\Delta t)}{\Delta t} + \lim_{\Delta t \rightarrow 0} \frac{\mathbf{T}(t)[\mathbf{a}(t+\Delta t) - \mathbf{a}(t)]}{\Delta t} \end{aligned}$$

Thus,

$$\frac{d}{dt}(\mathbf{T}\mathbf{a}) = \frac{d\mathbf{T}}{dt}\mathbf{a} + \mathbf{T} \frac{d\mathbf{a}}{dt}$$

#### Example 2C1.1

Show that in Cartesian coordinates the components of  $d\mathbf{T}/dt$ , i.e.,  $(d\mathbf{T}/dt)_{ij}$  are given by the derivatives of the components,  $dT_{ij}/dt$ .

*Solution.*

$$T_{ij} = \mathbf{e}_i \cdot \mathbf{T} \mathbf{e}_j \quad (i)$$

Since the base vectors are fixed,

$$\frac{d\mathbf{e}_1}{dt} = \frac{d\mathbf{e}_2}{dt} = \frac{d\mathbf{e}_3}{dt} = 0 \quad (ii)$$

Therefore,

$$\frac{dT_{ij}}{dt} = \mathbf{e}_i \cdot \frac{d}{dt}(\mathbf{T} \mathbf{e}_j) = \mathbf{e}_i \cdot \frac{d\mathbf{T}}{dt} \mathbf{e}_j = \left( \frac{d\mathbf{T}}{dt} \right)_{ij} \quad (iii)$$

### Example 2C1.2

Show that for an orthogonal tensor  $\mathbf{Q}(t)$ ,  $(d\mathbf{Q}/dt)\mathbf{Q}^T$  is an antisymmetric tensor.

*Solution.* Since  $\mathbf{Q}\mathbf{Q}^T = \mathbf{I}$ , we have

$$\mathbf{Q} \frac{d\mathbf{Q}^T}{dt} + \frac{d\mathbf{Q}}{dt} \mathbf{Q}^T = 0 \quad (i)$$

That is

$$\mathbf{Q} \frac{d\mathbf{Q}^T}{dt} = -\frac{d\mathbf{Q}}{dt} \mathbf{Q}^T \quad (ii)$$

Since

$$\frac{d\mathbf{Q}^T}{dt} = \left( \frac{d\mathbf{Q}}{dt} \right)^T \quad [\text{see Eq. (2C1.2e)}]$$

Therefore,

$$\mathbf{Q} \left( \frac{d\mathbf{Q}}{dt} \right)^T = -\frac{d\mathbf{Q}}{dt} \mathbf{Q}^T \quad (iii)$$

But

$$\mathbf{Q} \left( \frac{d\mathbf{Q}}{dt} \right)^T = \left( \frac{d\mathbf{Q}}{dt} \mathbf{Q}^T \right)^T \quad [\text{see Eq. (2B6.4)}]$$

therefore,

$$\left(\frac{d\mathbf{Q}}{dt}\mathbf{Q}^T\right)^T = -\frac{d\mathbf{Q}}{dt}\mathbf{Q}^T \quad (\text{iv})$$

### Example 2C1.3

A time-dependent rigid body rotation about a fixed point can be represented by a rotation tensor  $\mathbf{R}(t)$ , so that a position vector  $\mathbf{r}_o$  is transformed through rotation into  $\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_o$ . Derive the equation

$$\frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r} \quad (\text{i})$$

where  $\boldsymbol{\omega}$  is the dual vector of the antisymmetric tensor  $\frac{d\mathbf{R}}{dt}\mathbf{R}^T$ .

*Solution.* From  $\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_o$

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt}\mathbf{r}_o = \frac{d\mathbf{R}}{dt}\mathbf{R}^T\mathbf{r} \quad (\text{ii})$$

But,  $\frac{d\mathbf{R}}{dt}\mathbf{R}^T$  is an antisymmetric tensor (see Example 2C1.2) so that

$$\frac{d\mathbf{r}}{dt} = \left(\frac{d\mathbf{R}}{dt}\mathbf{R}^T\right)\mathbf{r} = \boldsymbol{\omega} \times \mathbf{r} \quad (\text{iii})$$

where  $\boldsymbol{\omega}$  is the dual vector of  $\frac{d\mathbf{R}}{dt}\mathbf{R}^T$ .

From the well-known equation in rigid body kinematics, we can identify  $\boldsymbol{\omega}$  as the angular velocity of the body.

## 2C2 Scalar Field, Gradient of a Scalar Function

Let  $\phi(\mathbf{r})$  be a scalar-valued function of the position vector  $\mathbf{r}$ . That is, for each position  $\mathbf{r}$ ,  $\phi(\mathbf{r})$  gives the value of a scalar, such as density, temperature or electric potential at the point. In other words,  $\phi(\mathbf{r})$  describes a scalar field. Associated with a scalar field, there is a vector field, called the **gradient** of  $\phi$ , which is of considerable importance. The gradient of  $\phi$  at a point  $\mathbf{r}$  is defined to be a vector, denoted by  $(\text{grad } \phi)$ , or by  $\nabla\phi$  such that its dot product with  $d\mathbf{r}$  gives the difference of the values of the scalar at  $\mathbf{r}+d\mathbf{r}$  and  $\mathbf{r}$ , i.e.,

$$d\phi = \phi(\mathbf{r}+d\mathbf{r}) - \phi(\mathbf{r}) \equiv \nabla\phi \cdot d\mathbf{r} \quad (2C2.1)$$

If  $dr$  denotes the magnitude of  $d\mathbf{r}$ , and  $\mathbf{e}$  the unit vector in the direction of  $d\mathbf{r}$  (note:  $\mathbf{e} = d\mathbf{r}/dr$ ), then the above equation gives, for  $d\mathbf{r}$  in the  $\mathbf{e}$  direction,

$$\frac{d\phi}{dr} = \nabla\phi \cdot \mathbf{e} \quad (2C2.2)$$

That is, the component of  $\nabla\phi$  in the direction of  $\mathbf{e}$  gives the rate of change of  $\phi$  in that direction (the directional derivative). In particular, the components of  $\nabla\phi$  in the  $\mathbf{e}_1$  direction is given by

$$\left(\frac{d\phi}{dr}\right)_{\text{in the } \mathbf{e}_1 \text{ direction}} \equiv \frac{\partial\phi}{\partial x_1} = \nabla\phi \cdot \mathbf{e}_1 = (\nabla\phi)_1$$

Similarly,

$$\left(\frac{d\phi}{dr}\right)_{\text{in the } \mathbf{e}_2 \text{ direction}} \equiv \frac{\partial\phi}{\partial x_2} = \nabla\phi \cdot \mathbf{e}_2 = (\nabla\phi)_2$$

$$\left(\frac{d\phi}{dr}\right)_{\text{in the } \mathbf{e}_3 \text{ direction}} \equiv \frac{\partial\phi}{\partial x_3} = \nabla\phi \cdot \mathbf{e}_3 = (\nabla\phi)_3$$

Therefore, the Cartesian components of  $\nabla\phi$  are  $\frac{\partial\phi}{\partial x_i}$ , that is,

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 \quad (2C2.3)$$

The gradient vector has a simple geometrical interpretation. For example, if  $\phi(\mathbf{r})$  describes a temperature field, then, on a surface of constant temperature (i.e., isothermal surface),  $\phi =$  a constant. Let  $\mathbf{r}$  be a point on this surface. Then, for any and all neighboring point  $\mathbf{r} + d\mathbf{r}$  on the same isothermal surface,  $d\phi = 0$ . Thus,  $\nabla\phi \cdot d\mathbf{r} = 0$ . In other words,  $\nabla\phi$  is a vector, perpendicular to the surface at the point  $\mathbf{r}$ . On the other hand, the dot product  $\nabla\phi \cdot d\mathbf{r}$  is a maximum when  $d\mathbf{r}$  is in the same direction as  $\nabla\phi$ . In other words,  $\nabla\phi$  is greatest if  $d\mathbf{r}$  is normal to the surface of constant  $\phi$  and in this case,

$$\frac{d\phi}{dr} = |\nabla\phi|, \text{ for } d\mathbf{r} \text{ in the normal direction.}$$

#### Example 2C2.1

If  $\phi = x_1x_2 + x_3$ , find a unit vector  $\mathbf{n}$  normal to the surface of a constant  $\phi$  passing through  $(2,1,0)$ .

*Solution.* We have

$$\nabla\phi = \frac{\partial\phi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\phi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\phi}{\partial x_3}\mathbf{e}_3 = x_2\mathbf{e}_1 + x_1\mathbf{e}_2 + \mathbf{e}_3$$

At the point  $(2,1,0)$ ,  $\nabla\phi = \mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$ . Thus,

$$\mathbf{n} = \frac{1}{\sqrt{6}}(\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)$$


---

## Example 2C.2

If  $\mathbf{q}$  denotes the heat flux vector (rate of heat flow/area), the Fourier heat conduction law states that

$$\mathbf{q} = -k\nabla\theta$$

where  $\theta$  is the temperature field and  $k$  is the thermal conductivity. If  $\theta = 2(x_1^2 + x_2^2)$ , find  $\theta$  at  $A(1,0)$  and  $B(1/\sqrt{2}, 1/\sqrt{2})$ . Sketch curves of constant  $\theta$  (isotherms) and indicate the vectors  $\mathbf{q}$  at the two points.

*Solution.* Since,

$$\nabla\theta = \frac{\partial\theta}{\partial x_1}\mathbf{e}_1 + \frac{\partial\theta}{\partial x_2}\mathbf{e}_2 + \frac{\partial\theta}{\partial x_3}\mathbf{e}_3 = 4x_1\mathbf{e}_1 + 4x_2\mathbf{e}_2$$

therefore,

$$\mathbf{q} = -4k(x_1\mathbf{e}_1 + x_2\mathbf{e}_2)$$

At point  $A$ ,

$$\mathbf{q}_A = -4k\mathbf{e}_1$$

and at point  $B$ ,

$$\mathbf{q}_B = -2\sqrt{2}k(\mathbf{e}_1 + \mathbf{e}_2)$$

Clearly, the isotherm, Fig.2C.1, are circles and the heat flux is an inward radial vector.

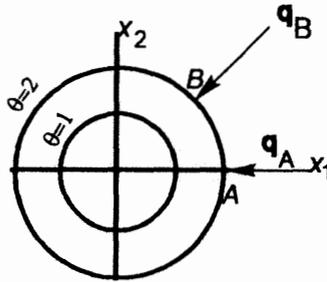


Fig. 2C.1

## Example 2C.3

A more general heat conduction law can be given in the following form:

$$\mathbf{q} = -K\nabla\theta$$

where  $\mathbf{K}$  is a tensor known as thermal conductivity tensor.

(a) What tensor  $\mathbf{K}$  corresponds to the Fourier heat conduction law mentioned in the previous example?

(b) If it is known that  $\mathbf{K}$  is symmetric, show that there are at least three directions in which heat flow is normal to the surface of constant temperature.

(c) If  $\theta = 2x_1 + 3x_2$  and

$$[\mathbf{K}] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

find  $\mathbf{q}$ .

*Solution.*

(a) Clearly,  $\mathbf{K} = k\mathbf{I}$ , so that  $\mathbf{q} = -k\mathbf{I}\nabla\theta = -k\nabla\theta$

(b) For symmetric  $\mathbf{K}$ , we know from Section 2B.18 that there exist at least three principal directions  $\mathbf{n}_1, \mathbf{n}_2$  and  $\mathbf{n}_3$  such that

$$\mathbf{K}\mathbf{n}_1 = k_1\mathbf{n}_1$$

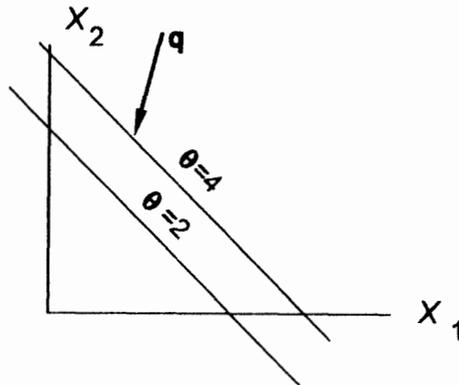


Fig. 2C.2

$$\mathbf{K}\mathbf{n}_2 = k_2\mathbf{n}_2$$

$$\mathbf{K}\mathbf{n}_3 = k_3\mathbf{n}_3$$

where  $k_1, k_2$  and  $k_3$  are eigenvalues of  $\mathbf{K}$ . Thus, for  $\nabla\theta$  in the direction of  $\mathbf{n}_1$ ,

$$\mathbf{q}_1 = -\mathbf{K}\nabla\theta = -\mathbf{K}|\nabla\theta|\mathbf{n}_1 = -|\nabla\theta|\mathbf{K}\mathbf{n}_1 = -k_1|\nabla\theta|\mathbf{n}_1$$

But  $\mathbf{n}_1$ , being in the same direction as  $\nabla\theta$ , is perpendicular to the surface of constant  $\theta$ . Thus,  $\mathbf{q}_1$  is normal to the surface of constant temperature. Similarly,  $\mathbf{q}_2$  is normal to the surface of constant temperature., etc. We note that if  $k_1, k_2$  and  $k_3$  are all distinct, the equations indicate that different thermal conductivities in the three principal directions.

(c) Since  $\theta = 2x_1 + 3x_2$ , we have

$$[\mathbf{q}] = - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 0 \end{bmatrix}$$

i.e.,

$$\mathbf{q} = -\mathbf{e}_1 - 4\mathbf{e}_2$$

which is clearly in a different direction from the normal.

### 2C3 Vector Field, Gradient of a Vector Field

Let  $\mathbf{v}(\mathbf{r})$  be a vector-valued function of position, describing, for example, a displacement or a velocity field. Associated with  $\mathbf{v}(\mathbf{r})$ , there is a tensor field, called the gradient of  $\mathbf{v}$ , which is of considerable importance. The **gradient** of  $\mathbf{v}$  (denoted by  $\nabla\mathbf{v}$  or  $\text{grad } \mathbf{v}$ ) is defined to be the second-order tensor which, when operating on  $d\mathbf{r}$  gives the difference of  $\mathbf{v}$  at  $\mathbf{r} + d\mathbf{r}$  and  $\mathbf{r}$ . That is,

$$d\mathbf{v} = \mathbf{v}(\mathbf{r} + d\mathbf{r}) - \mathbf{v}(\mathbf{r}) \equiv (\nabla\mathbf{v})d\mathbf{r} \tag{2C3.1}$$

Again, let  $d\mathbf{r}$  denote  $|d\mathbf{r}|$  and  $\mathbf{e}$  denote  $d\mathbf{r}/d\mathbf{r}$ , we have

$$\left( \frac{d\mathbf{v}}{d\mathbf{r}} \right)_{\text{in } \mathbf{e} \text{ direction}} = (\nabla\mathbf{v})\mathbf{e} \tag{2C3.2}$$

Thus, the second-order tensor  $(\nabla\mathbf{v})$  transforms the unit vector  $\mathbf{e}$  into the vector describing the rate of change  $\mathbf{v}$  in that direction.

Since

$$\left( \frac{d\mathbf{v}}{d\mathbf{r}} \right)_{\text{in } \mathbf{e}_1 \text{ direction}} \equiv \frac{\partial \mathbf{v}}{\partial x_1} = (\nabla\mathbf{v})\mathbf{e}_1$$

thus, in Cartesian coordinates,

$$(\nabla\mathbf{v})_{11} = \mathbf{e}_1 \cdot (\nabla\mathbf{v})\mathbf{e}_1 = \mathbf{e}_1 \cdot \frac{\partial \mathbf{v}}{\partial x_1} = \frac{\partial}{\partial x_1}(\mathbf{e}_1 \cdot \mathbf{v})$$

That is,

$$(\nabla\mathbf{v})_{11} = \frac{\partial v_1}{\partial x_1}$$

Or, in general

$$\left(\frac{d\mathbf{v}}{dr}\right)_{\text{in } \mathbf{e}_j \text{ direction}} \equiv \frac{\partial \mathbf{v}}{\partial x_j} = (\nabla \mathbf{v})\mathbf{e}_j \tag{2C3.3}$$

thus,

$$(\nabla \mathbf{v})_{ij} = \mathbf{e}_i \cdot (\nabla \mathbf{v})\mathbf{e}_j = \mathbf{e}_i \cdot \frac{\partial \mathbf{v}}{\partial x_j} = \frac{\partial}{\partial x_j}(\mathbf{e}_i \cdot \mathbf{v}) \tag{2C3.4}$$

so that the Cartesian components of  $(\nabla \mathbf{v})$  are

$$(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j} \tag{2C3.5a}$$

That is,

$$[\nabla \mathbf{v}] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \tag{2C3.5b}$$

A geometrical interpretation of  $\nabla \mathbf{v}$  will be given later in connection with the kinematics of deformation.

### 2C4 Divergence of a Vector Field and Divergence of a Tensor Field.

Let  $\mathbf{v}(\mathbf{r})$  be a vector field. The **divergence** of  $\mathbf{v}(\mathbf{r})$  is defined to be a scalar field given by the trace of the gradient of  $\mathbf{v}$ . That is,

$$\text{div} \mathbf{v} \equiv \text{tr}(\nabla \mathbf{v}) \tag{2C4.1}$$

With reference to rectangular Cartesian basis, the diagonal elements of  $\nabla \mathbf{v}$  are  $\frac{\partial v_1}{\partial x_1}$ ,  $\frac{\partial v_2}{\partial x_2}$  and

$\frac{\partial v_3}{\partial x_3}$ . Thus

$$\text{div} \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = \frac{\partial v_m}{\partial x_m} \tag{2C4.2}$$

Let  $\mathbf{T}(\mathbf{r})$  be a second order tensor field. The **divergence** of  $\mathbf{T}$  is defined to be a vector field, denoted by  $\text{div } \mathbf{T}$ , such that for any vector  $\mathbf{a}$

$$(\text{div} \mathbf{T}) \cdot \mathbf{a} \equiv \text{div}(\mathbf{T}^T \mathbf{a}) - \text{tr}(\mathbf{T}^T (\nabla \mathbf{a})) \tag{2C4.3}$$

To find the Cartesian components of the vector  $\text{div } \mathbf{T}$ , let  $\mathbf{b} = \text{div } \mathbf{T}$ , then (note  $\nabla \mathbf{e}_i = 0$  for Cartesian coordinates), from Eq. (2C4.3),

$$b_i = \mathbf{b} \cdot \mathbf{e}_i = \text{div}(\mathbf{T}^T \mathbf{e}_i) - \text{tr}(\mathbf{T}^T \nabla \mathbf{e}_i) = \text{div}(T_{im} \mathbf{e}_m) - 0 = \frac{\partial T_{im}}{\partial x_m} \quad (2C4.4)$$

In other words,

$$\text{div} \mathbf{T} = \frac{\partial T_{im}}{\partial x_m} \mathbf{e}_i \quad (2C4.5)$$

#### Example 2C4.1

If  $\alpha = \alpha(\mathbf{r})$  and  $\mathbf{a} = \mathbf{a}(\mathbf{r})$ , show that  $\text{div}(\alpha \mathbf{a}) = \alpha \text{div} \mathbf{a} + (\nabla \alpha) \cdot \mathbf{a}$ .

*Solution.* Let  $\mathbf{b} = \alpha \mathbf{a}$ . Then  $b_i = \alpha a_i$  and

$$\begin{aligned} \text{div} \mathbf{b} &= \frac{\partial b_i}{\partial x_i} = \alpha \frac{\partial a_i}{\partial x_i} + \frac{\partial \alpha}{\partial x_i} a_i \\ &= \alpha \text{div} \mathbf{a} + (\nabla \alpha) \cdot \mathbf{a} \end{aligned}$$


---

#### Example 2C4.2

Given  $\alpha(\mathbf{r})$  and  $\mathbf{T}(\mathbf{r})$ , show that

$$\text{div}(\alpha \mathbf{T}) = \mathbf{T}(\nabla \alpha) + \alpha \text{div} \mathbf{T}$$

*Solution.* We have, from Eq. (2C4.5),

$$\text{div}(\alpha \mathbf{T}) = \frac{\partial}{\partial x_j} (\alpha T_{ij}) \mathbf{e}_i = \frac{\partial \alpha}{\partial x_j} T_{ij} \mathbf{e}_i + \alpha \frac{\partial T_{ij}}{\partial x_j} \mathbf{e}_i$$

But

$$\frac{\partial \alpha}{\partial x_j} T_{ij} \mathbf{e}_i = \mathbf{T}(\nabla \alpha)$$

and

$$\alpha \frac{\partial T_{ij}}{\partial x_j} \mathbf{e}_i = \alpha \text{div} \mathbf{T}$$

Thus, the desired result follows.

### 2C5 Curl of a Vector Field

Let  $\mathbf{v}(\mathbf{r})$  be a vector field. The **curl** of  $\mathbf{v}$  is defined to be the vector field given by twice the dual vector of the antisymmetric part of  $(\nabla \mathbf{v})$ . That is

$$\text{curl } \mathbf{v} \equiv 2\mathbf{t}^A \quad (2C5.1)$$

where  $\mathbf{t}^A$  is the dual vector of  $(\mathbf{V}\mathbf{v})^A$ .

In a rectangular Cartesian basis,

$$[\mathbf{V}\mathbf{v}]^A = \begin{bmatrix} 0 & \frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \\ -\frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right) & 0 & \frac{1}{2} \left( \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} \right) \\ -\frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) & -\frac{1}{2} \left( \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} \right) & 0 \end{bmatrix}$$

Thus, the curl of  $\mathbf{v}$  is given by [see Eq. (2B16.2)]

$$\text{curl } \mathbf{v} = 2\mathbf{t}^A = \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) \mathbf{e}_1 + \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \mathbf{e}_2 + \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \mathbf{e}_3 \quad (2C5.2)$$

## Part D Curvilinear Coordinates

### 2D1 Polar Coordinates

In this section, the invariant definitions of  $\nabla f$ ,  $\nabla v$ ,  $\text{div}$  and  $\text{div}T$  will be utilized in order to determine their components in plane polar coordinates.

Let  $r, \theta$  denote, see Fig. 2D.1, plane polar coordinates such that

$$r = (x_1^2 + x_2^2)^{1/2}$$

$$\theta = \tan^{-1} \frac{x_2}{x_1}$$

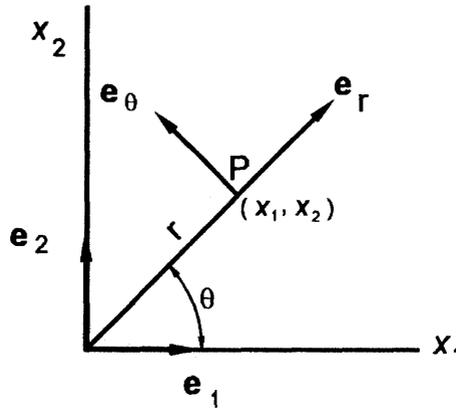


Fig. 2D.1

The unit base vectors  $e_r$  and  $e_\theta$  can be expressed in terms of the Cartesian base vectors  $e_1$  and  $e_2$  as:

$$e_r = \cos\theta e_1 + \sin\theta e_2 \quad (2D1.1a)$$

$$e_\theta = -\sin\theta e_1 + \cos\theta e_2 \quad (2D1.1b)$$

These unit base vectors vary in direction as  $\theta$  changes. In fact, from Eqs. (2D1.1a) and (2D1.1b), it is easily derived that

$$de_r = d\theta e_\theta \quad (2D1.2a)$$

$$de_\theta = -d\theta e_r \quad (2D1.2b)$$

The geometrical representation of  $de_r$  and  $de_\theta$  are shown in the following figure where one notes that  $e_r(P)$  has rotated an infinitesimal angle  $d\theta$  to become  $e_r(Q) = e_r(P) + de_r$ , where  $de_r$  is perpendicular to  $e_r(P)$  with a magnitude  $|de_r| = (1)(d\theta)$ . Similarly  $de_\theta$  is perpendicular to  $e_\theta(P)$  but is pointing in the negative  $e_r$  direction and its magnitude is also  $(1)d\theta$ .

From the position vector  $r = re_r$ , we have

$$dr = dre_r + rde_\theta$$

Using Eq. (2D1.2a), we get

$$dr = dre_r + rd\theta e_\theta \tag{2D1.3}$$

The geometrical representation of this equation is also easily seen if one notes that  $dr$  is the vector  $PQ$  in Fig. 2D.2. The components of  $\nabla f$ ,  $\nabla v$  etc. in polar coordinates will now be obtained.

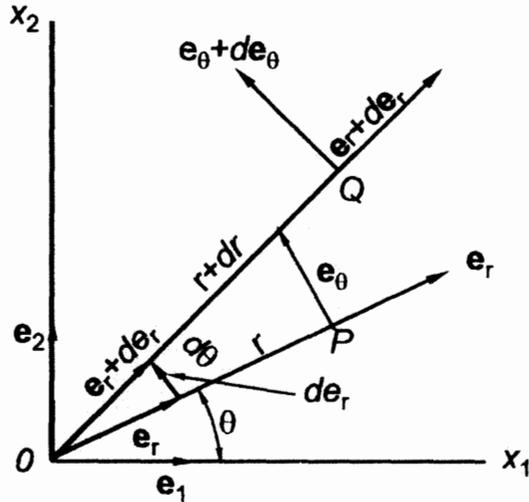


Fig. 2D.2

(i) Components of  $\nabla f$

Let  $f(r, \theta)$  be a scalar field. By definition of the gradient of  $f$ , we have

$$df = \nabla f \cdot dr = [a_r e_r + a_\theta e_\theta] \cdot [dr e_r + r d\theta e_\theta]$$

where  $a_r$  and  $a_\theta$  are components of  $\nabla f$  in the  $e_r$  and  $e_\theta$  direction respectively.

Thus,

$$df = a_r dr + a_\theta r d\theta \quad (2D1.4)$$

But from Calculus,

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta \quad (2D1.5)$$

Since Eqs. (2D1.4) and (2D1.5) must yield the same result for all increments  $dr, d\theta$ , we have

$$a_r = \frac{\partial f}{\partial r} \quad \text{and} \quad r a_\theta = \frac{\partial f}{\partial \theta}$$

Thus,

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta \quad (2D1.6)$$

(ii) *Components of  $\nabla \mathbf{v}$*

Let

$$\mathbf{v}(r, \theta) = v_r(r, \theta) \mathbf{e}_r + v_\theta(r, \theta) \mathbf{e}_\theta \quad (2D1.7)$$

By definition of  $\nabla \mathbf{v}$ , we have

$$d\mathbf{v} \equiv (\nabla \mathbf{v}) dr$$

Let  $\mathbf{T} \equiv \nabla \mathbf{v}$ , then

$$d\mathbf{v} = \mathbf{T} dr = \mathbf{T}(dr \mathbf{e}_r + r d\theta \mathbf{e}_\theta) = dr \mathbf{T} \mathbf{e}_r + r d\theta \mathbf{T} \mathbf{e}_\theta$$

Now,

$$\mathbf{T} \mathbf{e}_r = T_{rr} \mathbf{e}_r + T_{\theta r} \mathbf{e}_\theta \quad \text{and} \quad \mathbf{T} \mathbf{e}_\theta = T_{r\theta} \mathbf{e}_r + T_{\theta\theta} \mathbf{e}_\theta$$

Therefore,

$$d\mathbf{v} = (T_{rr} dr + T_{r\theta} r d\theta) \mathbf{e}_r + (T_{\theta r} dr + T_{\theta\theta} r d\theta) \mathbf{e}_\theta \quad (2D1.8)$$

But from Eq. (2D1.7)

$$d\mathbf{v} = dv_r \mathbf{e}_r + v_r d\mathbf{e}_r + dv_\theta \mathbf{e}_\theta + v_\theta d\mathbf{e}_\theta$$

and from calculus, we have,

$$dv_r = \frac{\partial v_r}{\partial r} dr + \frac{\partial v_r}{\partial \theta} d\theta \quad \text{and} \quad dv_\theta = \frac{\partial v_\theta}{\partial r} dr + \frac{\partial v_\theta}{\partial \theta} d\theta$$

From the above three equations and Eqs. (2D1.2), we have

$$d\mathbf{v} = \left[ \frac{\partial v_r}{\partial r} dr + \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right) d\theta \right] \mathbf{e}_r + \left[ \frac{\partial v_\theta}{\partial r} dr + \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) d\theta \right] \mathbf{e}_\theta \quad (2D1.9)$$

In order that Eqs. (2D1.8) and (2D1.9) agree for all increments  $dr, d\theta$ , we have

$$T_{rr} = \frac{\partial v_r}{\partial r}, \quad T_{r\theta} = \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right), \quad T_{\theta r} = \frac{\partial v_\theta}{\partial r}, \quad T_{\theta\theta} = \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right)$$

In matrix form,

$$[\mathbf{Vv}] = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right) \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) \end{bmatrix} \quad (2D1.10)$$

(iii)  $\text{div v}$

Using the components of  $\mathbf{Vv}$  obtained in (ii), we have

$$\text{div v} = \text{tr}(\mathbf{Vv}) = T_{rr} + T_{\theta\theta} = \frac{\partial v_r}{\partial r} + \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) \quad (2D1.11)$$

(iv)  $\text{curl v}$

From the definition that  $\text{curl v} \equiv$  twice the dual vector of  $(\mathbf{Vv})^A$ , we have

$$\text{curl v} = \left( \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) \mathbf{e}_3 \quad (2D1.12)$$

(v) *Components of  $\text{div T}$*

The definition of the divergence of a second-order tensor is

$$(\text{div T}) \cdot \mathbf{a} = \text{div}(\mathbf{T}^T \mathbf{a}) - \text{tr}((\mathbf{Va})\mathbf{T}^T)$$

for an arbitrary vector  $\mathbf{a}$ .

Take  $\mathbf{a} = \mathbf{e}_r$ , then, the above equation gives

$$(\text{div T})_r = \text{div}(\mathbf{T}^T \mathbf{e}_r) - \text{tr}((\mathbf{Ve}_r)\mathbf{T}^T) \quad (2D1.13)$$

To evaluate the first term on the right hand side, we note that

$$\mathbf{T}^T \mathbf{e}_r = T_{rr} \mathbf{e}_r + T_{r\theta} \mathbf{e}_\theta$$

so that according to Eq. (2D1.11), with  $v_r = T_{rr}$  and  $v_\theta = T_{r\theta}$

$$\text{div}(\mathbf{T}^T \mathbf{e}_r) = \text{div}(T_{rr} \mathbf{e}_r + T_{r\theta} \mathbf{e}_\theta) = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \left( \frac{\partial T_{r\theta}}{\partial \theta} + T_{rr} \right)$$

To evaluate the second term, we first use Eq. (2D1.10) to obtain  $\mathbf{Ve}_r$ . In fact, since  $\mathbf{e}_r = (1)\mathbf{e}_r + 0\mathbf{e}_\theta$ , we have, with  $v_r = 1$  and  $v_\theta = 0$  in Eq. (2D1.10),

$$[\mathbf{Ve}_r] = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix} \quad \text{and} \quad [\mathbf{Ve}_r][\mathbf{T}^T] = \begin{bmatrix} 0 & 0 \\ \frac{T_{r\theta}}{r} & \frac{T_{\theta\theta}}{r} \end{bmatrix}$$

so that  $\text{tr}(\nabla \mathbf{e}_r \mathbf{T}^T) = \frac{T_{\theta\theta}}{r}$ . Thus, from Eq. (2D1.13), we have

$$(\text{div} \mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} \quad (2D1.14)$$

In a similar manner, (see Prob. 2D1), one can derive

$$(\text{div} \mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r} \frac{T_{\theta\theta}}{\partial \theta} + \frac{T_{r\theta} + T_{\theta r}}{r} \quad (2D1.15)$$

## 2D2 Cylindrical Coordinates

In cylindrical coordinates, see Fig. 2D.3, the position of a point  $P$  is determined by  $(r, \theta, z)$  where  $r$  and  $\theta$  determine the position of the vertical projection of the point  $P$  on the  $xy$  plane (the point  $P'$  in the figure) and the coordinate  $z$  determines the height of the point  $P$  from the  $xy$  plane. In other words, the cylindrical coordinates is comprised of polar coordinates  $(r, \theta)$  in the  $xy$  plane plus a coordinate  $z$  perpendicular to the  $xy$  plane.

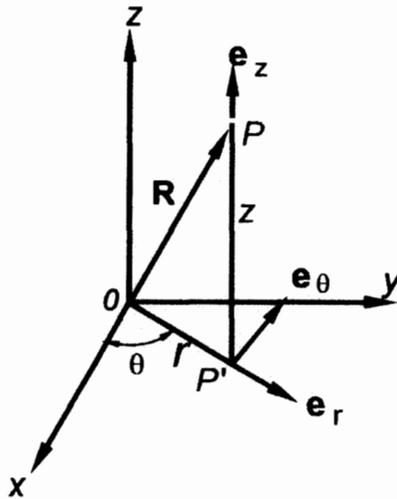


Fig.2D.3

We shall denote the position vector of  $P$  by  $\mathbf{R}$ , rather than  $\mathbf{r}$ , to avoid the possible confusion between the position vector  $\mathbf{R}$  and the coordinate  $r$  (which is a radial distance in the  $xy$  plane). The unit vector  $\mathbf{e}_r$  and  $\mathbf{e}_\theta$  are on the  $xy$  plane and it is clear from the above figure that

$$\mathbf{R} = r\mathbf{e}_r + z\mathbf{e}_z \quad (2D2.1)$$

and

$$d\mathbf{R} = dr\mathbf{e}_r + r d\mathbf{e}_r + dz\mathbf{e}_z + z d\mathbf{e}_z$$

In the above equation,  $d\mathbf{e}_r$  is given by exactly the same equation given earlier for the polar coordinates, i.e., Eq. (2D1.2a). We note also that  $\mathbf{e}_z$  never change its direction or magnitude regardless where the point  $P$  is, thus  $d\mathbf{e}_z = 0$ . Thus,

$$d\mathbf{R} = dr\mathbf{e}_r + r d\theta\mathbf{e}_\theta + dz\mathbf{e}_z \tag{2D2.2}$$

By retracing all the step used in the section on polar coordinates, we can easily obtain the following results:

(i) Components of  $\nabla f$

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z \tag{2D2.3}$$

(ii) Components of  $\nabla \mathbf{v}$

$$[\nabla \mathbf{v}] = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{2D2.4}$$

(iii)  $\text{div } \mathbf{v}$

$$\text{div } \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{\partial v_z}{\partial z} \tag{2D2.5}$$

(iv)  $\text{curl } \mathbf{v}$

$$\text{curl } \mathbf{v} = \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta + \left(\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)\mathbf{e}_z \tag{2D2.6}$$

(v) Components of  $\text{div } \mathbf{T}$

$$(\text{div } \mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} \tag{2D2.7a}$$

$$(\text{div } \mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{T_{r\theta} + T_{\theta r}}{r} + \frac{\partial T_{\theta z}}{\partial z} \tag{2D2.7b}$$

$$(\text{div } \mathbf{T})_z = \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r} \tag{2D2.7c}$$

We note that in dyadic notation,  $\text{div}\mathbf{T}^T$  is written as  $\mathbf{V}\cdot\mathbf{T}$ , so that  $(\text{div}\mathbf{T})_{r\theta} = (\mathbf{V}\cdot\mathbf{T})_{\theta r}$ , etc.

### 2D3 Spherical Coordinates

In Fig. 2D.4a, we show the spherical coordinates  $(r,\theta,\phi)$  of a general point  $P$ . In this figure,  $\mathbf{e}_r, \mathbf{e}_\theta$  and  $\mathbf{e}_\phi$  are unit vectors in the direction of increasing  $r, \theta, \phi$  respectively.

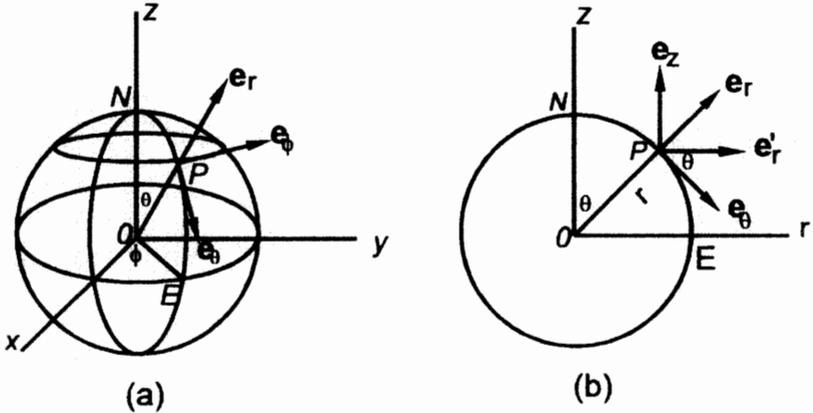


Fig. 2D.4

The position vector for the point  $P$  can be written as

$$\mathbf{r} = r\mathbf{e}_r \tag{2D3.1}$$

where  $r$  is the magnitude of the vector  $\mathbf{r}$ . Thus,

$$d\mathbf{r} = dr\mathbf{e}_r + r d\mathbf{e}_r \tag{2D3.2}$$

To evaluate  $d\mathbf{e}_r$ , we note from Fig. 2D.4b that

$$\mathbf{e}_r = \cos\theta\mathbf{e}_z + \sin\theta\mathbf{e}_{r'} \quad , \quad \mathbf{e}_\theta = \cos\theta\mathbf{e}_{r'} - \sin\theta\mathbf{e}_z \tag{2D3.3}$$

where  $\mathbf{e}_{r'}$  is the unit vector in the  $r'$  ( $OE$ ) direction ( $r'$  is in the  $xy$  plane). Thus,

$$\begin{aligned} d\mathbf{e}_r &= -\sin\theta d\theta\mathbf{e}_z + \cos\theta d\theta\mathbf{e}_{r'} + \sin\theta d\mathbf{e}_{r'} = d\theta(-\sin\theta\mathbf{e}_z + \cos\theta\mathbf{e}_{r'}) + \sin\theta d\mathbf{e}_{r'} \\ &= d\theta\mathbf{e}_\theta + \sin\theta d\mathbf{e}_{r'} \end{aligned} \tag{i}$$

But, just like in polar coordinates, due to  $d\phi$ ,  $d\mathbf{e}_{r'} = (1)d\phi\mathbf{e}_\phi$ , therefore,

$$d\mathbf{e}_r = (d\theta)\mathbf{e}_\theta + (\sin\theta d\phi)\mathbf{e}_\phi \tag{2D3.4a}$$

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Again, from Fig. 2D.4b, we have

$$\mathbf{e}_r' = \cos\theta\mathbf{e}_\theta + \sin\theta\mathbf{e}_r \quad (\text{ii})$$

therefore,

$$d\mathbf{e}_\theta = \cos\theta d\mathbf{e}_r' - \sin\theta d\theta\mathbf{e}_r' - \cos\theta d\theta\mathbf{e}_z = -d\theta(\sin\theta\mathbf{e}_r' + \cos\theta\mathbf{e}_z) + \cos\theta d\mathbf{e}_r' \quad (\text{iii})$$

that is,

$$d\mathbf{e}_\theta = -(d\theta)\mathbf{e}_r + (\cos\theta d\phi)\mathbf{e}_\phi \quad (2D3.4b)$$

From Fig. 2D.4a, it is clear that  $d\mathbf{e}_\phi = d\phi(-\mathbf{e}_r')$ , therefore,

$$d\mathbf{e}_\phi = -(\sin\theta d\phi)\mathbf{e}_r - (\cos\theta d\phi)\mathbf{e}_\theta \quad (2D3.4c)$$

Substituting Eq.(2D3.4a) into Eq.(2D3.2), we have

$$d\mathbf{r} = d\mathbf{r}_e + r(d\theta)\mathbf{e}_\theta + r(\sin\theta d\phi)\mathbf{e}_\phi \quad (2D3.5)$$

We are now in a position to obtain the components of  $\nabla f, \nabla \mathbf{v}$ ,  $\text{div } \mathbf{v}$ ,  $\text{curl } \mathbf{v}$  and  $\text{div } \mathbf{T}$  in spherical coordinates.

(i) *Components of  $\nabla f$*

Let  $(r, \theta, \phi)$  be a scalar field. By the definition of the gradient of  $f$ , we have,

$$df = (\nabla f) \cdot d\mathbf{r} = [(\nabla f)_r \mathbf{e}_r + (\nabla f)_\theta \mathbf{e}_\theta + (\nabla f)_\phi \mathbf{e}_\phi] \cdot [(dr)\mathbf{e}_r + (rd\theta)\mathbf{e}_\theta + (r\sin\theta d\phi)\mathbf{e}_\phi] \quad (2D3.6)$$

i.e.,

$$df = (\nabla f)_r dr + (\nabla f)_\theta r d\theta + (\nabla f)_\phi r \sin\theta d\phi \quad (2D3.7)$$

From calculus, the total derivative of  $f$  is

$$df = \frac{\partial f}{\partial r} dr + \frac{\partial f}{\partial \theta} d\theta + \frac{\partial f}{\partial \phi} d\phi \quad (2D3.8)$$

Comparing Eq. (2D3.7) with Eq. (2D3.8), we obtain

$$(\nabla f)_r = \frac{\partial f}{\partial r} \quad (\nabla f)_\theta = \frac{1}{r} \frac{\partial f}{\partial \theta} \quad (\nabla f)_\phi = \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \quad (2D3.9)$$

(ii) *Components of  $\nabla \mathbf{v}$*

Let the vector field  $\mathbf{v}$  be represented as:

$$\mathbf{v}(r, \theta, \phi) = v_r(r, \theta, \phi)\mathbf{e}_r + v_\theta(r, \theta, \phi)\mathbf{e}_\theta + v_\phi(r, \theta, \phi)\mathbf{e}_\phi \quad (2D3.10)$$

Letting  $\mathbf{T} \equiv \nabla \mathbf{v}$ , we have

$$d\mathbf{v} = \mathbf{T} d\mathbf{r} = \mathbf{T}(dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + r\sin\theta d\phi\mathbf{e}_\phi) = dr\mathbf{T}\mathbf{e}_r + rd\theta\mathbf{T}\mathbf{e}_\theta + r\sin\theta d\phi\mathbf{T}\mathbf{e}_\phi \quad (2D3.11)$$

Now by definition of the components of tensor  $\mathbf{T}$  in spherical coordinates

$$\begin{aligned}
\mathbf{T}_r &= T_{rr}\mathbf{e}_r + T_{r\theta}\mathbf{e}_\theta + T_{r\phi}\mathbf{e}_\phi \\
\mathbf{T}_\theta &= T_{r\theta}\mathbf{e}_r + T_{\theta\theta}\mathbf{e}_\theta + T_{\theta\phi}\mathbf{e}_\phi \\
\mathbf{T}_\phi &= T_{r\phi}\mathbf{e}_r + T_{\theta\phi}\mathbf{e}_\theta + T_{\phi\phi}\mathbf{e}_\phi
\end{aligned} \tag{2D3.12}$$

Substituting this equation into Eq. (2D3.11) and rearranging terms we have

$$\begin{aligned}
d\mathbf{v} &= (T_{rr}dr + rT_{r\theta}d\theta + r\sin\theta T_{r\phi}d\phi)\mathbf{e}_r \\
&\quad + (T_{r\theta}dr + rT_{\theta\theta}d\theta + r\sin\theta T_{\theta\phi}d\phi)\mathbf{e}_\theta \\
&\quad + (T_{r\phi}dr + rT_{\theta\phi}d\theta + r\sin\theta T_{\phi\phi}d\phi)\mathbf{e}_\phi
\end{aligned} \tag{2D3.13}$$

But from Eq. (2D3.10) we have,

$$d\mathbf{v} = dv_r\mathbf{e}_r + v_r d\mathbf{e}_r + dv_\theta\mathbf{e}_\theta + v_\theta d\mathbf{e}_\theta + dv_\phi\mathbf{e}_\phi + v_\phi d\mathbf{e}_\phi \tag{2D3.14}$$

and from calculus we have

$$\begin{aligned}
dv_r &= \frac{\partial v_r}{\partial r}dr + \frac{\partial v_r}{\partial \theta}d\theta + \frac{\partial v_r}{\partial \phi}d\phi \\
dv_\theta &= \frac{\partial v_\theta}{\partial r}dr + \frac{\partial v_\theta}{\partial \theta}d\theta + \frac{\partial v_\theta}{\partial \phi}d\phi \\
dv_\phi &= \frac{\partial v_\phi}{\partial r}dr + \frac{\partial v_\phi}{\partial \theta}d\theta + \frac{\partial v_\phi}{\partial \phi}d\phi
\end{aligned} \tag{2D3.15}$$

Thus, using Eqs. (2D3.15) and Eqs. (2D3.4), Eq. (2D3.14) becomes

$$\begin{aligned}
d\mathbf{v} &= \left[ \frac{\partial v_r}{\partial r}dr + \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right) d\theta + \left( \frac{\partial v_r}{\partial \phi} - v_\phi \sin\theta \right) d\phi \right] \mathbf{e}_r + \\
&\quad \left[ \frac{\partial v_\theta}{\partial r}dr + \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) d\theta + \left( \frac{\partial v_\theta}{\partial \phi} - v_\phi \cos\theta \right) d\phi \right] \mathbf{e}_\theta + \\
&\quad \left[ \frac{\partial v_\phi}{\partial r}dr + \frac{\partial v_\phi}{\partial \theta}d\theta + \left( \frac{\partial v_\phi}{\partial \phi} + v_r \sin\theta + v_\theta \cos\theta \right) d\phi \right] \mathbf{e}_\phi
\end{aligned} \tag{2D3.16}$$

In order that Eqs. (2D3.13) and (2D3.16) agree for all increments  $dr, d\theta, d\phi$ , we have

$$T_{rr} = \frac{\partial v_r}{\partial r}, \quad T_{r\theta} = \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right), \quad \dots$$

which we display in matrix form as

$$\mathbf{\nabla}\mathbf{v} = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right) & \frac{1}{r \sin \theta} \left( \frac{\partial v_r}{\partial \phi} - v_\phi \sin \theta \right) \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) & \frac{1}{r \sin \theta} \left( \frac{\partial v_\theta}{\partial \phi} - v_\phi \cos \theta \right) \\ \frac{\partial v_\phi}{\partial r} & \frac{1}{r} \frac{\partial v_\phi}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \end{bmatrix} \quad (2D3.17)$$

(iii)  $\text{div } \mathbf{v}$

Using the components of  $\mathbf{\nabla}\mathbf{v}$  obtained in (ii), we have

$$\begin{aligned} \text{div } \mathbf{v} = \text{tr}(\mathbf{\nabla}\mathbf{v}) &= \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + 2 \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \\ &= \frac{1}{r^2} \frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(v_\theta \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \end{aligned} \quad (2D3.18)$$

(iv)  $\text{curl } \mathbf{v}$

From the definition of the curl and Eq. (2D3.17) we have

$$\text{curl } \mathbf{v} = \left[ \frac{1}{r \sin \theta} \frac{\partial(v_\phi \sin \theta)}{\partial \theta} - \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right] \mathbf{e}_r + \left[ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{1}{r} \frac{\partial(r v_\phi)}{\partial r} \right] \mathbf{e}_\theta + \left[ \frac{1}{r} \frac{\partial(r v_\theta)}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \mathbf{e}_\phi \quad (2D3.19)$$

(v) *Components of  $\text{div } \mathbf{T}$*

Using the definition of the divergence of a tensor, Eq. (2C4.3), with the vector  $\mathbf{a}$  equal to the unit base vector  $\mathbf{e}_r$  gives

$$(\text{div } \mathbf{T})_r = \text{div}(\mathbf{T}^T \mathbf{e}_r) - \text{tr}((\mathbf{\nabla}\mathbf{e}_r) \mathbf{T}^T) \quad (2D3.20)$$

To evaluate the first term on the right-hand side, we note that

$$\mathbf{T}^T \mathbf{e}_r = T_{rr} \mathbf{e}_r + T_{r\theta} \mathbf{e}_\theta + T_{r\phi} \mathbf{e}_\phi$$

so that according to Eq. (2D3.18), with  $v_r = T_{rr}$ ,  $v_\theta = T_{r\theta}$ ,  $T_\phi = T_{r\phi}$

$$\text{div}(\mathbf{T}^T \mathbf{e}_r) = \frac{1}{r^2} \frac{\partial(r^2 T_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(T_{r\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial(T_{r\phi})}{\partial \phi} \quad (2D3.21)$$

To evaluate the second term on the right-hand side of Eq. (2D3.20) we first use Eq. (2D3.17) with  $\mathbf{v} = \mathbf{e}_r$  to obtain

$$[\mathbf{V}_{\mathbf{e}_r}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{r} & 0 \\ 0 & 0 & \frac{1}{r} \end{bmatrix} \quad \text{and} \quad [\mathbf{V}_{\mathbf{e}_r} \mathbf{T}^T] = \begin{bmatrix} 0 & 0 & 0 \\ \frac{T_{r\theta}}{r} & \frac{T_{\theta\theta}}{r} & \frac{T_{\phi\theta}}{r} \\ \frac{T_{r\phi}}{r} & \frac{T_{\theta\phi}}{r} & \frac{T_{\phi\phi}}{r} \end{bmatrix} \quad (2D3.22)$$

so that

$$\text{tr}(\mathbf{V}_{\mathbf{e}_r} \mathbf{T}^T) = \frac{T_{\theta\theta} + T_{\phi\phi}}{r} \quad (2D3.23)$$

From Eq. (2D3.20), we obtain

$$(\text{div} \mathbf{T})_r = \frac{1}{r^2} \frac{\partial(r^2 T_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(T_{r\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{r\phi}}{\partial \phi} - \frac{T_{\theta\theta} + T_{\phi\phi}}{r} \quad (2D3.24a)$$

In a similar manner, we can obtain (see Prob. 2D9)

$$(\text{div} \mathbf{T})_\theta = \frac{1}{r^3} \frac{\partial(r^3 T_{\theta r})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(T_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{\theta\phi}}{\partial \phi} + \frac{T_{r\theta} - T_{\theta r} - T_{\phi\phi} \cot \theta}{r} \quad (2D3.24b)$$

$$(\text{div} \mathbf{T})_\phi = \frac{1}{r^3} \frac{\partial(r^3 T_{\phi r})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(T_{\phi\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{T_{r\phi} - T_{\phi r} + T_{\theta\phi} \cot \theta}{r} \quad (2D3.24c)$$

## PROBLEMS

2A1. Given

$$[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix} \quad \text{and} \quad [a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

evaluate (a)  $S_{ii}$ , (b)  $S_{ij}S_{ij}$ , (c)  $S_{jk}S_{kj}$ , (d)  $a_m a_m$ , (e)  $S_{mn}a_m a_n$ .2A2. Determine which of these equations have an identical meaning with  $a_i = Q_{ij}a'_j$ 

(a)  $a_p = Q_{pm}a'_m$ ,

(b)  $a_p = Q_{qp}a'_q$ ,

(c)  $a_m = a'_n Q_{mn}$ .

2A3. Given the following matrices

$$[a_i] = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \quad [B_{ij}] = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad [C_{ij}] = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 0 & 2 \\ 2 & 4 & 3 \end{bmatrix}$$

Demonstrate the equivalence of the following subscripted equations and the corresponding matrix equations.

(a)  $D_{ji} = B_{ij} \quad [D] = [B]^T$ ,

(b)  $b_i = B_{ij}a_j \quad [b] = [B][a]$ ,

(c)  $c_j = B_{ji}a_i \quad [c] = [B][a]$ ,

(d)  $s = B_{ij}a_i a_j \quad s = [a]^T [B][a]$ ,

(e)  $D_{ik} = B_{ij}C_{jk} \quad [D] = [B][C]$ ,

(f)  $D_{ik} = B_{ij}C_{kj} \quad [D] = [B][C]^T$ .

2A4. Given that  $T_{ij} = 2\mu E_{ij} + \lambda(E_{kk})\delta_{ij}$ , show that

(a)

$$W = \frac{1}{2}T_{ij}E_{ij} = \mu E_{ij}E_{ij} + \frac{\lambda}{2}(E_{kk})^2$$

(b)

$$P = T_{ij}T_{ij} = 4\mu^2 E_{ij}E_{ij} + (E_{kk})^2(4\mu\lambda + 3\lambda^2)$$

2A5. Given

$$[a_i] = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad [b_i] = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \quad [S_{ij}] = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix}$$

- (a) Evaluate  $[T_{ij}]$  if  $T_{ij} = \varepsilon_{ijk}a_k$   
 (b) Evaluate  $[c_i]$  if  $c_i = \varepsilon_{ijk}S_{jk}$   
 (c) Evaluate  $[d_i]$  if  $d_k = \varepsilon_{ijk}a_i b_j$  and show that this result is the same as  $d_k = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_k$

**2A6.**

- (a) If  $\varepsilon_{ijk}T_{jk} = 0$ , show that  $T_{ij} = T_{ji}$   
 (b) Show that  $\delta_{ij}\varepsilon_{ijk} = 0$

**2A7.** (a) Verify that

$$\varepsilon_{ijm}\varepsilon_{klm} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$$

By contracting the result of part (a) show that

$$(b) \varepsilon_{ilm}\varepsilon_{jlm} = 2\delta_{ij}$$

$$(c) \varepsilon_{ijk}\varepsilon_{ijk} = 6$$

**2A8.** Using the relation of Problem 2A7a, show that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

**2A9.** (a) If  $T_{ij} = -T_{ji}$  show that  $T_{ij}a_i a_j = 0$ 

$$(b) \text{ If } T_{ij} = -T_{ji} \text{ and } S_{ij} = S_{ji}, \text{ show that } T_{kl}S_{kl} = 0$$

**2A10.** Let  $T_{ij} = \frac{1}{2}(S_{ij} + S_{ji})$  and  $R_{ij} = \frac{1}{2}(S_{ij} - S_{ji})$ , show that

$$S_{ij} = T_{ij} + R_{ij}, \quad T_{ij} = T_{ji}, \quad \text{and} \quad R_{ij} = -R_{ji}$$

**2A11.** Let  $f(x_1, x_2, x_3)$  be a function of  $x_i$  and  $v_i(x_1, x_2, x_3)$  represent three functions of  $x_i$ . By expanding the following equations, show that they correspond to the usual formulas of differential calculus.

$$(a) df = \frac{\partial f}{\partial x_i} dx_i$$

$$(b) dv_i = \frac{\partial v_i}{\partial x_j} dx_j$$

**2A12.** Let  $|A_{ij}|$  denote the determinant of the matrix  $[A_{ij}]$ . Show that  $|A_{ij}| = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$ .

**2B1.** A transformation  $\mathbf{T}$  operates on a vector  $\mathbf{a}$  to give  $\mathbf{T}\mathbf{a} = \frac{\mathbf{a}}{|\mathbf{a}|}$ , where  $|\mathbf{a}|$  is the magnitude of  $\mathbf{a}$ . Show that  $\mathbf{T}$  is not a linear transformation.

**2B2.** (a) A tensor  $\mathbf{T}$  transforms every vector  $\mathbf{a}$  into a vector  $\mathbf{T}\mathbf{a} = \mathbf{m} \times \mathbf{a}$ , where  $\mathbf{m}$  is a specified vector. Prove that  $\mathbf{T}$  is a linear transformation.

(b) If  $\mathbf{m} = \mathbf{e}_1 + \mathbf{e}_2$ , find the matrix of the tensor  $\mathbf{T}$

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**2B3.** A tensor  $\mathbf{T}$  transforms the base vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$  so that

$$\mathbf{T}\mathbf{e}_1 = \mathbf{e}_1 + \mathbf{e}_2$$

$$\mathbf{T}\mathbf{e}_2 = \mathbf{e}_1 - \mathbf{e}_2$$

If  $\mathbf{a} = 2\mathbf{e}_1 + 3\mathbf{e}_2$  and  $\mathbf{b} = 3\mathbf{e}_1 + 2\mathbf{e}_2$ , use the linear property of  $\mathbf{T}$  to find

(a)  $\mathbf{T}\mathbf{a}$  (b)  $\mathbf{T}\mathbf{b}$  and (c)  $\mathbf{T}(\mathbf{a} + \mathbf{b})$ .

**2B4.** Obtain the matrix for the tensor  $\mathbf{T}$  which transforms the base vectors as follows:

$$\mathbf{T}\mathbf{e}_1 = 2\mathbf{e}_1 + \mathbf{e}_3$$

$$\mathbf{T}\mathbf{e}_2 = \mathbf{e}_2 + 3\mathbf{e}_3$$

$$\mathbf{T}\mathbf{e}_3 = -\mathbf{e}_1 + 3\mathbf{e}_2$$

**2B5.** Find the matrix of the tensor  $\mathbf{T}$  which transforms any vector  $\mathbf{a}$  into a vector  $\mathbf{b} = \mathbf{m}(\mathbf{a} \cdot \mathbf{n})$  where

$$\mathbf{m} = \frac{\sqrt{2}}{2}(\mathbf{e}_1 + \mathbf{e}_2) \quad \text{and} \quad \mathbf{n} = \frac{\sqrt{2}}{2}(-\mathbf{e}_1 + \mathbf{e}_3)$$

**2B6.** (a) A tensor  $\mathbf{T}$  transforms every vector into its mirror image with respect to the plane whose normal is  $\mathbf{e}_2$ . Find the matrix of  $\mathbf{T}$ .

b) Do part (a) if the plane has a normal in the  $\mathbf{e}_3$  direction instead.

**2B7.** a) Let  $\mathbf{R}$  correspond to a right-hand rotation of angle  $\theta$  about the  $x_1$ -axis. Find the matrix of  $\mathbf{R}$ .

b) Do part (a) if the rotation is about the  $x_2$ -axis.

**2B8.** Consider a plane of reflection which passes through the origin. Let  $\mathbf{n}$  be a unit normal vector to the plane and let  $\mathbf{r}$  be the position vector for a point in space

(a) Show that the reflected vector for  $\mathbf{r}$  is given by  $\mathbf{T}\mathbf{r} = \mathbf{r} - 2(\mathbf{r} \cdot \mathbf{n})\mathbf{n}$ , where  $\mathbf{T}$  is the transformation that corresponds to the reflection.

(b) Let  $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ , find the matrix of the linear transformation  $\mathbf{T}$  that corresponds to this reflection.

(c) Use this linear transformation to find the mirror image of a vector  $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ .

**2B9.** A rigid body undergoes a right hand rotation of angle  $\theta$  about an axis which is in the direction of the unit vector  $\mathbf{m}$ . Let the origin of the coordinates be on the axis of rotation and  $\mathbf{r}$  be the position vector for a typical point in the body.

(a) Show that the rotated vector of  $\mathbf{r}$  is given by  $\mathbf{R}\mathbf{r} = (1 - \cos\theta)(\mathbf{m} \cdot \mathbf{r})\mathbf{m} + \cos\theta\mathbf{r} + \sin\theta\mathbf{m} \times \mathbf{r}$ , where  $\mathbf{R}$  is the transformation that corresponds to the rotation.

(b) Let  $\mathbf{m} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ , find the matrix of the linear transformation that corresponds to this rotation.

(c) Use this linear transformation to find the rotated vector of  $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ .

**2B10.** (a) Find the matrix of the tensor  $\mathbf{S}$  that transforms every vector into its mirror image in a plane whose normal is  $\mathbf{e}_2$  and then by a  $45^\circ$  right-hand rotation about the  $\mathbf{e}_1$ -axis.

(b) Find the matrix of the tensor  $\mathbf{T}$  that transforms every vector by the combination of first the rotation and then the reflection of part (a).

(c) Consider the vector  $\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ , find the transformed vector by using the transformations  $\mathbf{S}$ . Also, find the transformed vector by using the transformation  $\mathbf{T}$ .

**2B11.** a) Let  $\mathbf{R}$  correspond to a right-hand rotation of angle  $\theta$  about the  $x_3$ -axis.

(a) Find the matrix of  $\mathbf{R}^2$ .

(b) Show that  $\mathbf{R}^2$  corresponds to a rotation of angle  $2\theta$  about the same axis.

(c) Find the matrix of  $\mathbf{R}^n$  for any integer  $n$ .

**2B12.** Rigid body rotations that are small can be described by an orthogonal transformation  $\mathbf{R} = \mathbf{I} + \varepsilon\mathbf{R}^*$ , where  $\varepsilon \rightarrow 0$  as the rotation angle approaches zero. Considering two successive rotations  $\mathbf{R}_1$  and  $\mathbf{R}_2$ , show that for small rotations (so that terms containing  $\varepsilon^2$  can be neglected) the final result does not depend on the order of the rotations.

**2B13.** Let  $\mathbf{T}$  and  $\mathbf{S}$  be any two tensors. Show that

(a)  $\mathbf{T}^T$  is a tensor.

(b)  $\mathbf{T}^T + \mathbf{S}^T = (\mathbf{T} + \mathbf{S})^T$

(c)  $(\mathbf{TS})^T = \mathbf{S}^T \mathbf{T}^T$ .

**2B14.** Using the form for the reflection in an arbitrary plane of Prob. 2B8, write the reflection tensor in terms of dyadic products.

**2B15.** For arbitrary tensors  $\mathbf{T}$  and  $\mathbf{S}$ , without relying on the component form, prove that

(a)  $(\mathbf{T}^{-1})^T = (\mathbf{T}^T)^{-1}$ .

(b)  $(\mathbf{TS})^{-1} = \mathbf{S}^{-1} \mathbf{T}^{-1}$ .

**2B16.** Let  $\mathbf{Q}$  define an orthogonal transformation of coordinates, so that  $\mathbf{e}'_i = Q_{mi} \mathbf{e}_m$ . Consider  $\mathbf{e}'_i \cdot \mathbf{e}'_j$  and verify that  $Q_{mi} Q_{mj} = \delta_{ij}$ .

**2B17.** The basis  $\mathbf{e}'_i$  is obtained by a  $30^\circ$  counterclockwise rotation of the  $\mathbf{e}_i$  basis about  $\mathbf{e}_3$ .

(a) Find the orthogonal transformation  $\mathbf{Q}$  that defines this change of basis, i.e.,  $\mathbf{e}'_i = Q_{mi} \mathbf{e}_m$

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(b) By using the vector transformation law, find the components of  $\mathbf{a} = \sqrt{3}\mathbf{e}_1 + \mathbf{e}_2$  in the primed basis (i.e., find  $a'_i$ )

(c) Do part (b) geometrically.

**2B18.** Do the previous problem with  $\mathbf{e}'_i$  obtained by a  $30^\circ$  clockwise rotation of the  $\mathbf{e}_i$ -basis about  $\mathbf{e}_3$ .

**2B19.** The matrix of a tensor  $\mathbf{T}$  in respect to the basis  $\{\mathbf{e}_i\}$  is

$$[\mathbf{T}] = \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

Find  $T'_{11}$ ,  $T'_{12}$  and  $T'_{31}$  in respect to a right-hand basis  $\mathbf{e}'_i$  where  $\mathbf{e}'_1$  is in the direction of  $-\mathbf{e}_2 + 2\mathbf{e}_3$  and  $\mathbf{e}'_2$  is in the direction of  $\mathbf{e}_1$

**2B20.** (a) For the tensor of the previous problem, find  $[T'_{ij}]$  if  $\mathbf{e}'_i$  is obtained by a  $90^\circ$  right-hand rotation about the  $\mathbf{e}_3$ -axis.

(b) Compare both the sum of the diagonal elements and the determinants of  $[\mathbf{T}]$  and  $[\mathbf{T}]'$ .

**2B21.** The dot product of two vectors  $\mathbf{a} = a_i\mathbf{e}_i$  and  $\mathbf{b}_i = b_i\mathbf{e}_i$  is equal to  $a_i b_i$ . Show that the dot product is a scalar invariant with respect to an orthogonal transformation of coordinates.

**2B22.** (a) If  $T_{ij}$  are the components of a tensor, show that  $T_{ij}T_{ij}$  is a scalar invariant with respect to an orthogonal transformation of coordinates.

(b) Evaluate  $T_{ij}T_{ij}$  if in respect to the basis  $\mathbf{e}_i$

$$[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{\mathbf{e}_i}$$

(c) Find  $[\mathbf{T}]'$  if  $\mathbf{e}'_i = \mathbf{Q}\mathbf{e}_i$  and

$$[\mathbf{Q}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{\mathbf{e}_i}$$

(d) Show for this specific  $[\mathbf{T}]$  and  $[\mathbf{T}]'$  that

$$T'_{mn}T'_{mn} = T_{ij}T_{ij}$$

**2B23.** Let  $[\mathbf{T}]$  and  $[\mathbf{T}]'$  be two matrices of the same tensor  $\mathbf{T}$ , show that

$$\det [\mathbf{T}] = \det [\mathbf{T}]'$$

**2B24.** (a) The components of a third-order tensor are  $R_{ijk}$ . Show that  $R_{iik}$  are components of a vector.

(b) Generalize the result of part (a) by considering the components of a tensor of  $n^{\text{th}}$  order  $R_{ijk\dots}$ . Show that  $R_{ijk\dots}$  are components of an  $(n-2)^{\text{th}}$  order tensor.

**2B25.** The components of an arbitrary vector  $\mathbf{a}$  and an arbitrary second-order tensor  $\mathbf{T}$  are related by a triply subscripted quantity  $R_{ijk}$  in the manner  $a_i = R_{ijk}T_{jk}$  for any rectangular Cartesian basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ . Prove that  $R_{ijk}$  are the components of a third-order tensor.

**2B26.** For any vector  $\mathbf{a}$  and any tensor  $\mathbf{T}$ , show that

$$(a) \mathbf{a} \cdot \mathbf{T}^A \mathbf{a} = 0,$$

$$(b) \mathbf{a} \cdot \mathbf{T} \mathbf{a} = \mathbf{a} \cdot \mathbf{T}^S \mathbf{a}.$$

**2B27.** Any tensor may be decomposed into a symmetric and antisymmetric part. Prove that the decomposition is unique. (Hint: Assume that it is not unique.)

**2B28.** Given that a tensor  $\mathbf{T}$  has a matrix

$$[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

(a) find the symmetric and antisymmetric part of  $\mathbf{T}$ .

(b) find the dual vector of the antisymmetric part of  $\mathbf{T}$ .

**2B29** From the result of part (a) of Prob. 2B9, for the rotation about an arbitrary axis  $\mathbf{m}$  by an angle  $\theta$ ,

(a) Show that the rotation tensor is given by  $\mathbf{R} = (1 - \cos\theta)(\mathbf{m}\mathbf{m}) + \sin\theta\mathbf{E}$ , where  $\mathbf{E}$  is the antisymmetric tensor whose dual vector is  $\mathbf{m}$ . [note  $\mathbf{m}\mathbf{m}$  denotes the dyadic product of  $\mathbf{m}$  with  $\mathbf{m}$ ].

(b) Find  $\mathbf{R}^A$ , the antisymmetric part of  $\mathbf{R}$ .

(c) Show that the dual vector for  $\mathbf{R}^A$  is given by  $\sin\theta\mathbf{m}$

**2B30.** Prove that the only possible real eigenvalues of an orthogonal tensor are  $\lambda = \pm 1$ .

**2B31.** Tensors  $\mathbf{T}$ ,  $\mathbf{R}$ , and  $\mathbf{S}$  are related by  $\mathbf{T} = \mathbf{R}\mathbf{S}$ . Tensors  $\mathbf{R}$  and  $\mathbf{S}$  have the same eigenvector  $\mathbf{n}$  and corresponding eigenvalues  $r_1$  and  $s_1$ . Find an eigenvalue and the corresponding eigenvector of  $\mathbf{T}$ .

**2B32.** If  $\mathbf{n}$  is a real eigenvector of an antisymmetric tensor  $\mathbf{T}$ , then show that the corresponding eigenvalue vanishes.

**2B33.** Let  $\mathbf{F}$  be an arbitrary tensor. It can be shown (Polar Decomposition Theorem) that any invertible tensor  $\mathbf{F}$  can be expressed as  $\mathbf{F} = \mathbf{V}\mathbf{Q} = \mathbf{Q}\mathbf{U}$ , where  $\mathbf{Q}$  is an orthogonal tensor and  $\mathbf{U}$  and  $\mathbf{V}$  are symmetric tensors.

(b) Show that  $\mathbf{V}\mathbf{V} = \mathbf{F}\mathbf{F}^T$  and  $\mathbf{U}\mathbf{U} = \mathbf{F}^T\mathbf{F}$ .

(c) If  $\lambda_i$  and  $\mathbf{n}_i$  are the eigenvalues and eigenvectors of  $\mathbf{U}$ , find the eigenvectors and eigenvectors of  $\mathbf{V}$ .

**2B34.** (a) By inspection find an eigenvector of the dyadic product  $\mathbf{ab}$

(b) What vector operation does the first scalar invariant of  $\mathbf{ab}$  correspond to?

(c) Show that the second and the third scalar invariants of  $\mathbf{ab}$  vanish. Show that this indicates that zero is a double eigenvalue of  $\mathbf{ab}$ . What are the corresponding eigenvectors?

**2B35.** A rotation tensor  $\mathbf{R}$  is defined by the relations

$$\mathbf{R}\mathbf{e}_1 = \mathbf{e}_2, \quad \mathbf{R}\mathbf{e}_2 = \mathbf{e}_3, \quad \mathbf{R}\mathbf{e}_3 = \mathbf{e}_1$$

(a) Find the matrix of  $\mathbf{R}$  and verify that  $\mathbf{R}\mathbf{R}^T = \mathbf{I}$  and  $\det|\mathbf{R}| = 1$ .

(b) Find the angle of rotation that could have been used to effect this particular rotation.

**2B36.** For any rotation transformation a basis  $\mathbf{e}'_i$  may be chosen so that  $\mathbf{e}'_3$  is along the axis of rotation.

(a) Verify that for a right-hand rotation angle  $\theta$ , the rotation matrix in respect to the  $\mathbf{e}'_i$  basis is

$$[\mathbf{R}]' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}'_i}$$

(b) Find the symmetric and antisymmetric parts of  $[\mathbf{R}]'$ .

(c) Find the eigenvalues and eigenvectors of  $\mathbf{R}^S$ .

(d) Find the first scalar invariant of  $\mathbf{R}$ .

(e) Find the dual vector of  $\mathbf{R}^A$ .

(f) Use the result of (d) and (e) to find the angle of rotation and the axis of rotation for the previous problem.

**2B37.** (a) If  $\mathbf{Q}$  is an improper orthogonal transformation (corresponding to a reflection), what are the eigenvalues and corresponding eigenvectors of  $\mathbf{Q}$ ?

(b) If the matrix  $\mathbf{Q}$  is

$$[\mathbf{Q}] = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

find the normal to the plane of reflection.

**2B38.** Show that the second scalar invariant of  $\mathbf{T}$  is

$$I_2 = \frac{T_{ii}T_{jj}}{2} - \frac{T_{ij}T_{ji}}{2}$$

by expanding this equation.

**2B39.** Using the matrix transformation law for second-order tensors, show that the third scalar invariant is indeed independent of the particular basis.

**2B40.** A tensor  $\mathbf{T}$  has a matrix

$$[\mathbf{T}] = \begin{bmatrix} 5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Find the scalar invariants, the principle values and corresponding principal directions of the tensor  $\mathbf{T}$ .

(b) If  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  are the principal directions, write  $[\mathbf{T}]_{\mathbf{n}_i}$ .

(c) Could the following matrix represent the tensor  $\mathbf{T}$  in respect to some basis?

$$\begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

**2B41.** Do the previous Problem for the matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

**2B42.** A tensor  $\mathbf{T}$  has a matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Find the principal values and three mutually orthogonal principal directions.

**2B43.** The inertia tensor  $\bar{\mathbf{I}}_o$  of a rigid body with respect to a point  $o$ , is defined by

$$\bar{\mathbf{I}}_o = \int (r^2 \mathbf{I} - \mathbf{r}\mathbf{r}) \rho dV$$

where  $\mathbf{r}$  is the position vector,  $r = |\mathbf{r}|$ ,  $\rho =$  mass density,  $\mathbf{I}$  is the identity tensor, and  $dV$  is a differential volume. The moment of inertia, with respect to an axis pass through  $o$ , is given by  $\bar{I}_{nn} = \mathbf{n} \cdot \bar{\mathbf{I}}_o \mathbf{n}$ , (no sum on  $n$ ), where  $\mathbf{n}$  is a unit vector in the direction of the axis of interest.

(a) Show that  $\bar{\mathbf{I}}_o$  is symmetric.

(b) Letting  $\mathbf{r} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ , write out all components of the inertia tensor  $\bar{\mathbf{I}}_o$ .

(c) The diagonal terms of the inertia matrix are the moments of inertia and the off-diagonal terms the products of inertia. For what axes will the products of inertia be zero? For which axis will the moments of inertia be greatest (or least)?

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Let a coordinate frame  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  be attached to a rigid body which is spinning with an angular velocity  $\boldsymbol{\omega}$ . Then, the angular momentum vector  $\mathbf{H}_c$ , in respect to the mass center, is given by

$$\mathbf{H}_c = \bar{\mathbf{I}}_c \boldsymbol{\omega}$$

and

$$\frac{d\mathbf{e}_i}{dt} = \boldsymbol{\omega} \times \mathbf{e}_i.$$

(d) Let  $\boldsymbol{\omega} = \omega_i \mathbf{e}_i$  and demonstrate that

$$\dot{\boldsymbol{\omega}} = \frac{d\boldsymbol{\omega}}{dt} = \frac{d\omega_i}{dt} \mathbf{e}_i$$

and that

$$\dot{\mathbf{H}}_c = \frac{d}{dt} \mathbf{H}_c = \bar{\mathbf{I}}_c \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\bar{\mathbf{I}}_c \boldsymbol{\omega})$$

**2C1.** Prove the identities (2C1.2a-e) of Section 2C1.

**2C2.** Consider the scalar field defined by  $\phi = x^2 + 3xy + 2z$ .

- (a) Find a unit normal to the surface of constant  $\phi$  at the origin (0,0,0).
- (b) What is the maximum value of the directional derivative of  $\phi$  at the origin?
- (c) Evaluate  $d\phi/dr$  at the origin if  $d\mathbf{r} = ds(\mathbf{e}_1 + \mathbf{e}_3)$ .

**2C3.** Consider the ellipsoid defined by the equation  $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ .

Find the unit normal vector at a given position  $(x, y, z)$ .

**2C4.** Consider a temperature field given by  $\theta = 3xy$ .

- (a) Find the heat flux at the point  $A(1,1,1)$  if  $\mathbf{q} = -k\nabla\theta$ .
- (b) Find the heat flux at the same point as part (a) if  $\mathbf{q} = -\mathbf{K}\nabla\theta$ , where

$$[\mathbf{K}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix}$$

**2C5.** Consider an electrostatic potential given by  $\phi = \alpha[x\cos\theta + y\sin\theta]$ , where  $\alpha$  and  $\theta$  are constants.

- (a) Find the electric field  $\mathbf{E}$  if  $\mathbf{E} = -\nabla\phi$ .
- (b) Find the electric displacement  $\mathbf{D}$  if  $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$ , where the matrix of  $\boldsymbol{\varepsilon}$  is

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{bmatrix}$$

- (c) Find the angle  $\theta$  for which the magnitude of  $\mathbf{D}$  is a maximum.

**2C6.** Let  $\phi(x,y,z)$  and  $\psi(x,y,z)$  be scalar fields, and let  $\mathbf{v}(x,y,z)$  and  $\mathbf{w}(x,y,z)$  be vector fields. By writing the subscripted component form, verify the following identities:

(a)  $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$

Sample solution:

$$[\nabla(\phi+\psi)]_i = \frac{\partial}{\partial x_i}(\phi+\psi) = \frac{\partial\phi}{\partial x_i} + \frac{\partial\psi}{\partial x_i} = (\nabla\phi)_i + (\nabla\psi)_i$$

(b)  $\operatorname{div}(\mathbf{v}+\mathbf{w}) = \operatorname{div}\mathbf{v} + \operatorname{div}\mathbf{w}$ ,

(c)  $\operatorname{div}(\phi\mathbf{v}) = (\nabla\phi) \cdot \mathbf{v} + \phi(\operatorname{div}\mathbf{v})$ ,

(d)  $\operatorname{curl}(\nabla\phi) = 0$ ,

(e)  $\operatorname{div}(\operatorname{curl}\mathbf{v}) = 0$ .

**2C7.** Consider the vector field  $\mathbf{v} = x^2\mathbf{e}_1 + z^2\mathbf{e}_2 + y^2\mathbf{e}_3$ . For the point  $(1, 1, 0)$ :

(a) Find the matrix of  $\nabla\mathbf{v}$ .

(b) Find the vector  $(\nabla\mathbf{v})\mathbf{v}$ .

(c) Find  $\operatorname{div}\mathbf{v}$  and  $\operatorname{curl}\mathbf{v}$ .

(d) if  $d\mathbf{r} = ds(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ , find the differential  $d\mathbf{v}$ .

**2D1.** Obtain Eq. (2D1.15)

**2D2.** Calculate  $\operatorname{div}\mathbf{u}$  for the following vector field in cylindrical coordinates:

(a)  $u_r = u_\theta = 0$ ,  $u_z = A + Br^2$ ,

(b)  $u_r = \frac{\sin\theta}{r}$ ,  $u_\theta = 0$ ,  $u_z = 0$ ,

(c)  $u_r = \frac{1}{2}\sin\theta r^2$ ,  $u_\theta = \frac{1}{2}\cos\theta r^2$ ,  $u_z = 0$ ,

(d)  $u_r = \frac{\sin\theta}{r^2}$ ,  $u_\theta = -\frac{\cos\theta}{r^2}$ ,  $u_z = 0$ .

**2D3.** Calculate  $\operatorname{div}\mathbf{u}$  for the following vector field in spherical coordinates:

$$u_r = Ar + \frac{B}{r^2}, \quad u_\theta = u_\phi = 0$$

**2D4.** Calculate  $\nabla\mathbf{u}$  for the following vector field in cylindrical coordinate

$$u_r = \frac{A}{r}, \quad u_\theta = Br, \quad v_z = 0$$

**2D5.** Calculate  $\nabla\mathbf{u}$  for the following vector field in spherical coordinate

$$u_r = Ar + \frac{B}{r^2}, \quad u_\theta = u_\phi = 0$$

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2D6. Calculate  $\text{div } \mathbf{T}$  for the following tensor field in cylindrical coordinates:

$$T_{rr} = \frac{Az}{R^3} - \frac{3r^2z}{R^5}, \quad T_{\theta\theta} = \frac{Az}{R^3}, \quad T_{zz} = -\left[\frac{Az}{R^3} + \frac{3z^3}{R^5}\right], \quad T_{rz} = -\left[\frac{Ar}{R^3} + \frac{3rz^2}{R^5}\right]$$

$$T_{z\theta} = T_{r\theta} = 0, \quad \text{where } R^2 = r^2 + z^2$$

2D7. Calculate  $\text{div } \mathbf{T}$  for the following tensor field in cylindrical coordinates:

$$T_{rr} = A + \frac{B}{r^2}, \quad T_{\theta\theta} = A - \frac{B}{r^2}, \quad T_{zz} = \text{constant}, \quad T_{r\theta} = T_{rz} = T_{\theta z} = 0$$

2D8. Calculate  $\text{div } \mathbf{T}$  for the following tensor field in spherical coordinates:

$$T_{rr} = A - \frac{2B}{r^3}, \quad T_{\theta\theta} = T_{\phi\phi} = A + \frac{B}{r^3}$$

$$T_{\theta r} = T_{\phi r} = T_{\phi\theta} = 0$$

2D9. Derive Eq. (2D3.24b) and Eq. (2D3.24c).

### 3

## Kinematics of a Continuum

The branch of mechanics in which materials are treated as continuous is known as **continuum mechanics**. Thus, in this theory, one speaks of an infinitesimal volume of material, the totality of which forms a **body**. One also speaks of a **particle** in a continuum, meaning, in fact an infinitesimal volume of material. This chapter is concerned with the kinematics of such particles.

### 3.1 Description of Motions of a Continuum

In particle kinematics, the path line of a particle is described by a vector function of time, i.e.,

$$\mathbf{r} = \mathbf{r}(t) \quad (\text{ia})$$

where  $\mathbf{r}(t) = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2 + z(t)\mathbf{e}_3$  is the position vector. In component form, the above equation reads:

$$x = x(t), \quad y = y(t) \quad \text{and} \quad z = z(t) \quad (\text{ib})$$

If there are  $N$  particles, there are  $N$  pathlines, each is described by one of the equations:

$$\mathbf{r}_n = \mathbf{r}_n(t), \quad n = 1, 2, \dots, N \quad (\text{ii})$$

That is, for the particle number 1, the path line is given by  $\mathbf{r}_1(t)$ , for the particle number 2, it is given by  $\mathbf{r}_2(t)$ , etc.

For a continuum, there are not only infinitely many particles, but within each and every neighborhood of a particle there are infinitely many other particles. Therefore, it is not possible to identify particles by assigning each of them a number in the same way as in the kinematics of particles. However, it is possible to identify them by the positions they occupy at some reference time  $t_0$ . For example, if a particle of a continuum was at the position (1,2,3) at the reference time  $t_0$ , the set of coordinates (1,2,3) can be used to identify this particle. In general, therefore, if a particle of a continuum was at the position  $(X_1, X_2, X_3)$  at the reference time  $t_0$ , the set of coordinate  $(X_1, X_2, X_3)$  can be used to identify this particle. Thus, in general,

the path lines of every particle in a continuum can be described by a vector equation of the

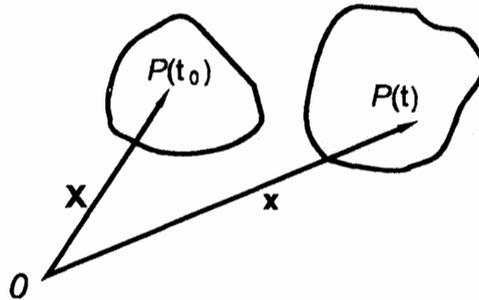


Fig. 3.1

form

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \text{ with } \mathbf{x}(\mathbf{X}, t_0) = \mathbf{X} \tag{3.1.1}$$

where  $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$  is the position vector at time  $t$  for the particle  $P$  which was at  $\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3$  (see Fig. F3.1).

In component form, Eq. (3.1.1) takes the form:

$$\begin{aligned} x_1 &= x_1(X_1, X_2, X_3, t) \\ x_2 &= x_2(X_1, X_2, X_3, t) \\ x_3 &= x_3(X_1, X_2, X_3, t) \end{aligned} \tag{3.1.2a}$$

or

$$x_i = x_i(X_1, X_2, X_3, t) \text{ with } x_i(X_1, X_2, X_3, t_0) = X_i \tag{3.1.2b}$$

In Eqs. (3.1.2), the triple  $(X_1, X_2, X_3)$  serves to identify the different particles of the body and is known as **material coordinates**. Equation (3.1.1) or Eqs. (3.1.2) is said to define a motion for a continuum; these equations describe the **pathline** for every particle in the continuum. They may also be called the **kinematic equations of motion**.

Example 3.1.1

Consider the motion

$$\mathbf{x} = \mathbf{X} + ktX_2\mathbf{e}_1 \tag{i}$$

where  $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$  is the position vector at time  $t$  for a particle which was at  $\mathbf{X} = X_1\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3$  at  $t=0$ . Sketch the configuration at time  $t$  for a body which at  $t=0$  has the shape of a cube of unit sides as shown in Fig. 3.2.

*Solution.* In component form, Eq. (i) becomes

$$x_1 = X_1 + ktX_2 \quad (\text{ia})$$

$$x_2 = X_2 \quad (\text{ib})$$

$$x_3 = X_3 \quad (\text{ic})$$

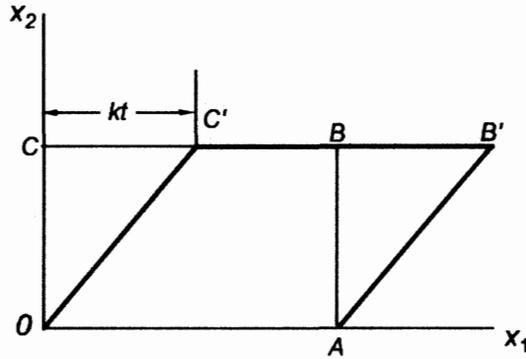


Fig. 3.2

At  $t=0$ , the particle  $O$  is located at  $(0,0,0)$ . Thus, for this particle, the material coordinates are

$$X_1=0, X_2=0 \text{ and } X_3=0.$$

Substituting these values for  $X_i$  in Eq. (ii), we get, for all time  $t$ ,  $(x_1, x_2, x_3) = (0,0,0)$ . In other words, this particle remains at  $(0,0,0)$  at all times.

Similarly, the material coordinates for the particle  $A$  are

$$(X_1, X_2, X_3) = (1,0,0)$$

and the position for  $A$  at time  $t$  is

$$(x_1, x_2, x_3) = (1,0,0)$$

Thus, the particle  $A$  also does not move with time. In fact, since the material coordinates for the points along the material line  $OA$  are

$$(X_1, X_2, X_3) = (X_1, 0, 0)$$

Therefore, for them, the positions at time  $t$  are

$$(x_1, x_2, x_3) = (X_1, 0, 0)$$

so that the whole material line  $OA$  is motionless.

On the other hand, the material coordinates for the material line  $CB$  are

$$(X_1, X_2, X_3) = (X_1, 1, 0)$$

so that according Eq. (ii)

$$(x_1, x_2, x_3) = (X_1 + kt, 1, 0)$$

In other words, the material line has moved horizontally through a distance of  $kt$  (see Fig. 3.2).

The material coordinates for the material line  $OC$  are  $(X_1, X_2, X_3) = (0, X_2, 0)$ , so that for the particles along this line  $(x_1, x_2, x_3) = (ktX_2, X_2, 0)$ . The fact that  $x_1 = ktX_2$  means that the straight material line  $OC$  remains a straight line  $OC'$  at time  $t$  as shown in Fig. 3.2. The situation for the material line  $AB$  is similar. Thus, at time  $t$ , the side view of the cube changes from that of a square to a parallelogram as shown. Since  $x_3 = X_3$  at all time for all particles, it is clear that all motions are parallel to the plane  $x_3 = 0$ . The motion given in this example is known as **simple shearing motion**.

### Example 3.1.2

Let

$$Y_1 = -X_1, \quad Y_2 = X_2, \quad \text{and} \quad Y_3 = X_3. \tag{i}$$

Express the simple shearing motion given in Example 3.1.1 in terms of  $(Y_1, Y_2, Y_3)$

*Solution.* Straight forward substitutions give

$$\begin{aligned} x_1 &= -Y_1 + ktY_2 \\ x_2 &= Y_2 \\ x_3 &= Y_3. \end{aligned} \tag{ii}$$

These equations, i.e.,

$$x_i = x_i(Y_1, Y_2, Y_3, t) \tag{iii}$$

obviously also describe the simple shearing motion just as the equations given in the previous example. The triples  $(Y_1, Y_2, Y_3)$  are also material coordinates in that they also identify the particles in the continuum although they are not the coordinates of the particles at any time. This example demonstrates the fact that while the positions of the particles at some reference time  $t_0$  can be used as the material coordinates, the material coordinates need not be the positions of the particle at any time. However, within this book, all material coordinates will be coordinates of the particles at some reference time.

Example 3.1.3

The position at time  $t$ , of a particle initially at  $(X_1, X_2, X_3)$ , is given by the equations:

$$x_1 = X_1 + (X_1 + X_2)t, \quad x_2 = X_2 + (X_1 + X_2)t, \quad x_3 = X_3 \quad (i)$$

(a) Find the velocity at  $t=2$  for the particle which was at  $(1,1,0)$  at the reference time.

(b) Find the velocity at  $t=2$  for the particle which is at the position  $(1,1,0)$  at  $t=2$ .

*Solution.* (a)

$$v_1 = \left( \frac{\partial x_1}{\partial t} \right)_{X_1\text{-fixed}} = X_1 + X_2, \quad v_2 = \left( \frac{\partial x_2}{\partial t} \right)_{X_1\text{-fixed}} = X_1 + X_2, \quad v_3 = 0 \quad (ii)$$

For the particle  $(X_1, X_2, X_3) = (1,1,0)$ , the velocity at  $t = 2$  (and any time  $t$ ) is

$$v_1 = 1 + 1 = 2, \quad v_2 = 1 + 1 = 2, \quad v_3 = 0 \quad (iii)$$

i.e.,

$$\mathbf{v} = 2\mathbf{e}_1 + 2\mathbf{e}_2 \quad (iv)$$

(b) To calculate the reference position  $(X_1, X_2, X_3)$  which was occupied by the particle which is at  $(x_1, x_2, x_3) = (1,1,0)$  at  $t = 2$ , we substitute the value of  $(x_1, x_2, x_3) = (1,1,0)$  and  $t = 2$  in Eq. (i) and solve for  $(X_1, X_2, X_3)$ , i.e.,

$$1 = 3X_1 + 2X_2, \quad 1 = 2X_1 + 3X_2 \quad (v)$$

Thus,  $X_1 = \frac{1}{5}$ ,  $X_2 = \frac{1}{5}$ . Substituting these values in Eq. (ii), we obtain

$$v_1 = \frac{2}{5}, \quad v_2 = \frac{2}{5}, \quad v_3 = 0 \quad (vi)$$

### 3.2 Material Description and Spatial Description

When a continuum is in motion, its temperature  $\Theta$ , its velocity  $\mathbf{v}$ , its stress tensor  $\mathbf{T}$  (to be defined in the next chapter) may change with time. We can describe these changes by:

I. Following the particles, i.e., we express  $\Theta$ ,  $\mathbf{v}$ ,  $\mathbf{T}$  as functions of the particles (identified by the material coordinates,  $(X_1, X_2, X_3)$ ) and time  $t$ . In other words, we express

$$\Theta = \hat{\Theta}(X_1, X_2, X_3, t) \quad (3.2.1a)$$

$$\mathbf{v} = \hat{\mathbf{v}}(X_1, X_2, X_3, t) \quad (3.2.1b)$$

$$\mathbf{T} = \hat{\mathbf{T}}(X_1, X_2, X_3, t) \quad (3.2.1c)$$

Such a description is known as the **material description**. Other names for it are: **Lagrangian description** and **reference description**.

II. Observing the changes at fixed locations, i.e., we express,  $\Theta, \mathbf{v}, \mathbf{T}$  etc. as functions of fixed position and time. Thus,

$$\Theta = \tilde{\Theta}(x_1, x_2, x_3, t) \quad (3.2.2a)$$

$$\mathbf{v} = \tilde{\mathbf{v}}(x_1, x_2, x_3, t) \quad (3.2.2b)$$

$$\mathbf{T} = \tilde{\mathbf{T}}(x_1, x_2, x_3, t) \quad (3.2.2c)$$

Such a description is known as a **spatial description** or **Eulerian description**. The triple  $(x_1, x_2, x_3)$  locates the fixed position of points in the physical space and is known as the **spatial coordinates**. The spatial coordinates  $x_i$  of a particle at any time  $t$  are related to the material coordinates  $X_i$  of the particle by Eq. (3.1.2). We note that in this description, what is described (or measured) is the change of quantities at a fixed location as a function of time. Spatial positions are occupied by different particles at different times. Therefore, the spatial description does not provide direct information regarding changes in particle properties as they move about. The material and spatial descriptions are, of course, related by the motion. That is, if the motion is known then, one description can be obtained from the other as illustrated by the following example.

#### Example 3.2.1

Given the motion of a continuum to be

$$x_1 = X_1 + kt, \quad x_2 = X_2, \quad x_3 = X_3 \quad (i)$$

If the temperature field is given by the spatial description

$$\Theta = x_1 + x_2 \quad (ii)$$

(a) find the material description of temperature and (b) obtain the velocity and rate of change of temperature for particular material particles and express the answer in both a material and a spatial description.

*Solution.* (a) Substituting (i) into (ii), we obtain

$$\Theta = X_1 + (kt + 1)X_2. \quad (iii)$$

(b) Since a particular material particle is designated by a specific  $\mathbf{X}$ , its velocity will be given by

---

† Note: the superposed  $\hat{\phantom{x}}$  and the superposed  $\tilde{\phantom{x}}$  are used to distinguish different functions for the same dependent variable.

$$v_i = \left( \frac{\partial x_i}{\partial t} \right)_{X_i\text{-fixed}} \tag{iv}$$

so that from Eq. (i)

$$v_1 = kX_2, \quad v_2 = v_3 = 0 \tag{v}$$

This is the material description of the velocity field. To obtain the spatial description, we make use of Eq. (i) again, where we have  $x_2 = X_2$ , so that

$$v_1 = kx_2, \quad v_2 = v_3 = 0 \tag{vi}$$

From Eq. (iii), the rate of change of temperature for particular material particles is given by

$$\left( \frac{\partial \Theta}{\partial t} \right)_{X_i\text{-fixed}} = kX_2 = kx_2 \tag{vii}$$

We note that even though the given temperature field is independent of time, each particle experiences changes of temperature, since it flows from one spatial position to another.

### 3.3 Material Derivative

The time rate of change of a quantity (such as temperature or velocity or stress tensor) of a material particle, is known as a material derivative. We shall denote the material derivative by  $D/Dt$ .

(i) When a material description of the quantity is used, we have

$$\Theta = \hat{\Theta}(X_1, X_2, X_3, t) \tag{3.3.1}$$

Thus,

$$\frac{D\Theta}{Dt} = \left( \frac{\partial \hat{\Theta}}{\partial t} \right)_{X_i\text{-fixed}} \tag{3.3.2}$$

(ii) When a spatial description of the quantity is used, we have

$$\Theta = \tilde{\Theta}(x_1, x_2, x_3, t) \tag{3.3.3}$$

where  $x_i$ , the positions of material particles at time  $t$ , are related to the material coordinates by the motion  $x_i = \hat{x}_i(X_1, X_2, X_3, t)$ . Then,

$$\left( \frac{D\Theta}{Dt} \right) = \left( \frac{\partial \hat{\Theta}}{\partial t} \right)_{X_i\text{-fixed}} = \frac{\partial \tilde{\Theta}}{\partial x_1} \frac{\partial x_1}{\partial t} + \frac{\partial \tilde{\Theta}}{\partial x_2} \frac{\partial x_2}{\partial t} + \frac{\partial \tilde{\Theta}}{\partial x_3} \frac{\partial x_3}{\partial t} + \left( \frac{\partial \tilde{\Theta}}{\partial t} \right)_{x_i\text{-fixed}} \tag{i}$$

where  $\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, \frac{\partial x_3}{\partial t}$  are to be obtained with fixed values of the  $X_i$ 's. When rectangular Cartesian coordinates are used, these are the velocity components  $v_i$  of the particle  $X_i$ . Thus, the material derivative in rectangular coordinates is

$$\frac{D\Theta}{Dt} = \frac{\partial\Theta}{\partial t} + v_1 \frac{\partial\Theta}{\partial x_1} + v_2 \frac{\partial\Theta}{\partial x_2} + v_3 \frac{\partial\Theta}{\partial x_3} \quad (3.3.4a)$$

or,

$$\frac{D\Theta}{Dt} = \frac{\partial\Theta}{\partial t} + \mathbf{v} \cdot \nabla\Theta \quad (3.3.4b)$$

where it should be emphasized that these equations are for  $\Theta$  in a spatial description, i.e.,  $\Theta = \tilde{\Theta}(x_1, x_2, x_3, t)$ . Note that if the temperature field is independent of time and if the velocity of a particle is perpendicular to  $\nabla\Theta$  (i.e, the particle is moving along the path of constant  $\Theta$ ) then, as expected  $\frac{D\Theta}{Dt} = 0$ .

Note again that Eq. (3.3.4a) is valid only for rectangular Cartesian coordinates, whereas Eq. (3.3.4b) has the advantage that it is valid for all coordinate systems. For a specific coordinate system, all that is needed is the appropriate expression for the gradient. For example, in cylindrical coordinate  $(r, \theta, z)$ ,

$$\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_z \mathbf{e}_z \quad (3.3.5)$$

and from Eq. (2D2.3)

$$\nabla\Theta = \frac{\partial\Theta}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial\Theta}{\partial\theta} \mathbf{e}_\theta + \frac{\partial\Theta}{\partial z} \mathbf{e}_z \quad (3.3.6)$$

Thus,

$$\frac{D\Theta}{Dt} = \frac{\partial\Theta}{\partial t} + v_r \frac{\partial\Theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial\Theta}{\partial\theta} + v_z \frac{\partial\Theta}{\partial z} \quad (3.3.7)$$

In spherical coordinates  $(r, \theta, \phi)$

$$\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta + v_\phi \mathbf{e}_\phi \quad (3.3.8)$$

and from Eq. (2D3.9)

$$(\nabla\Theta)_r = \frac{\partial\Theta}{\partial r} \quad (\nabla\Theta)_\theta = \frac{1}{r} \frac{\partial\Theta}{\partial\theta} \quad (\nabla\Theta)_\phi = \frac{1}{r \sin\theta} \frac{\partial\Theta}{\partial\phi} \quad (3.3.9)$$

Thus,

$$\frac{D\Theta}{Dt} = \frac{\partial\Theta}{\partial t} + v_r \frac{\partial\Theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial\Theta}{\partial\theta} + \frac{v_\phi}{r \sin\theta} \frac{\partial\Theta}{\partial\phi} \quad (3.3.10)$$

## Example 3.3.1

Use Eq. (3.3.4), obtain  $\frac{D\Theta}{Dt}$  for the motion and temperature field given in the previous example.

*Solution.* From Example 3.2.1, we have

$$\mathbf{v} = (kx_2)\mathbf{e}_1 \quad (\text{i})$$

and

$$\Theta = x_1 + x_2 \quad (\text{ii})$$

The gradient of  $\Theta$  is simply

$$\nabla\Theta = \mathbf{e}_1 + \mathbf{e}_2 \quad (\text{iii})$$

Therefore,

$$\frac{D\Theta}{Dt} = 0 + (kx_2)\mathbf{e}_1 \cdot (\mathbf{e}_1 + \mathbf{e}_2) = kx_2 \quad (\text{iv})$$

which agrees with the previous example.

### 3.4 Acceleration of a Particle in a Continuum

The acceleration of a particle is the rate of change of velocity of the particle. It is therefore the material derivative of velocity. If the motion of a continuum is given by Eq. (3.1.1), i.e.,

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \quad \text{with } \mathbf{x}(\mathbf{X}, t_0) = \mathbf{X}$$

then the velocity  $\mathbf{v}$ , at time  $t$ , of a particle  $\mathbf{X}$  is given by

$$\mathbf{v} = \left( \frac{\partial \mathbf{x}}{\partial t} \right)_{X_i \text{-fixed}} \equiv \frac{D\mathbf{x}}{Dt} \quad (3.4.1)$$

and the acceleration  $\mathbf{a}$ , at time  $t$ , of a particle  $\mathbf{X}$  is given by

$$\mathbf{a} = \left( \frac{\partial \mathbf{v}}{\partial t} \right)_{X_i \text{-fixed}} \equiv \frac{D\mathbf{v}}{Dt} \quad (3.4.2)$$

Thus, if the material description of velocity,  $\mathbf{v}(\mathbf{X}, t)$  is known (or is obtained from Eq. (3.4.1), then the acceleration is very easily computed, simply taking the partial derivative with respect to time of the function  $\mathbf{v}(\mathbf{X}, t)$ . On the other hand, if only the spatial description of velocity [i.e.,  $\mathbf{v} = \mathbf{v}(x, t)$ ] is known, the computation of acceleration is not as simple.

**(A) Rectangular Cartesian Coordinates**  $(x_1, x_2, x_3)$ . With

$$\mathbf{v} = v_1(x_1, x_2, x_3, t)\mathbf{e}_1 + v_2(x_1, x_2, x_3, t)\mathbf{e}_2 + v_3(x_1, x_2, x_3, t)\mathbf{e}_3 \quad (3.4.3)$$

we have, since the base vectors  $\mathbf{e}_1, \mathbf{e}_2$ , and  $\mathbf{e}_3$  do not change with time

$$\frac{D\mathbf{v}}{Dt} = \frac{Dv_1}{Dt} \mathbf{e}_1 + \frac{Dv_2}{Dt} \mathbf{e}_2 + \frac{Dv_3}{Dt} \mathbf{e}_3 \tag{3.4.4}$$

where

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_1 \frac{\partial v_i}{\partial x_1} + v_2 \frac{\partial v_i}{\partial x_2} + v_3 \frac{\partial v_i}{\partial x_3} \tag{3.4.5a}$$

i.e.,

$$a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \tag{3.4.5b}$$

Or, in a form valid for all coordinate systems:

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v}) \mathbf{v} \tag{3.4.5c}$$

In dyadic notation, the above equation is written as

$$\mathbf{a} = \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \tilde{\nabla} \mathbf{v} \tag{3.4.5d}$$

where  $\tilde{\nabla} = \mathbf{e}_m \frac{\partial}{\partial x_m}$ .

**(B)Cylindrical Coordinates  $(r, \theta, z)$ .** With

$$\mathbf{v} = v_r(r, \theta, z) \mathbf{e}_r + v_\theta(r, \theta, z) \mathbf{e}_\theta + v_z(r, \theta, z) \mathbf{e}_z \tag{3.4.6}$$

and, [ see Eq. (2D2.4) ]

$$[\nabla \mathbf{v}] = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right) & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r} \frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix} \tag{3.4.7}$$

we have,

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right) + v_z \frac{\partial v_r}{\partial z} \tag{3.4.8a}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) + v_z \frac{\partial v_\theta}{\partial z} \tag{3.4.8b}$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta \partial v_z}{r \partial \theta} + v_z \frac{\partial v_z}{\partial z} \quad (3.4.8c)$$

(C) **Spherical Coordinates**  $(r, \theta, \phi)$ . With

$$\mathbf{v} = v_r(r, \theta, \phi) \mathbf{e}_r + v_\theta(r, \theta, \phi) \mathbf{e}_\theta + v_\phi(r, \theta, \phi) \mathbf{e}_\phi \quad (3.4.9)$$

and, [see Eq. (2D3.17)]

$$[\nabla \mathbf{v}] = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right) & \frac{1}{r \sin \theta} \left( \frac{\partial v_r}{\partial \phi} - v_\phi \sin \theta \right) \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) & \frac{1}{r \sin \theta} \left( \frac{\partial v_\theta}{\partial \phi} - v_\phi \cos \theta \right) \\ \frac{\partial v_\phi}{\partial r} & \frac{1}{r} \frac{\partial v_\phi}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \end{bmatrix} \quad (3.4.10)$$

we have,

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \left( \frac{\partial v_r}{\partial \theta} - v_\theta \right) + \frac{v_\phi}{r \sin \theta} \left( \frac{\partial v_r}{\partial \phi} - v_\phi \sin \theta \right) \quad (3.4.11a)$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \left( \frac{\partial v_\theta}{\partial \theta} + v_r \right) + \frac{v_\phi}{r \sin \theta} \left( \frac{\partial v_\theta}{\partial \phi} - v_\phi \cos \theta \right) \quad (3.4.11b)$$

$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \left( \frac{\partial v_\phi}{\partial \phi} + v_r \sin \theta + v_\theta \cos \theta \right) \quad (3.4.11c)$$

### Example 3.4.1

(a) Find the velocity field associated with the motion of a rigid body rotating with angular velocity  $\boldsymbol{\omega} = \omega \mathbf{e}_3$  in Cartesian and in cylindrical coordinates.

(b) Using the velocity field of part (a), evaluate the acceleration field.

*Solution.* (a) For a rigid body rotation

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x} \quad (i)$$

In Cartesian coordinates

$$\mathbf{v} = \omega \mathbf{e}_3 \times (x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3) \quad (ii)$$

i.e.,

$$v_1 = -\omega x_2, \quad v_2 = \omega x_1, \quad v_3 = 0 \quad (iii)$$

In cylindrical coordinates

$$\mathbf{v} = \omega \mathbf{e}_z \times (r \mathbf{e}_r) = \omega r \mathbf{e}_\theta \quad (iv)$$

i.e.,

$$v_r = 0, \quad v_\theta = \omega r, \quad v_z = 0 \tag{v}$$

(b) We can use either Eq. (iii) or Eq. (v) to find the acceleration.

Using Eq. (iii) and Eq. (3.4.5b), we obtain

$$\begin{aligned} a_1 &= 0 + (-\omega x_2)(0) + (\omega x_1)(-\omega) + (0)(0) = -\omega^2 x_1 \\ a_2 &= 0 + (-\omega x_2)(\omega) + (\omega x_1)(0) + (0)(0) = -\omega^2 x_2 \\ a_3 &= 0 \end{aligned}$$

i.e.,

$$\mathbf{a} = -\omega^2(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \tag{vi}$$

Or, using Eq. (v) and Eqs. (3.4.8), we obtain

$$\begin{aligned} a_r &= 0 + (0)(0) + \frac{v_\theta}{r}(0 - v_\theta) + (0)(0) = -\frac{v_\theta^2}{r} = -\omega^2 r \\ a_\theta &= 0 + (0)(\omega) + \frac{v_\theta}{r}(0 + 0) + (0)(0) = 0 \\ a_z &= 0 \end{aligned}$$

i.e.,

$$\mathbf{a} = -\omega^2 r \mathbf{e}_r \tag{vii}$$

We note that  $(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = r\mathbf{e}_r$ , so that (vi) and (vii) are the same. We also note that in this example, even though at every spatial point there is no change of velocity with time, for every material point, there is a rate of change of velocity due to a change of direction at every point as it moves along a circular path giving rise to a centripetal acceleration.

### Example 3.4.2

Given the velocity field

$$v_1 = \frac{x_1}{1+t}, \quad v_2 = \frac{x_2}{1+t}, \quad v_3 = \frac{x_3}{1+t} \tag{i}$$

(a) Find the acceleration field and (b) find the pathline  $x_i = x_i(\mathbf{X}, t)$

*Solution.* (a) With

$$v_i = \frac{x_i}{1+t}, \quad i = 1, 2, 3 \tag{ii}$$

we have

$$\frac{\partial v_i}{\partial t} = - \frac{x_i}{(1+t)^2} \quad (\text{iii})$$

Also, since

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

so that

$$\frac{\partial v_i}{\partial x_j} = \frac{\delta_{ij}}{1+t} \quad (\text{iv})$$

therefore,

$$v_j \frac{\partial v_i}{\partial x_j} = v_j \frac{\delta_{ij}}{1+t} = \frac{v_i}{1+t} = \frac{x_i}{(1+t)^2} \quad (\text{v})$$

Thus,

$$a_i = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = - \frac{x_i}{(1+t)^2} + \frac{x_i}{(1+t)^2} = 0 \quad (\text{vi})$$

i.e.,

$$\mathbf{a} = \mathbf{0} \quad (\text{vii})$$

We note that in this example, even though at any spatial position (except the origin), the velocity is observed to be changing with time, but the actual velocity of a particular particle is a constant, with zero acceleration.

(b) Since

$$v_i = \left( \frac{\partial x_i}{\partial t} \right)_{\mathbf{x} - \text{fixed}} = \frac{x_i}{1+t} \quad i = 1, 2, 3 \quad (\text{viii})$$

therefore,

$$\int_{X_1}^{x_1} \frac{dx_1}{x_1} = \int_0^t \frac{dt}{1+t} \quad (\text{ix})$$

so that

$$\ln x_1 - \ln X_1 = \ln(1+t) \quad (\text{x})$$

Thus,

$$x_1 = X_1(1+t) \quad (\text{xi})$$

Similarly,

$$x_2 = X_2(1+t) \quad \text{and} \quad x_3 = X_3(1+t) \quad (\text{xii})$$

### 3.5 Displacement Field

The displacement of a particle from position  $P$  to position  $Q$  is the vector  $PQ$ . Thus, the displacement vector of a particle, from the reference position to the position at time  $t$ , is given by

$$\mathbf{u} = \mathbf{x}(\mathbf{X}, t) - \mathbf{X} \quad (3.5.1)$$

From the above equation, it is clear that whenever the pathline  $\mathbf{x}(\mathbf{X}, t)$  of a particle is known, its displacement field is also known. Thus, the motion of a continuum can be described either by the pathlines equation Eq. (3.1.1) or by its displacement vector field as given by Eq. (3.5.1).

#### Example 3.5.1

The position at time  $t$ , of a particle initially at  $(X_1, X_2, X_3)$  is given by

$$x_1 = X_1 + (X_1 + X_2)t, \quad x_2 = X_2 + (X_1 + X_2)t, \quad x_3 = X_3 \quad (\text{i})$$

Find the displacement field.

*Solution.*

$$u_1 = x_1 - X_1 = (X_1 + X_2)t, \quad u_2 = x_2 - X_2 = (X_1 + X_2)t, \quad u_3 = x_3 - X_3 = 0 \quad (\text{ii})$$

#### Example 5.2

The deformed configuration of a continuum is given by

$$x_1 = \frac{1}{2}X_1, \quad x_2 = X_2, \quad x_3 = X_3 \quad (\text{i})$$

Find the displacement field.

*Solution.* The displacement components are:

$$u_1 = \frac{1}{2}X_1 - X_1 = -\frac{1}{2}X_1, \quad u_2 = X_2 - X_2 = 0, \quad u_3 = X_3 - X_3 = 0 \quad (\text{ii})$$

This displacement field represents a *uniaxial contraction* (the state of confined compression).

### 3.6 Kinematic Equation For Rigid Body Motion

(a) *Rigid body translation*: For this motion, the kinematic equation of motion is given by

$$\mathbf{x} = \mathbf{X} + \mathbf{c}(t) \quad (3.6.1)$$

where  $\mathbf{c}(0) = \mathbf{0}$ . We note that the displacement vector,  $\mathbf{u} = \mathbf{x} - \mathbf{X} = \mathbf{c}(t)$ , is independent of  $\mathbf{X}$ . That is, every material point is displaced in an identical manner, with the same magnitude and the same direction at time  $t$ .

(b) *Rigid body rotation about a fixed point*: For this motion, the kinematic equation of motion is given by

$$\mathbf{x} - \mathbf{b} = \mathbf{R}(t)(\mathbf{X} - \mathbf{b}) \quad (3.6.2)$$

where  $\mathbf{R}(t)$  is a proper orthogonal tensor (i.e., a rotation tensor, see Sect. 2B.10) with  $\mathbf{R}(0) = \mathbf{I}$ , and  $\mathbf{b}$  is a constant vector. We note that the material point  $\mathbf{X} = \mathbf{b}$  is always at the spatial point  $\mathbf{x} = \mathbf{b}$  so that the rotation is about the fixed point  $\mathbf{x} = \mathbf{b}$ .

If the rotation is about the origin, then  $\mathbf{b} = \mathbf{0}$  and  $\mathbf{x} = \mathbf{R}(t)\mathbf{X}$ .

#### Example 3.6.1

Show that for motions given by Eq. (3.6.2) there is no change in distance between any pair of material points.

*Solution.* Consider two material points  $\mathbf{X}^{(1)}$  and  $\mathbf{X}^{(2)}$ , we have, from Eq. (3.6.2)

$$\mathbf{x}^{(1)} - \mathbf{x}^{(2)} = \mathbf{R}(t)(\mathbf{X}^{(1)} - \mathbf{X}^{(2)}) \quad (i)$$

That is, the material vector  $\Delta\mathbf{X} \equiv \mathbf{X}^{(1)} - \mathbf{X}^{(2)}$  changes to  $\Delta\mathbf{x} \equiv \mathbf{x}^{(1)} - \mathbf{x}^{(2)}$  where

$$\Delta\mathbf{x} = \mathbf{R}(t)\Delta\mathbf{X}. \quad (ii)$$

Now, the square of the length of  $\Delta\mathbf{x}$  is given by

$$\Delta\mathbf{x} \cdot \Delta\mathbf{x} = \mathbf{R}(t)\Delta\mathbf{X} \cdot \mathbf{R}(t)\Delta\mathbf{X} \quad (iii)$$

The right side of the above equation is, according to the definition of transpose of a tensor  $\Delta\mathbf{X} \cdot \mathbf{R}(t)\mathbf{R}^T(t)\Delta\mathbf{X}$ . and for an orthogonal tensor,  $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ , so that

$$\Delta\mathbf{x} \cdot \Delta\mathbf{x} = \Delta\mathbf{X} \cdot \Delta\mathbf{X} \quad (iv)$$

In other words, the length of  $\Delta\mathbf{X}$  does not change.

(c) *General rigid body motion*: The kinematic equation describing a general rigid body motion is given by

$$\mathbf{x} = \mathbf{R}(t)(\mathbf{X} - \mathbf{b}) + \mathbf{c}(t) \quad (3.6.3)$$

where  $\mathbf{R}(t)$  is a rotation tensor with  $\mathbf{R}(0) = \mathbf{I}$  and  $\mathbf{c}(t)$  is a vector with  $\mathbf{c}(0) = \mathbf{b}$ .

Equation (3.6.3) states that the motion is described by a translation  $\mathbf{c}(t)$ , of an arbitrary chosen material base point  $\mathbf{X}=\mathbf{b}$  plus a rotation  $\mathbf{R}(t)$ .

Example 3.6.2

From Eq. (3.6.3) derive the relation between the velocity of a general material point in the rigid body with the angular velocity of the body and the velocity of the arbitrary chosen material point.

*Solution.* Taking the material derivative of Eq. (3.6.3), we obtain

$$\mathbf{v} = \dot{\mathbf{R}}(\mathbf{X}-\mathbf{b}) + \dot{\mathbf{c}}(t)$$

Now, from Eq. (3.6.3), we have

$$(\mathbf{X}-\mathbf{b}) = \mathbf{R}^T(\mathbf{x}-\mathbf{c}) \tag{ii}$$

Thus

$$\mathbf{v} = \dot{\mathbf{R}} \mathbf{R}^T(\mathbf{x}-\mathbf{c}) + \dot{\mathbf{c}}(t) \tag{iii}$$

Since  $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ ,  $\dot{\mathbf{R}}\mathbf{R}^T + \mathbf{R}\dot{\mathbf{R}}^T = \mathbf{0}$ , so that  $\dot{\mathbf{R}}\mathbf{R}^T$  is antisymmetric which is equivalent to a dual (or axial) vector  $\boldsymbol{\omega}$  [see Sect. 2B16], thus,

$$\mathbf{v} = \boldsymbol{\omega} \times (\mathbf{x}-\mathbf{c}) + \dot{\mathbf{c}}(t) \tag{iv}$$

If we measure the position vector  $\mathbf{r}$  for the general material point from the position at time  $t$  of the chosen material base point, i.e.,  $\mathbf{r} = (\mathbf{x}-\mathbf{c})$ , then

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{c}}(t) \tag{v}$$

### 3.7 Infinitesimal Deformations

There are many important engineering problems which involves structural members or machine parts, for which the displacement of every material point is very small (mathematically infinitesimal) under design loadings. In this section, we derive the tensor which characterizes the deformation of such bodies.

Consider a body, having a particular configuration at some reference time  $t_0$ , changes to another configuration at time  $t$ . Referring to Fig. 3.3, a typical material point  $P$  undergoes a displacement  $\mathbf{u}$ , so that it arrives at the position

$$\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}, t) \tag{i}$$

A neighboring point  $Q$  at  $\mathbf{X} + d\mathbf{X}$  arrives at  $\mathbf{x} + d\mathbf{x}$  which is related to  $\mathbf{X} + d\mathbf{X}$  by:

$$\mathbf{x} + d\mathbf{x} = \mathbf{X} + d\mathbf{X} + \mathbf{u}(\mathbf{X} + d\mathbf{X}, t) \tag{ii}$$

Subtracting Eq. (i) from Eq. (ii), we obtain

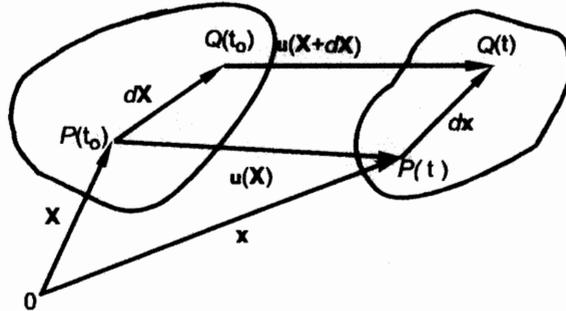


Fig. 3.3

$$dx = dX + u(X + dX, t) - u(X, t) \tag{iii}$$

Using the definition of gradient of a vector function [see Eq. (2C3.1)], Eq. (iii) becomes

$$dx = dX + (\nabla u) dX \tag{3.7.1a}$$

where  $\nabla u$  is a second-order tensor known as the **displacement gradient**. The matrix of  $\nabla u$  with respect to rectangular Cartesian coordinates ( with  $\mathbf{X} = X_i \mathbf{e}_i$  and  $\mathbf{u} = u_i \mathbf{e}_i$ ) is

$$[\nabla u] = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{\partial u_1}{\partial X_2} & \frac{\partial u_1}{\partial X_3} \\ \frac{\partial u_2}{\partial X_1} & \frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_3} \\ \frac{\partial u_3}{\partial X_1} & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{3.7.1b}$$

Example 3.7.1

Given the following displacement components

$$u_1 = kX_2^2, \quad u_2 = u_3 = 0. \tag{i}$$

- (a) Sketch the deformed shape of the unit square  $OABC$  in Fig. 3.4
- (b) Find the deformed vector (i.e.,  $d\mathbf{x}^{(1)}$  and  $d\mathbf{x}^{(2)}$ ) of the material elements  $d\mathbf{X}^{(1)} = dX_1 \mathbf{e}_1$  and  $d\mathbf{X}^{(2)} = dX_2 \mathbf{e}_2$  which were at the point  $C$ .
- (c) determine the ratio of the deformed to the undeformed lengths of the differential elements (known as **stretch**) of part (b) and the change in angle between these elements.

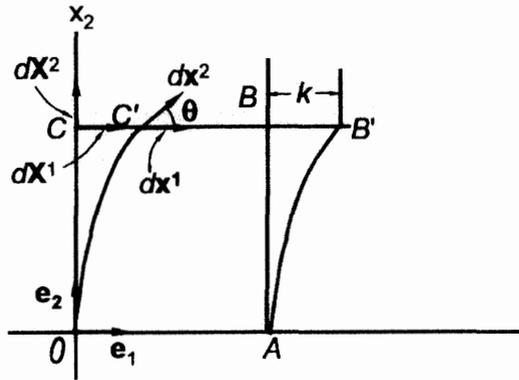


Fig. 3.4

*Solution.* (a) For the material line  $OA$ ,  $X_2 = 0$ , therefore,  $u_1 = u_2 = u_3 = 0$ . That is, the line is not displaced. For the material  $CB$ ,  $X_2 = 1$ ,  $u_1 = k$ , the line is displaced by  $k$  units to the right. For the material line  $OC$  and  $AB$ ,  $u_1 = kX_2^2$ , the lines become parabolic in shape. Thus, the deformed shape is given by  $OAB'C'$  in Fig. 3.4.

(b) For the material point  $C$ , the matrix of the displacement gradient is

$$[\nabla \mathbf{u}] = \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{X_2=1} = \begin{bmatrix} 0 & 2k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

Therefore, from Eq. (3.7.1a)

$$d\mathbf{x}^{(1)} = d\mathbf{X}^{(1)} + (\nabla \mathbf{u})d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1 + 0 = dX_1\mathbf{e}_1 \tag{iii}$$

$$d\mathbf{x}^{(2)} = d\mathbf{X}^{(2)} + (\nabla \mathbf{u})d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2 + 2kdX_2\mathbf{e}_1 = dX_2(\mathbf{e}_2 + 2k\mathbf{e}_1) \tag{iv}$$

(c) From Eqs. (iii) and (iv), we have  $|d\mathbf{x}^{(1)}| = dX_1$ ,  $|d\mathbf{x}^{(2)}| = dX_2(1+4k^2)^{1/2}$ , thus,

$$\frac{|d\mathbf{x}^{(1)}|}{|d\mathbf{X}^{(1)}|} = 1 \quad \text{and} \quad \frac{|d\mathbf{x}^{(2)}|}{|d\mathbf{X}^{(2)}|} = (1+4k^2)^{1/2} \tag{v}$$

and

$$\cos\theta = \frac{d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}}{|d\mathbf{x}^{(1)}| |d\mathbf{x}^{(2)}|} = \frac{2k}{(1+4k^2)^{1/2}} \tag{vi}$$

If  $k$  is very small, we have the case of small deformations and by the binomial theorem, we have, keeping only the first power of  $k$ ,

$$\frac{|d\mathbf{x}^{(1)}|}{|d\mathbf{X}^{(1)}|} = 1 \quad \text{and} \quad \frac{|d\mathbf{x}^{(2)}|}{|d\mathbf{X}^{(2)}|} = (1+2k^2) \approx 1 \quad (\text{vii})$$

and

$$\cos\theta = 2k \quad (\text{viii})$$

if  $\gamma$  denote the decrease in angle, then

$$\cos\theta = \cos\left(\frac{\pi}{2} - \gamma\right) = \sin\gamma = 2k$$

That is, for small  $k$ ,

$$\gamma = 2k \quad (\text{ix})$$

We can write Eq. (3.7.1a), i.e.,  $d\mathbf{x} = d\mathbf{X} + (\nabla\mathbf{u})d\mathbf{X}$  as

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \quad (\text{3.7.2})$$

where

$$\mathbf{F} = \mathbf{I} + \nabla\mathbf{u} \quad (\text{3.7.3})$$

To find the relationship between  $ds$ , the length of  $d\mathbf{x}$  and  $dS$ , the length of  $d\mathbf{X}$ , we take the dot product of Eq. (3.7.2) with itself:

$$d\mathbf{x} \cdot d\mathbf{x} = \mathbf{F}d\mathbf{X} \cdot \mathbf{F}d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^T \mathbf{F} d\mathbf{X} \quad (\text{3.7.4a})$$

i.e.,

$$(ds)^2 = d\mathbf{X} \cdot \mathbf{F}^T \mathbf{F} d\mathbf{X} \quad (\text{3.7.4b})$$

If  $\mathbf{F}$  is an orthogonal tensor, then  $\mathbf{F}^T \mathbf{F} = \mathbf{I}$ , and

$$(ds)^2 = (dS)^2$$

Thus, an orthogonal  $\mathbf{F}$  corresponds to a rigid body motion (translation and/or rotation).

Now, from Eq. (3.7.3),

$$\mathbf{F}^T \mathbf{F} = (\mathbf{I} + \nabla\mathbf{u})^T (\mathbf{I} + \nabla\mathbf{u}) = \mathbf{I} + \nabla\mathbf{u} + (\nabla\mathbf{u})^T + (\nabla\mathbf{u})^T \nabla\mathbf{u} \quad (\text{3.7.5})$$

In this section, we shall consider only cases where the components of the displacement vector as well as their partial derivatives are all very small (mathematically, infinitesimal) so that the absolute value of every component of  $(\nabla\mathbf{u})^T \nabla\mathbf{u}$  is a small quantity of higher order than those of the components of  $\nabla\mathbf{u}$ . For such a case, the above equation becomes:

$$\mathbf{F}^T \mathbf{F} \approx \mathbf{I} + \nabla \mathbf{u} + (\nabla \mathbf{u})^T \equiv \mathbf{I} + 2\mathbf{E} \tag{3.7.6}$$

where

$$\mathbf{E} = \frac{1}{2} [(\nabla \mathbf{u})^T + \nabla \mathbf{u}] \equiv \text{symmetric part of } \nabla \mathbf{u} \tag{3.7.7}$$

From Eq. (3.7.4b) and (3.7.6), it is clear that the tensor  $\mathbf{E}$  characterizes the changes of lengths in the continuum undergoing small deformations. This tensor  $\mathbf{E}$  is known as the **infinitesimal strain tensor**.

Consider two material elements  $d\mathbf{X}^{(1)}$  and  $d\mathbf{X}^{(2)}$ . Due to motion, they become  $d\mathbf{x}^{(1)}$  and  $d\mathbf{x}^{(2)}$  at time  $t$  with  $d\mathbf{x}^{(1)} = \mathbf{F}d\mathbf{X}^{(1)}$  and  $d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)}$ . Taking the dot product of  $d\mathbf{x}^{(1)}$  and  $d\mathbf{x}^{(2)}$ , we obtain

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = d\mathbf{X}^{(1)} \cdot \mathbf{F}^T \mathbf{F} d\mathbf{X}^{(2)} \tag{3.7.8}$$

Thus, using Eq. (3.7.6), we have the important equation

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} + 2d\mathbf{X}^{(1)} \cdot \mathbf{E}d\mathbf{X}^{(2)} \tag{3.7.9}$$

This equation will be used in the next section to establish the meanings of the components of the infinitesimal strain tensor  $\mathbf{E}$ .

The components of the infinitesimal strain tensor  $\mathbf{E}$  can be obtained easily from the components of the gradient of  $\mathbf{u}$  given in Chapter 2. We have

(a) *In rectangular coordinates:*

$$E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) \tag{3.7.10a}$$

or,

$$[\mathbf{E}] = \begin{bmatrix} \frac{\partial u_1}{\partial X_1} & \frac{1}{2} \left( \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \right) & \frac{1}{2} \left( \frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} \right) \\ \frac{1}{2} \left( \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \right) & \frac{\partial u_2}{\partial X_2} & \frac{1}{2} \left( \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \right) \\ \frac{1}{2} \left( \frac{\partial u_1}{\partial X_3} + \frac{\partial u_3}{\partial X_1} \right) & \frac{1}{2} \left( \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \right) & \frac{\partial u_3}{\partial X_3} \end{bmatrix} \tag{3.7.10b}$$

(B) *In cylindrical coordinates:*

$$[\mathbf{E}] = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2} \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right] & \frac{1}{2} \left[ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] \\ \frac{1}{2} \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right] & \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right) & \frac{1}{2} \left[ \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right] \\ \frac{1}{2} \left[ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right] & \frac{1}{2} \left[ \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right] & \frac{\partial u_z}{\partial z} \end{bmatrix} \quad (3.7.11)$$

(c) In spherical coordinates:

$$[\mathbf{E}] = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{2} \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right] & \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right] \\ \frac{1}{2} \left[ \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right] & \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right) & \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi \cot \theta}{r} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} \right] \\ \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right] & \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi \cot \theta}{r} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} \right] & \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r} \end{bmatrix} \quad (3.7.12)$$

### 3.8 Geometrical Meaning of the Components of the Infinitesimal Strain Tensor

(a) Diagonal elements of  $\mathbf{E}$

Consider the single material element  $d\mathbf{X}^{(1)} = d\mathbf{X}^{(2)} = d\mathbf{X} = (dS)\mathbf{n}$ , where  $\mathbf{n}$  is a unit vector and  $dS$  is the length of  $d\mathbf{X}$ . Let  $ds$  denote the deformed length of  $d\mathbf{x}^{(1)}$ , i.e.,  $ds = |d\mathbf{x}^{(1)}|$ . Then, Eq. (3.7.9) gives

$$(ds)^2 - (dS)^2 = 2(dS)^2 \mathbf{n} \cdot \mathbf{E} \mathbf{n}$$

Now, for small deformation  $(ds)^2 - (dS)^2 = (ds + dS)(ds - dS) \approx 2dS(ds - dS)$ . Thus

$$\frac{ds - dS}{dS} = \mathbf{n} \cdot \mathbf{E} \mathbf{n} = E_{nn} \quad (\text{no sum on } n) \quad (3.8.1)$$

This equation states that the **unit elongation** (i.e., the increase in length per unit original length) for the element which was in the direction  $\mathbf{n}$ , is given by  $\mathbf{n} \cdot \mathbf{E} \mathbf{n}$ . In particular, if the element was in the  $\mathbf{e}_1$  direction in the reference state, then  $\mathbf{n} = \mathbf{e}_1$ , and  $E_{11} = \mathbf{e}_1 \cdot \mathbf{E} \mathbf{e}_1$  so that

$E_{11}$  is the unit elongation for an element originally in the  $x_1$ -direction. Similarly,

$E_{22}$  is the unit elongation for an element originally in the  $x_2$ -direction and

$E_{33}$  is the unit elongation for an element originally in the  $x_3$ -direction.

These components (the diagonal elements of the tensor  $\mathbf{E}$ ) are also known as the **normal strains**.

(b) *The off diagonal elements:*

Let  $d\mathbf{X}^{(1)} = dS_1\mathbf{m}$  and  $d\mathbf{X}^{(2)} = dS_2\mathbf{n}$ , where  $\mathbf{m}$  and  $\mathbf{n}$  are unit vectors perpendicular to each other. Then Eq. (3.7.9) gives

$$(ds_1)(ds_2)\cos\theta = 2(dS_1)(dS_2)\mathbf{m} \cdot \mathbf{E}\mathbf{n}$$

where  $\theta$  is the angle between  $d\mathbf{x}^{(1)}$  and  $d\mathbf{x}^{(2)}$ . If we let  $\theta = (\pi/2) - \gamma$ , then  $\gamma$  will measure the small decrease in angle between  $d\mathbf{X}^{(1)}$  and  $d\mathbf{X}^{(2)}$ , known as the **shear strain**. Since

$$\cos(\frac{\pi}{2} - \gamma) = \sin\gamma$$

and for small strain

$$\sin\gamma \approx \gamma, \frac{ds_1}{dS_1} \approx 1, \frac{ds_2}{dS_2} \approx 1$$

therefore,

$$\gamma = 2\mathbf{m} \cdot \mathbf{E}\mathbf{n} \tag{3.8.2}$$

If the elements were in the direction of  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , then  $\mathbf{m} \cdot \mathbf{E}\mathbf{n} = E_{12}$  so that according to Eq. (3.8.2):

$2E_{12}$  gives the decrease in angle between two elements initially in the  $x_1$  and  $x_2$  directions. Similarly,

$2E_{13}$  gives the decrease in angle between two elements initially in the  $x_1$  and  $x_3$  directions, and

$2E_{23}$  gives the decrease in angle between two elements initially in the  $x_2$  and  $x_3$  directions.

Example 3.8.1

Given the displacement components

$$u_1 = kX_2^2 \text{ and } u_2 = u_3 = 0, k = 10^{-4} \tag{i}$$

(a) Obtain the infinitesimal strain tensor  $\mathbf{E}$ .

(b) Using the strain tensor  $\mathbf{E}$ , find the unit elongation for the material elements  $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$  and  $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$ , which were at the point  $C(0,1,0)$  of Fig. 3.4 (which is reproduced here for convenience). Also, find the decrease in angle between these two elements.

(c) Compare the results with those of Example 3.7.1.

*Solution.* (a) We have

$$[\nabla \mathbf{u}] = \begin{bmatrix} 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{ii})$$

Therefore

$$[\mathbf{E}] = [\nabla \mathbf{u}]^s = \begin{bmatrix} 0 & kX_2 & 0 \\ kX_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{iii})$$

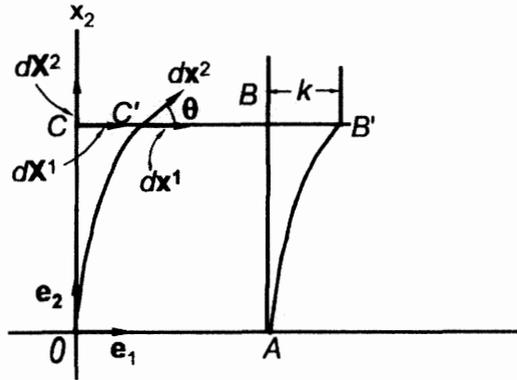


Fig. 3.4 (repeated)

(b) At the point  $C$ ,  $X_2=1$ , therefore

$$[\mathbf{E}] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{iv})$$

For the element  $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$ , the unit elongation is  $E_{11}$ , which is zero. For the element  $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$ , the unit elongation is  $E_{22}$  which is also zero. The decrease in angle between these elements is given by  $2E_{12}$ , which is equal to  $2k$ , i.e.,  $2 \times 10^{-4}$  radians.

(c) In Example 3.7.1, we found that

$$\frac{|d\mathbf{x}^{(1)}| - |d\mathbf{X}^{(1)}|}{|d\mathbf{X}^{(1)}|} = 0, \quad \frac{|d\mathbf{x}^{(2)}| - |d\mathbf{X}^{(2)}|}{|d\mathbf{X}^{(2)}|} = (1+4k^2)^{1/2} - 1 = 1+2k^2 - 1 = 2k^2 (\approx 0) \quad (\text{v})$$

and  $\sin \gamma = 2k = 2 \times 10^{-4}$  so that  $\gamma \approx 2 \times 10^{-4}$ .

We see that the results of this example is accurate up to the order of  $k$ .

Example 3.8.2

Given the displacement field

$$u_1 = k(2X_1 + X_2^2), \quad u_2 = k(X_1^2 - X_2^2), \quad u_3 = 0; \quad k = 10^{-4} \quad (i)$$

(a) Find the unit elongation and the change of angle for the two material elements  $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$  and  $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$  that emanate from a particle designated by  $\mathbf{X} = \mathbf{e}_1 - \mathbf{e}_2$ .

(b) Find the deformed position of these two elements  $d\mathbf{x}^{(1)}$  and  $d\mathbf{x}^{(2)}$ .

*Solution.* (a) We evaluate  $[\nabla\mathbf{u}]$  at  $(X_1, X_2, X_3) = (1, -1, 0)$  as

$$[\nabla\mathbf{u}] = k \begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (ii)$$

and therefore the strain matrix is

$$[\mathbf{E}] = k \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (iii)$$

Since  $E_{11} = E_{22} = 2k$ , both elements have a unit elongation of  $2 \times 10^{-4}$ . Further, since  $E_{12} = 0$ , these line elements remain perpendicular to each other.

(b) From Eq. (3.7.1a)

$$[d\mathbf{x}^{(1)}] = [d\mathbf{X}^{(1)}] + [\nabla\mathbf{u}][d\mathbf{X}^{(1)}] = \begin{bmatrix} dX_1 \\ 0 \\ 0 \end{bmatrix} + k \begin{bmatrix} 2 & -2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dX_1 \\ 0 \\ 0 \end{bmatrix} = dX_1 \begin{bmatrix} 1+2k \\ 2k \\ 0 \end{bmatrix} \quad (iv)$$

and similarly

$$[d\mathbf{x}^{(2)}] = [d\mathbf{X}^{(2)}] + [\nabla\mathbf{u}][d\mathbf{X}^{(2)}] = dX_2 \begin{bmatrix} -2k \\ 1+2k \\ 0 \end{bmatrix} \quad (v)$$

The deformed position of these elements is sketched in Fig. 3.5. Note from the diagram that

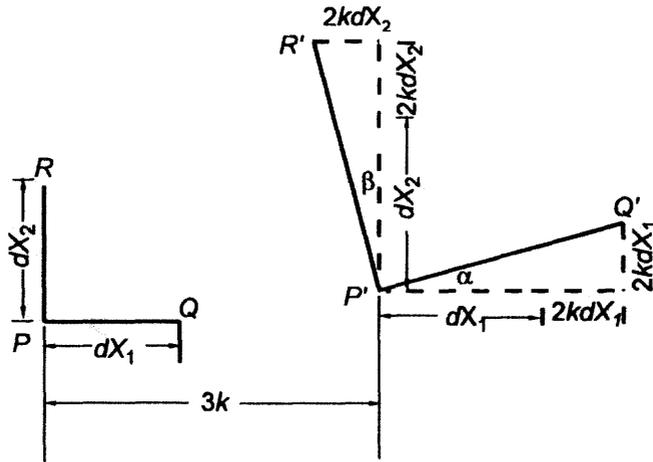


Fig. 3.5

$$\alpha \approx \tan \alpha = \frac{2kdX_1}{dX_1(1+k)} = \frac{2k}{1+k} \approx 2k \quad (\text{vi})$$

and

$$\beta \approx \tan \beta = \frac{2kdX_2}{dX_2} = 2k \quad (\text{vii})$$

Thus, as previously obtained, there is no change of angle between  $d\mathbf{X}^{(1)}$  and  $d\mathbf{X}^{(2)}$ .

### Example 3.8.3

A unit cube, with edges parallel to the coordinates axes, is given a displacement field

$$u_1 = kX_1, \quad u_2 = u_3 = 0, \quad k = 10^{-4} \quad (\text{i})$$

Find the increase in length of the diagonal  $AB$  (see Fig. 3.6) (a) by using the infinitesimal strain tensor  $\mathbf{E}$  and (b) by geometry

*Solution.* (a) The strain tensor is easily obtained to be

$$[\mathbf{E}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

Since the diagonal  $AB$  was originally in the direction  $\mathbf{n} = \frac{\sqrt{2}}{2}(\mathbf{e}_1 + \mathbf{e}_2)$ , its unit elongation is given by

$$E_{nn} = \mathbf{n} \cdot \mathbf{E} \mathbf{n} = [\sqrt{2}/2, \sqrt{2}/2, 0] \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \\ 0 \end{bmatrix} = \frac{k}{2} \quad (\text{no sum on } n) \tag{iii}$$

Since  $AB = \sqrt{2}$ ,

$$\Delta AB = \left(\frac{k}{2}\right) \sqrt{2} \tag{iv}$$

(b) Geometrically,

$$AB' - AB = [1 + (1+k)^2]^{1/2} - \sqrt{2}$$

or,

$$\Delta AB = \sqrt{2} [(1 + k + k^2/2)^{1/2} - 1] \tag{v}$$

To take advantage of the smallness of  $k$ , we expand the first term in the right hand side of Eq. (v) as

$$\left(1 + k + \frac{k^2}{2}\right) = 1 + \frac{1}{2} \left(k + \frac{k^2}{2}\right) + \dots \approx 1 + \frac{k}{2} \tag{vi}$$

Therefore, in agreement with Part (a), Eq. (iv),

$$\Delta AB = \sqrt{2} \left(\frac{k}{2}\right) \tag{vii}$$

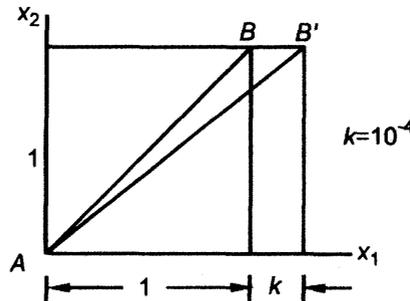


Fig.3.6

### 3.9 Principal Strain

Since the strain tensor  $\mathbf{E}$  is symmetric, therefore, (see Section 2B.18) there exists at least three mutually perpendicular directions  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  with respect to which the matrix of  $\mathbf{E}$  is diagonal. That is

$$[\mathbf{E}]_{\mathbf{n}_i} = \begin{bmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{bmatrix} \quad (3.9.1)$$

Geometrically, this means that infinitesimal line elements in the directions of  $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$  remain mutually perpendicular after deformation. These directions are known as the **principal directions** of strain. The unit elongation along the principal direction (i.e.,  $E_1, E_2, E_3$ ) are the eigenvalues of  $\mathbf{E}$ , or **principal strains**, they include the maximum and the minimum normal strains among all directions emanating from the particle. For a given  $\mathbf{E}$ , the principal strains are to be found from the characteristic equation of  $\mathbf{E}$ , i.e.,

$$\lambda^3 - I_1 \lambda^2 + I_2 \lambda - I_3 = 0 \quad (3.9.2)$$

where

$$I_1 = E_{11} + E_{22} + E_{33} \quad (3.9.3a)$$

$$I_2 = \begin{vmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{vmatrix} + \begin{vmatrix} E_{11} & E_{13} \\ E_{31} & E_{33} \end{vmatrix} + \begin{vmatrix} E_{22} & E_{23} \\ E_{32} & E_{33} \end{vmatrix} \quad (3.9.3b)$$

$$I_3 = |E_{ij}| \quad (3.9.3c)$$

The coefficients  $I_1, I_2$ , and  $I_3$  are called the **principal scalar invariants** of the strain tensor.

### 3.10 Dilatation

The first scalar invariant of the infinitesimal strain tensor has a simple geometric meaning. For a specific deformation, consider the three material lines that emanate from a single point  $P$  and are in the principal directions. These lines define a rectangular parallelepiped whose sides have been elongated from the initial dimension

$$dS_1, dS_2 \text{ and } dS_3$$

to

$$dS_1(1+E_1), dS_2(1+E_2) \text{ and } dS_3(1+E_3)$$

where  $E_1, E_2$  and  $E_3$  are the principal strains. Hence the change  $\Delta(dV)$  in this material volume  $dV$  is

$$\begin{aligned} \Delta(dV) &= (dS_1)(dS_2)(dS_3)(1+E_1)(1+E_2)(1+E_3) - (dS_1)(dS_2)(dS_3) \\ &= (dV)(E_1+E_2+E_3) + \text{higher order terms in the } E_i\text{'s.} \end{aligned}$$

Thus, for small deformation

$$e \equiv \frac{\Delta(dV)}{dV} = E_1 + E_2 + E_3 = E_{11} + E_{22} + E_{33} \quad (3.10.1)$$

This unit volume change is known as **dilatation**. Note also that

$$e = E_{ii} = \frac{\partial u_i}{\partial X_i} = \text{div} \mathbf{u} \quad (3.10.2a)$$

In cylindrical coordinates,

$$e = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} \quad (3.10.2b)$$

In spherical coordinates,

$$e = \frac{\partial u_r}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{2u_r}{r} + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta \cot \theta}{r} \quad (3.10.2c)$$

### 3.11 The Infinitesimal Rotation Tensor

Equation (3.7.1), i.e.,  $d\mathbf{x} = d\mathbf{X} + (\nabla \mathbf{u})d\mathbf{X}$ , can be written

$$d\mathbf{x} = d\mathbf{X} + (\mathbf{E} + \mathbf{\Omega})d\mathbf{X} \quad (3.11.1)$$

where  $\mathbf{\Omega}$ , the antisymmetric part of  $\nabla \mathbf{u}$ , is known as the **infinitesimal rotation tensor**. We see that the change of direction for  $d\mathbf{X}$  in general comes from two sources, the infinitesimal deformation tensor  $\mathbf{E}$  and the infinitesimal rotation tensor  $\mathbf{\Omega}$ . However, for any  $d\mathbf{X}$  which is in the direction of an eigenvector of  $\mathbf{E}$ , there is no change of direction due to  $\mathbf{E}$ , only that due to  $\mathbf{\Omega}$ . Therefore, the tensor  $\mathbf{\Omega}$  represents the infinitesimal rotation of the triad of the eigenvectors of  $\mathbf{E}$ . It can be described by a vector  $\mathbf{r}^A$  in the sense that

$$\mathbf{r}^A \times d\mathbf{X} = \mathbf{\Omega} d\mathbf{X} \quad (3.11.2)$$

where (see Section 2B.16)

$$\mathbf{r}^A = \Omega_{32}\mathbf{e}_1 + \Omega_{13}\mathbf{e}_2 + \Omega_{21}\mathbf{e}_3 \quad (3.11.3)$$

Thus,  $\Omega_{32}, \Omega_{13}, \Omega_{21}$  are the infinitesimal angles of rotation about  $\mathbf{e}_1, \mathbf{e}_2,$  and  $\mathbf{e}_3$ -axes, of the triad of material elements which are in the principal direction of  $\mathbf{E}$ .

### 3.12 Time Rate of Change of a Material Element

Let us consider a material element  $d\mathbf{x}$  emanating from a material point  $\mathbf{X}$  located at  $\mathbf{x}$  at time  $t$ . We wish to compute  $(D/Dt)d\mathbf{x}$ , the rate of change of length and direction of the material element  $d\mathbf{x}$ . From  $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ , we have

$$dx = \mathbf{x}(\mathbf{X}+d\mathbf{X},t) - \mathbf{x}(\mathbf{X},t) \quad (i)$$

Taking the material derivative of Eq. (i), we obtain

$$\left(\frac{D}{Dt}\right)dx = \left(\frac{D}{Dt}\right)\mathbf{x}(\mathbf{X}+d\mathbf{X},t) - \left(\frac{D}{Dt}\right)\mathbf{x}(\mathbf{X},t) \quad (ii)$$

Now,

$$(D/Dt)\mathbf{x}(\mathbf{X},t) = \widehat{\mathbf{v}}(\mathbf{X},t) = \widetilde{\mathbf{v}}(\mathbf{x},t) \quad (3.12.1)$$

where  $\widehat{\mathbf{v}}(\mathbf{X},t)$  and  $\widetilde{\mathbf{v}}(\mathbf{x},t)$  are the material and the spatial description of the velocity of the particle  $\mathbf{X}$ , therefore Eq. (ii) becomes

$$\left(\frac{D}{Dt}\right)dx = \widehat{\mathbf{v}}(\mathbf{X}+d\mathbf{X},t) - \widehat{\mathbf{v}}(\mathbf{X},t) = \widetilde{\mathbf{v}}(\mathbf{x}+d\mathbf{x},t) - \widetilde{\mathbf{v}}(\mathbf{x},t) \quad (iii)$$

Thus, from the definition (see Section 2C3.1) of the gradient of a vector function, we have

$$\left(\frac{D}{Dt}\right)dx = (\nabla_{\mathbf{X}}\widehat{\mathbf{v}})d\mathbf{X} \quad (3.12.2)$$

and

$$\left(\frac{D}{Dt}\right)dx = (\nabla_{\mathbf{x}}\widetilde{\mathbf{v}})d\mathbf{x} \quad (3.12.3)$$

In Eq. (3.12.2) the subscript  $\mathbf{X}$  in  $(\nabla_{\mathbf{X}}\widehat{\mathbf{v}})$  emphasizes that  $(\nabla_{\mathbf{X}}\widehat{\mathbf{v}})$  is the gradient of the material description of the velocity field  $\mathbf{v}$  and in Eq. (3.12.3) the subscript  $\mathbf{x}$  in  $(\nabla_{\mathbf{x}}\widetilde{\mathbf{v}})$  emphasizes that  $(\nabla_{\mathbf{x}}\widetilde{\mathbf{v}})$  is the gradient of the spatial description of  $\mathbf{v}$ .

In the following, the spatial description of the velocity function will be used exclusively so that *the notation*  $(\nabla\mathbf{v})$  *will be understood to mean*  $(\nabla_{\mathbf{x}}\widetilde{\mathbf{v}})$ . Thus we write Eq. (3.12.3) simply as

$$\left(\frac{D}{Dt}\right)dx = (\nabla\mathbf{v})d\mathbf{x} \quad (3.12.4)$$

With respect to rectangular Cartesian coordinates, the components of  $(\nabla\mathbf{v})$  are given by

$$[\nabla\mathbf{v}] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \quad (3.12.5)$$

### 3.13 The Rate of Deformation Tensor

The velocity gradient  $(\nabla \mathbf{v})$  can be decomposed into a symmetric part and an antisymmetric part as follows:

$$(\nabla \mathbf{v}) = \mathbf{D} + \mathbf{W} \quad (3.13.1)$$

where  $\mathbf{D}$  is the symmetric part, i.e.,

$$\mathbf{D} = \frac{1}{2}[(\nabla \mathbf{v}) + (\nabla \mathbf{v})^T] \quad (3.13.2)$$

and  $\mathbf{W}$  is the antisymmetric part, i.e.,

$$\mathbf{W} = \frac{1}{2}[(\nabla \mathbf{v}) - (\nabla \mathbf{v})^T] \quad (3.13.3)$$

The symmetric tensor  $\mathbf{D}$  is known as the **rate of deformation tensor** and the antisymmetric tensor  $\mathbf{W}$  is known as the **spin tensor**. The reason for these names will be apparent soon.

With respect to rectangular Cartesian coordinates, the components of  $\mathbf{D}$  and  $\mathbf{W}$  are given by:

$$[\mathbf{D}] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) \\ \frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) & \frac{\partial v_2}{\partial x_2} & \frac{1}{2} \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right) \\ \frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right) & \frac{\partial v_3}{\partial x_3} \end{bmatrix} \quad (3.13.4)$$

$$[\mathbf{W}] = \begin{bmatrix} 0 & \frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right) & \frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \\ -\frac{1}{2} \left( \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \right) & 0 & \frac{1}{2} \left( \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} \right) \\ -\frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) & -\frac{1}{2} \left( \frac{\partial v_2}{\partial x_3} - \frac{\partial v_3}{\partial x_2} \right) & 0 \end{bmatrix} \quad (3.13.5)$$

With respect to cylindrical and spherical coordinates the matrices take the form given in Eq. (3.7.11) and Eq. (3.7.12).

We now show that the rate of change of length of  $d\mathbf{x}$  is described by the tensor  $\mathbf{D}$  whereas the rate of rotation of  $d\mathbf{x}$  is described by the tensor  $\mathbf{W}$ .

Let  $d\mathbf{x} = ds\mathbf{n}$ , where  $\mathbf{n}$  is a unit vector, then

$$d\mathbf{x} \cdot d\mathbf{x} = (ds)^2 \quad (i)$$

Taking the material derivatives of the above equation gives

$$2dx \cdot \frac{D}{Dt}(dx) = 2ds \frac{D}{Dt}(ds) \tag{ii}$$

Now, from Eq. (3.12.4) and (3.13.1)

$$dx \cdot \frac{D}{Dt}(dx) = dx \cdot (\nabla v)dx = dx \cdot Ddx + dx \cdot Wdx \tag{iii}$$

and by the definition of transpose of a tensor and the fact that  $W$  is an antisymmetric tensor (i.e.,  $W = -W^T$ ), we have

$$dx \cdot Wdx = dx \cdot W^T dx = -dx \cdot Wdx \tag{iv}$$

Thus,

$$dx \cdot Wdx = 0 \tag{v}$$

Therefore,

$$dx \cdot \frac{D}{Dt}(dx) = dx \cdot Ddx \tag{vi}$$

Equation (ii) then gives

$$ds \frac{D(ds)}{Dt} = dx \cdot D dx \tag{3.13.6a}$$

With  $dx = ds n$ , Eq. (3.13.6a) can also be written:

$$\frac{1}{ds} \frac{D(ds)}{Dt} = n \cdot Dn = D_{nn} \quad (\text{no sum on } n) \tag{3.13.6b}$$

Eq. (3.13.6b) states that for a material element in the direction of  $n$ , its **rate of extension** (i.e., rate of change of length per unit length) is given by  $D_{nn}$ (no sum on  $n$ ). The rate of extension is also known as **stretching**. In particular

$D_{11}$  = rate of extension for an element which is in the  $e_1$  direction

$D_{22}$  = rate of extension for an element which is in the  $e_2$  direction and

$D_{33}$  = rate of extension for an element which is in the  $e_3$  direction

We note that since  $v dt$  gives the infinitesimal displacement undergone by a particle during the time interval  $dt$ , the interpretation just given can be inferred from those for the infinitesimal strain components. Thus, we obviously will have the following results: [see also Prob. 3. 45(b)]:

$2 D_{12}$  = rate of decrease of angle (from  $\frac{\pi}{2}$ ) of two elements in  $e_1$  and  $e_2$  directions

$2 D_{13}$  = rate of decrease of angle (from  $\frac{\pi}{2}$ ) of two elements in  $e_1$  and  $e_3$  directions and

$2 D_{23}$  = rate of decrease of angle (from  $\frac{\pi}{2}$ ) of two elements in  $\mathbf{e}_2$  and  $\mathbf{e}_3$  directions.

These rates of decrease of angle are also known as the **rates of shear**, or **shearings**.

Also, the first scalar invariant of the rate of deformation tensor  $\mathbf{D}$  gives the rate of change of volume per unit volume (see also Prob. 3.46). That is,

$$D_{11} + D_{22} + D_{33} = \frac{1}{dV} \frac{D(dV)}{Dt} \tag{3.13.7a}$$

Or, in terms of the velocity components, we have

$$\frac{1}{dV} \frac{D(dV)}{Dt} = \frac{\partial v_i}{\partial x_i} = \text{div} \mathbf{v} \tag{3.13.7b}$$

Since  $\mathbf{D}$  is symmetric, we also have the result that there always exists three mutually perpendicular directions (eigenvectors of  $\mathbf{D}$ ) along which the stretchings (eigenvalues of  $\mathbf{D}$ ) include a maximum and a minimum value among all differential elements extending from a material point.

Example 3.13.1

Given the velocity field:

$$v_1 = kx_2 \quad v_2 = v_3 = 0 \tag{i}$$

- (a) Find the rate of deformation and spin tensor.
- (b) Determine the rate of extension of the material elements:

$$d\mathbf{x}^{(1)} = (ds_1)\mathbf{e}_1, \quad d\mathbf{x}^{(2)} = (ds_2)\mathbf{e}_2, \quad \text{and} \quad d\mathbf{x} = \frac{ds}{\sqrt{5}}(\mathbf{e}_1 + 2\mathbf{e}_2) \tag{ii}$$

- (c) Find the maximum and minimum rates of extension.

*Solution.* (a) The matrix of the velocity gradient is

$$[\nabla \mathbf{v}] = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

so that

$$[\mathbf{D}] = [\nabla \mathbf{v}]^s = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iv}$$

and

$$[\mathbf{W}] = [\nabla\mathbf{v}]^A = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ -\frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{v})$$

(b) The material element  $d\mathbf{x}^{(1)}$  is currently in the  $\mathbf{e}_1$ -direction and therefore its rate of extension is equal to  $D_{11} = 0$ . Similarly, the rate of extension of  $d\mathbf{x}^{(2)}$  is equal to  $D_{22} = 0$ . For the element  $d\mathbf{x} = (ds)\mathbf{n}$ , where  $\mathbf{n} = \left(\frac{1}{\sqrt{5}}\right)(\mathbf{e}_1 + 2\mathbf{e}_2)$

$$\frac{1}{ds} \frac{D}{Dt}(ds) = \mathbf{n} \cdot \mathbf{D}\mathbf{n} = \frac{1}{5}[1, 2, 0] \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \frac{2}{5}k \quad (\text{vi})$$

(c) From the characteristic equation

$$|\mathbf{D} - \lambda\mathbf{I}| = -\lambda(\lambda^2 - k^2/4) = 0 \quad (\text{vii})$$

we determine the eigenvalues of the tensor  $\mathbf{D}$  as  $\lambda = 0, \pm k/2$ , therefore,  $k/2$  is the maximum and  $-k/2$  is the minimum rate of extension. The eigenvectors  $\mathbf{n}_1 = \left(\frac{\sqrt{2}}{2}\right)(\mathbf{e}_1 + \mathbf{e}_2)$  and  $\mathbf{n}_2 = \left(\frac{\sqrt{2}}{2}\right)(\mathbf{e}_1 - \mathbf{e}_2)$  give the directions of the elements having the maximum and the minimum stretching respectively.

### 3.14 The Spin Tensor and the Angular Velocity Vector

In section 2B.16 of Chapter 2, it was shown that an antisymmetric tensor  $\mathbf{W}$  is equivalent to a vector  $\boldsymbol{\omega}$  in the sense that for any vector  $\mathbf{a}$

$$\mathbf{W}\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a} \quad (3.14.1)$$

The vector  $\boldsymbol{\omega}$  is called the dual vector or axial vector of the tensor  $\mathbf{W}$  and is related to the three nonzero components of  $\mathbf{W}$  by the relation:

$$\boldsymbol{\omega} = -(W_{23}\mathbf{e}_1 + W_{31}\mathbf{e}_2 + W_{12}\mathbf{e}_3) \quad (3.14.2)$$

Now, since the spin tensor  $\mathbf{W}$  is an antisymmetric tensor (by definition, the antisymmetric part of  $\nabla\mathbf{v}$ ), therefore

$$\mathbf{W}d\mathbf{x} = \boldsymbol{\omega} \times d\mathbf{x} \quad (3.14.3)$$

and

$$\frac{Dd\mathbf{x}}{Dt} = (\nabla\mathbf{v})d\mathbf{x} = (\mathbf{D}+\mathbf{W})d\mathbf{x} = \mathbf{D}d\mathbf{x}+\boldsymbol{\omega}\times d\mathbf{x} \quad (3.14.4)$$

We have already seen in the previous section that  $\mathbf{W}$  does not contribute to the rate of change of length of the material vector  $d\mathbf{x}$ . Thus, Eq. (3.14.3) shows that its effect on  $d\mathbf{x}$  is simply to rotate it (without changing its length) with an angular velocity  $\boldsymbol{\omega}$ .

It should be noted however, that the rate of deformation tensor  $\mathbf{D}$  also contributes to the rate of change in direction of  $d\mathbf{x}$  as well so that in general, most material vectors  $d\mathbf{x}$  rotate with an angular velocity different from  $\boldsymbol{\omega}$  (while changing their lengths). Indeed, it can be proven that in general, only the three material vectors which are in the principal direction of  $\mathbf{D}$  do rotate with the angular velocity  $\boldsymbol{\omega}$ , (while changing their length). (see Prob. 3.47)

We also note that in fluid mechanics literature,  $2\mathbf{W}$  is called the **vorticity tensor**.

### 3.15 Equation of Conservation of Mass

If we follow an infinitesimal volume of material through its motion, its volume  $dV$  and density  $\rho$  may change, but its total mass  $\rho dV$  will remain unchanged. That is,

$$\frac{D}{Dt}(\rho dV) = 0 \quad (3.15.1)$$

i.e.,

$$\rho \frac{D(dV)}{Dt} + dV \frac{D\rho}{Dt} = 0$$

Using Eq. (3.13.7), we obtain

$$\rho \frac{\partial v_i}{\partial x_i} + \frac{D\rho}{Dt} = 0 \quad (3.15.2a)$$

Or, in invariant form,

$$\rho \operatorname{div}\mathbf{v} + \frac{D\rho}{Dt} = 0 \quad (3.15.2b)$$

where in spatial description,

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \mathbf{v} \cdot \nabla\rho \quad (3.15.3)$$

Equation (3.15.2) is the equation of conservation of mass, also known as the equation of continuity.

In Cartesian coordinates, Eq. (3.15.2b) reads:

$$\rho \left( \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) + \frac{\partial\rho}{\partial t} + v_1 \frac{\partial\rho}{\partial x_1} + v_2 \frac{\partial\rho}{\partial x_2} + v_3 \frac{\partial\rho}{\partial x_3} = 0 \quad (3.15.4)$$

In cylindrical coordinates, it reads:

$$\rho \left( \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} \right) + \frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \frac{v_\theta}{r} \frac{\partial \rho}{\partial \theta} + v_z \frac{\partial \rho}{\partial z} = 0 \quad (3.15.5)$$

In spherical coordinates it reads:

$$\rho \left( \frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{2v_r}{r} + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\theta \cot \theta}{r} \right) + \frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + \frac{v_\theta}{r} \frac{\partial \rho}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial \rho}{\partial \phi} = 0 \quad (3.15.6)$$

For an **incompressible material**, the material derivative of the density is zero, and the mass conservation of equation reduces to simply:

$$\text{div} \mathbf{v} = 0 \quad (3.15.7)$$

or, in Cartesian coordinates

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0. \quad (3.15.7a)$$

in cylindrical coordinates

$$\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = 0. \quad (3.15.7b)$$

and in spherical coordinates

$$\frac{\partial v_r}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{2v_r}{r} + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\theta \cot \theta}{r} = 0. \quad (3.15.7c)$$

### Example 3.15.1

For the velocity field of Example 3.4.2,

$$v_i = \frac{x_i}{(1+t)}$$

find the density of a material particle as a function of time.

*Solution.* From the mass conservation equation

$$\frac{D\rho}{Dt} = -\rho \frac{\partial v_i}{\partial x_i} = -\rho \left[ \frac{1}{1+t} + \frac{1}{1+t} + \frac{1}{1+t} \right] = -\frac{3\rho}{1+t} \quad (i)$$

Thus,

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = - \int_0^t \frac{3dt}{1+t}$$

from which we obtain

$$\rho = \frac{\rho_0}{(1+t)^3} \quad (\text{iii})$$

### 3.16 Compatibility Conditions for Infinitesimal Strain Components

When any three displacement functions  $u_1$ ,  $u_2$ , and  $u_3$  are given, one can always determine the six strain components in any region where the partial derivatives  $\frac{\partial u_i}{\partial X_j}$  exist. On the other hand, when the six strain components ( $E_{11}, E_{22}, E_{33}, E_{12}, E_{13}, E_{23}$ ) are arbitrarily prescribed in some region, in general, there may not exist three displacement functions ( $u_1, u_2, u_3$ ), satisfying the six equations

$$\frac{\partial u_1}{\partial X_1} = E_{11} \quad (3.16.1)$$

$$\frac{\partial u_2}{\partial X_2} = E_{22} \quad (3.16.2)$$

$$\frac{\partial u_3}{\partial X_3} = E_{33} \quad (3.16.3)$$

$$\frac{1}{2} \left( \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \right) = E_{12} \quad (3.16.4)$$

$$\frac{1}{2} \left( \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \right) = E_{23} \quad (3.16.5)$$

$$\frac{1}{2} \left( \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3} \right) = E_{31} \quad (3.16.6)$$

For example, if we let

$$E_{11} = X_2^2, \quad E_{22} = E_{33} = E_{12} = E_{13} = E_{23} = 0 \quad (\text{i})$$

then, from Eq. (3.16.1)  $\frac{\partial u_1}{\partial X_1} = X_2^2$  and from Eq. (3.16.2),  $\frac{\partial u_2}{\partial X_2} = 0$ , so that

$$u_1 = X_1 X_2^2 + f(X_2, X_3) \quad (\text{ii})$$

and

$$u_2 = g(X_1, X_3) \quad (\text{iii})$$

where  $f$  and  $g$  are arbitrary integration functions. Now, since  $E_{12} = 0$ , we must have, from Eq. (3.16.4)

$$\frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} = 0 \quad (\text{iv})$$

Using Eqs. (ii) and (iii), we get from Eq. (iv)

$$2X_1X_2 + \frac{\partial f(X_2, X_3)}{\partial X_2} + \frac{\partial g(X_1, X_3)}{\partial X_1} = 0 \quad (\text{v})$$

Since the second or third term cannot have terms of the form  $X_1X_2$ , the above equation can never be satisfied. In other words, there is no displacement field corresponding to this given  $E_{ij}$ . That is, the given six strain components are not compatible with the three displacement-strain equations.

We now state the following theorem: If  $E_{ij}(X_1, X_2, X_3)$  are continuous functions having continuous second partial derivatives in a simply connected region, then the necessary and sufficient conditions for the existence of single-valued continuous solutions  $u_1, u_2$  and  $u_3$  of the six equation Eq. (3.16.1) to Eq. (3.16.6) are

$$\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = 2 \frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} \quad (\text{3.16.7})$$

$$\frac{\partial^2 E_{22}}{\partial X_3^2} + \frac{\partial^2 E_{33}}{\partial X_2^2} = 2 \frac{\partial^2 E_{23}}{\partial X_2 \partial X_3} \quad (\text{3.16.8})$$

$$\frac{\partial^2 E_{33}}{\partial X_1^2} + \frac{\partial^2 E_{11}}{\partial X_3^2} = 2 \frac{\partial^2 E_{31}}{\partial X_3 \partial X_1} \quad (\text{3.16.9})$$

$$\frac{\partial^2 E_{11}}{\partial X_2 \partial X_3} = \frac{\partial}{\partial X_1} \left( \frac{-\partial E_{23}}{\partial X_1} + \frac{\partial E_{31}}{\partial X_2} + \frac{\partial E_{12}}{\partial X_3} \right) \quad (\text{3.16.10})$$

$$\frac{\partial^2 E_{22}}{\partial X_3 \partial X_1} = \frac{\partial}{\partial X_2} \left( \frac{-\partial E_{31}}{\partial X_2} + \frac{\partial E_{12}}{\partial X_3} + \frac{\partial E_{23}}{\partial X_1} \right) \quad (\text{3.16.11})$$

$$\frac{\partial^2 E_{33}}{\partial X_1 \partial X_2} = \frac{\partial}{\partial X_3} \left( \frac{-\partial E_{12}}{\partial X_3} + \frac{\partial E_{23}}{\partial X_1} + \frac{\partial E_{31}}{\partial X_2} \right) \quad (\text{3.16.12})$$

These six equations are known as the equations of compatibility (or integrability conditions).

That these conditions are necessary can be easily proved as follows:

From

$$\frac{\partial u_1}{\partial X_1} = E_{11} \quad \text{and} \quad \frac{\partial u_2}{\partial X_2} = E_{22} \quad (\text{i})$$

we get

$$\frac{\partial^2 E_{11}}{\partial X_2^2} = \frac{\partial^3 u_1}{\partial X_2^2 \partial X_1} \quad \text{and} \quad \frac{\partial^2 E_{22}}{\partial X_1^2} = \frac{\partial^3 u_2}{\partial X_1^2 \partial X_2} \quad (\text{ii})$$

Now, since the left-hand sides of the above equations are, by postulate, continuous, therefore, the right-hand sides are continuous, and so the order of the differentiation is immaterial, so that

$$\frac{\partial^2 E_{11}}{\partial X_2^2} = \frac{\partial^2}{\partial X_1 \partial X_2} \left( \frac{\partial u_1}{\partial X_2} \right) \quad \text{and} \quad \frac{\partial^2 E_{22}}{\partial X_1^2} = \frac{\partial^2}{\partial X_1 \partial X_2} \left( \frac{\partial u_2}{\partial X_1} \right) \quad (\text{iii})$$

Thus, from Eqs. (iii) and Eq. (3.16.4)

$$\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = \frac{\partial^2}{\partial X_1 \partial X_2} \left( \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \right) = 2 \frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} \quad (\text{iv})$$

The other five conditions can be similarly established. We omit the proof that the conditions are also sufficient (under the conditions stated in the theorem). In Example 3.16.3 below, we shall give an instance where the conditions are not sufficient for a region which is not simply-connected. (A region of space is said to be simply-connected if every closed curve drawn in the region can be shrunk to a point, by continuous deformation, without passing out of the boundaries of the region. For example, the solid prismatical bar represented in Fig. 3.7 is simply-connected whereas, the prismatical tube represented in Fig. 3.8 is not simply-connected).

It is worth noting the following two special cases of strain components where the compatibility conditions need not be considered because they are obviously satisfied:

- (1) The strain components are obtained from given displacement components.
- (2) The strain components are linear functions of coordinates.

#### Example 3.16.1

Will the strain components obtained from the displacements

$$u_1 = X_1^3, \quad u_2 = e^{X_1}, \quad u_3 = \sin X_2 \quad (\text{i})$$

be compatible?

*Solution.* Yes. There is no need to check, because the displacement  $\mathbf{u}$  is given (and therefore exists!)

Example 3.16.2

Does the following strain field:

$$[\mathbf{E}] = \begin{bmatrix} 2X_1 & X_1+2X_2 & 0 \\ X_1+2X_2 & 2X_1 & 0 \\ 0 & 0 & 2X_3 \end{bmatrix} \quad (i)$$

represent a compatible strain field?

*Solution.* Since each term of the compatibility equations involves second derivatives of the strain components with respect to the coordinates, the above strain tensor with each component a linear function of  $X_1, X_2, X_3$  will obviously satisfy them. The given strain components are obviously continuous functions having continuous second derivatives (in fact continuous derivatives of all orders) in any bounded region. Thus, the existence of single valued continuous displacement field in any bounded simply-connected region is ensured by the theorem stated above. In fact, it can be easily verified that

$$u_1 = X_1^2 + X_2^2, \quad u_2 = 2X_1X_2 + X_1^2, \quad u_3 = X_3^2 \quad (ii)$$

(to which of course, can be added any rigid body displacements) which is a single-valued continuous displacement field in any bounded region, including multiply-connected region.

Example 16.3

For the following strain field

$$E_{11} = \frac{-X_2}{X_1^2 + X_2^2}, \quad E_{12} = \frac{X_1}{2(X_1^2 + X_2^2)}, \quad E_{22} = E_{33} = E_{23} = E_{13} = 0 \quad (i)$$

does there exist single-valued continuous displacement fields for (a) the cylindrical body with the normal cross-section shown in Fig. 3.7 and (b) for the body with the normal cross-section shown in Fig. 3.8 and with the origin of the axis inside the hole of the cross-section.

*Solution.* Out of the six compatibility conditions, only the first one needs to be checked, the others are automatically satisfied. Now,

$$\frac{\partial E_{11}}{\partial X_2} = -\frac{(X_1^2 + X_2^2) - X_2(2X_2)}{(X_1^2 + X_2^2)^2} = \frac{X_2^2 - X_1^2}{(X_1^2 + X_2^2)^2} \quad (ii)$$

$$2 \frac{\partial E_{12}}{\partial X_1} = \frac{(X_1^2 + X_2^2) - X_1(2X_1)}{(X_1^2 + X_2^2)^2} = \frac{X_2^2 - X_1^2}{(X_1^2 + X_2^2)^2} \tag{iii}$$

and

$$\frac{\partial^2 E_{22}}{\partial X_1^2} = 0 \tag{iv}$$

Thus, the equation

$$\frac{\partial^2 E_{11}}{\partial X_2^2} + \frac{\partial^2 E_{22}}{\partial X_1^2} = 2 \frac{\partial^2 E_{12}}{\partial X_1 \partial X_2} \tag{v}$$

is satisfied, and the existence of solution is assured. In fact it can be easily verified that for the given  $E_{ij}$ ,

$$u_1 = \arctan \frac{X_2}{X_1}, \quad u_2 = 0, \quad u_3 = 0 \tag{vi}$$

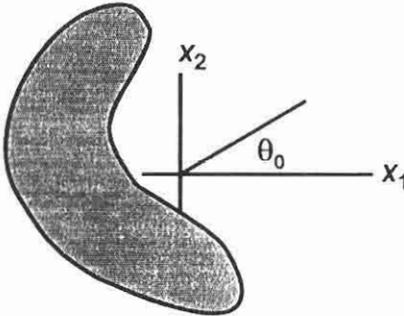


Fig. 3.7

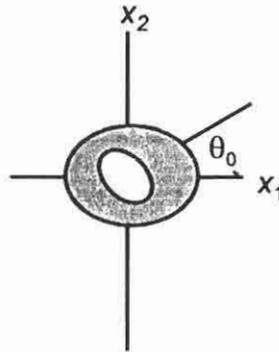


Fig. 3.8

(to which, of course, any rigid body displacement field can be added). Now  $\arctan X_2/X_1$  is a multiple-valued function, having infinitely many values corresponding to a point  $(X_1, X_2, X_3)$ .

For example, for the point  $(X_1, X_2, X_3) = (1, 0, 0)$ ,  $\arctan X_2/X_1 = 0, 2\pi, 4\pi$ , etc. It can be made a single-valued function by the restriction  $\theta_o \leq \arctan X_2/X_1 < \theta_o + 2\pi$  for any  $\theta_o$ . For a simply-connected region as that shown in Fig. 3.7, a  $\theta_o$  can be chosen so that such a restriction makes Eq. (vi) a single-valued continuous displacement for the region. But for the body shown in Fig. 3.8, the function  $u_1 = \arctan X_2/X_1$ , under the same restriction is discontinuous along the line  $\theta = \theta_o$  in the body ( in fact,  $u_1$  jumps by the value of  $2\pi$  in crossing the line). Thus, for this so-called doubly-connected region, there does not exist single-valued continuous  $u_1$  corresponding to the given  $E_{ij}$ , even though the compatibility equations are satisfied.

**3.17 Compatibility Conditions For Rate Of Deformation**

When any three velocity functions  $v_1, v_2$ , and  $v_3$  are given, one can always determine the six rate of deformation components in any region where the partial derivatives  $\partial v_i/\partial x_j$  exist. On the other hand, when the six components  $D_{ij}$  are arbitrarily prescribed in some region, in general, there does not exist any velocity field  $v_i$ , satisfying the six equations

$$\frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = D_{ij} \tag{3.17.1}$$

The compatibility conditions for the rate of deformation components are similar to those of the infinitesimal strain components [Eqs. (3.16.7-12)], i.e.,

$$\frac{\partial^2 D_{11}}{\partial x_2^2} + \frac{\partial^2 D_{22}}{\partial x_1^2} = 2 \frac{\partial^2 D_{12}}{\partial x_1 \partial x_2} \tag{3.17.2a}$$

$$\frac{\partial^2 D_{22}}{\partial x_3^2} + \frac{\partial^2 D_{33}}{\partial x_2^2} = 2 \frac{\partial^2 D_{23}}{\partial x_2 \partial x_3} \tag{3.17.2b}$$

$$\frac{\partial^2 D_{33}}{\partial x_1^2} + \frac{\partial^2 D_{11}}{\partial x_3^2} = 2 \frac{\partial^2 D_{13}}{\partial x_1 \partial x_3} \tag{3.17.2c}$$

etc. It should be emphasized that if one deals directly with differentiable velocity functions  $v_i(x_1, x_2, x_3, t)$ , (as is often the case in fluid mechanics), the question of compatibility does not arise.

### 3.18 Deformation Gradient

We recall that the general motion of a continuum is described by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \quad (3.18.1)$$

where  $\mathbf{x}$  is the spatial position at time  $t$ , of a material particle with a material coordinate  $\mathbf{X}$ . A material element  $d\mathbf{X}$  at the reference configuration is transformed, through motion, into a material element  $d\mathbf{x}$  at time  $t$ . The relation between  $d\mathbf{X}$  and  $d\mathbf{x}$  is given by

$$d\mathbf{x} = \mathbf{x}(\mathbf{X} + d\mathbf{X}, t) - \mathbf{x}(\mathbf{X}, t) = (\nabla \mathbf{x})d\mathbf{X} \quad (3.18.2)$$

i.e.,

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \quad (3.18.3)$$

where the tensor

$$\mathbf{F} = \nabla \mathbf{x} \quad (3.18.4)$$

is called the **deformation gradient** at  $\mathbf{X}$ . The notation  $\nabla \mathbf{x}$  is an abbreviation for the notation  $\nabla_{\mathbf{X}} \mathbf{x}$  where the subscript  $\mathbf{X}$  indicates that the gradient is with respect to  $\mathbf{X}$  for the function  $\mathbf{x}(\mathbf{X}, t)$ . We note that with  $\mathbf{x} = \mathbf{X} + \mathbf{u}$ , where  $\mathbf{u}$  is the displacement vector,

$$\mathbf{F} = \mathbf{I} + \nabla \mathbf{u} \quad (3.18.5)$$

#### Example 3.18.1

Given the following motion:

$$x_1 = X_1 + X_1^2 t, \quad x_2 = X_2 - X_2 t - X_3 t, \quad x_3 = X_3 + X_2 t - X_3 t \quad (i)$$

where both  $x_i$  and  $X_i$  are rectangular Cartesian coordinates. Find the deformation gradient at  $t = 0$  and at  $t = 1$ .

*Solution.* For rectangular Cartesian coordinates,

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \quad (ii)$$

Thus, from Eq. (i) and (ii),

$$[\mathbf{F}] = \begin{bmatrix} 1 + 2X_1 t & 0 & 0 \\ 0 & 1 - t & -t \\ 0 & t & 1 - t \end{bmatrix} \quad (iii)$$

From Eq. (iii) we have at  $t = 0$ ,  $\mathbf{F} = \mathbf{I}$ , and  $dx = d\mathbf{X}$ .

At  $t = 1$ , for all elements

$$[\mathbf{F}] = \begin{bmatrix} 1+2X_1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{iv})$$

### 3.19 Local Rigid Body Displacements

In Section 3.6, we discussed the case where the entire body undergoes rigid body displacements from the configuration at a reference time  $t_0$  to that at a particular time  $t$ . For a body in a general motion, however, it is possible that the body as a whole undergoes deformations while some (infinitesimally) small volumes of material inside the body undergo rigid body displacements. For example, for the motion given in the last example, at  $t = 1$  and  $X_1 = 0$ ,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{i})$$

It is easily to verify that the above  $\mathbf{F}$  is a rotation tensor  $\mathbf{R}$  (i.e.,  $\mathbf{F}\mathbf{F}^T = \mathbf{I}$  and  $\det \mathbf{F} = +1$ ).

Thus, all infinitesimal material volumes with material coordinates  $(0, X_2, X_3)$  undergo a rigid body displacement from the reference position to the position at  $t = 1$ .

### 3.20 Finite Deformation

Deformations at a material point  $\mathbf{X}$  of a body are characterized by changes of distances between any pair of material points within the small neighborhood of  $\mathbf{X}$ . Since, through motion, a material element  $d\mathbf{X}$  becomes  $dx = \mathbf{F}d\mathbf{X}$ , whatever deformation there may be at  $\mathbf{X}$ , is embodied in the deformation gradient  $\mathbf{F}$ . We have already seen that if  $\mathbf{F}$  is a proper orthogonal tensor, then there is no deformation at  $\mathbf{X}$ . In the following, we first consider the case where the deformation gradient  $\mathbf{F}$  is a symmetric tensor before going to more general cases.

We shall use the notation  $\mathbf{U}$  for a deformation gradient  $\mathbf{F}$  that is symmetric. Thus, for a symmetric deformation gradient, we write

$$dx = \mathbf{U}d\mathbf{X} \quad (3.20.1)$$

In this case, the material within a small neighborhood of  $\mathbf{X}$  is said to be in a state of **pure stretch** deformation (from the reference configuration). Of course, Eq. (3.20.1) includes the special case where the motion is homogeneous, i.e.,  $\mathbf{x} = \mathbf{U}\mathbf{X}$ , ( $\mathbf{U} = \text{constant tensor}$ ) in which case the entire body is in a state of pure stretch.

Since  $\mathbf{U}$  is real and symmetric, there exists three mutually perpendicular directions, with respect to which, the matrix of  $\mathbf{U}$  is diagonal. Thus, if  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  are these principal directions, with eigenvalues  $\lambda_1, \lambda_2, \lambda_3$ , then, for  $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$ , Eq. (3.20.1) gives  $d\mathbf{x}^{(1)} = \lambda_1 dX_1\mathbf{e}_1$ , i.e.,

$$d\mathbf{x}^{(1)} = \lambda_1 d\mathbf{X}^{(1)} \tag{3.20.2a}$$

Similarly, for  $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$  and  $d\mathbf{X}^{(3)} = dX_3\mathbf{e}_3$ , we have

$$d\mathbf{x}^{(2)} = \lambda_2 d\mathbf{X}^{(2)} \tag{3.20.2b}$$

$$d\mathbf{x}^{(3)} = \lambda_3 d\mathbf{X}^{(3)} \tag{3.20.2c}$$

We see that along each of these three directions, the deformed element is in the same direction as the undeformed element. If the eigenvalues are distinct, these will be the only elements which do not change their directions. The ratio of the deformed length to the original length is called the **stretch**, i.e.,

$$\text{Stretch} \equiv \frac{|d\mathbf{x}|}{|d\mathbf{X}|} \tag{3.20.3}$$

Thus, the eigenvalues of  $\mathbf{U}$  are the principal stretches; they include the maximum and the minimum stretches.

Example 3.20.1

Given that at time  $t$ ,

$$\begin{aligned} x_1 &= 3X_1 \\ x_2 &= 4X_2 \\ x_3 &= X_3 \end{aligned} \tag{i}$$

Referring to Fig. 3.9, find the stretches for the following material line (a)  $OP$  (b)  $OQ$  and (c)  $OB$ .

*Solution.* The matrix of the deformation gradient for this given motion is

$$[\mathbf{F}] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is a symmetric matrix and is independent of  $X_i$  (i.e., the same for all material points). Thus, the given deformation is a homogeneous pure stretch deformation. The eigenvectors are obviously (see Sect. 2B.17, Example 2B17.2)  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  with corresponding eigenvalues, 3, 4 and 1. Thus:

(a) At the deformed state, the line  $OP$  triples its original length and remains parallel to the  $x_1$ -axis, i.e., stretch  $\equiv \lambda_1 = 3$ .

(b) At the deformed state, the line  $OQ$  quadruple its original length and remains parallel to the  $x_2$ -axis; stretch  $\equiv \lambda_2 = 4$ .

(c) The line  $OB$  has an original length of 1.414. In the deformed state, it has a length of 5, thus, the stretch is  $5/1.414$ . Originally, the line  $OB$  makes an angle of  $45^\circ$  with the  $x_1$ -axis; in the deformed state, it makes an angle of  $\tan^{-1}(4/3)$ . In other words, the material line  $OB$  changes its direction from  $OB$  to  $OB'$  (see Fig. 3.9).

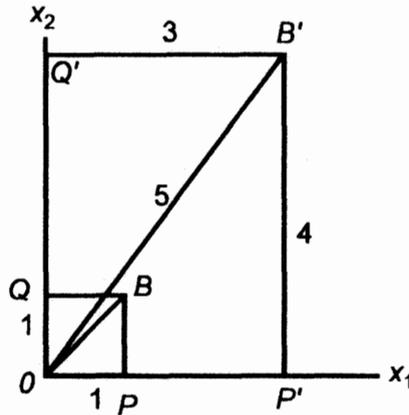


Fig. 3.9

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### Example 3.20.2

For a material sphere with center at  $\mathbf{X}$  and described by  $|d\mathbf{X}| = \varepsilon$ , under a symmetric deformation gradient  $\mathbf{U}$ , what does the sphere become after the deformation?

*Solution.* Let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  be the principal directions for  $\mathbf{U}$ , then with respect  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  a material element  $d\mathbf{X}$  can be written

$$d\mathbf{X} = dX_1\mathbf{e}_1 + dX_2\mathbf{e}_2 + dX_3\mathbf{e}_3 \quad (\text{i})$$

In the deformed state, this material vector becomes

$$d\mathbf{x} = dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2 + dx_3\mathbf{e}_3 \quad (\text{ii})$$

Since  $\mathbf{F}$  is diagonal, with diagonal element  $\lambda_1, \lambda_2, \lambda_3$ , therefore  $d\mathbf{x} = \mathbf{F}d\mathbf{X}$  gives

$$dx_1 = \lambda_1 dX_1, \quad dx_2 = \lambda_2 dX_2, \quad dx_3 = \lambda_3 dX_3, \quad (\text{iii})$$

thus, the sphere :

$$(dX_1)^2 + (dX_2)^2 + (dX_3)^2 = \varepsilon^2 \quad (\text{iv})$$

becomes

$$\left(\frac{dx_1}{\lambda_1}\right)^2 + \left(\frac{dx_2}{\lambda_2}\right)^2 + \left(\frac{dx_3}{\lambda_3}\right)^2 = \varepsilon^2 \tag{v}$$

This is the equation of an ellipsoid with its axis parallel to the eigenvectors of  $U$ . (see Fig. 3.10).

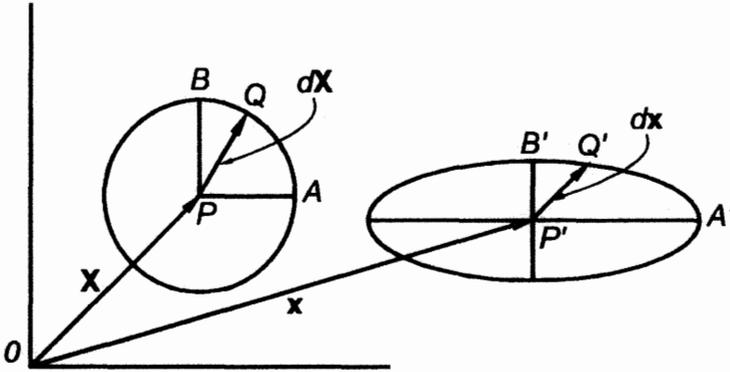


Fig. 3.10

### 3.21 Polar Decomposition Theorem

In the previous two sections, we considered two special deformation gradients  $F$ : a proper orthogonal  $F$  (denoted by  $R$ ) describing rigid body displacements and a symmetric  $F$  (denoted by  $U$ ) describing pure stretch deformation tensor. It can be shown that for any real tensor  $F$  with a nonzero determinant (i.e.,  $F^{-1}$  exists), one can always decompose it into the product of a proper orthogonal tensor and a symmetric tensor. That is

$$F = RU \tag{3.21.1}$$

or,

$$F = VR \tag{3.21.2}$$

In the above two equations,  $U$  and  $V$  are positive definite symmetric tensors and  $R$  (the same in both equations) is a proper orthogonal tensor. Eqs. (3.21.1) and (3.21.2) are known as the **polar decomposition theorem**. The decomposition is unique in that there is only one  $R$ , one  $U$  and one  $V$  for the above equations. The proof of this theorem consists of two steps: (1) Establishing a procedure which always enables one to obtain a symmetric tensor  $U$  and a proper orthogonal tensor  $R$  (or a symmetric tensor  $V$  and a proper orthogonal tensor  $R$ ) which satisfies Eq. (3.21.1) (or, Eq. (3.21.2)) and (2) proving that the  $U$ ,  $V$  and  $R$  so obtained are unique. The procedures for obtaining the tensors  $U$ ,  $V$ , and  $R$  for a given  $F$  will be

demonstrated in Example 3.22.1 and 3.23.1. The proof of the uniqueness of the decompositions will be given in Example 3.22.2.

For any material element  $d\mathbf{X}$  at  $\mathbf{X}$ , the deformation gradient transforms it (i.e.,  $d\mathbf{X}$ ) into a vector  $d\mathbf{x}$ :

$$d\mathbf{x} = \mathbf{F} d\mathbf{X} = \mathbf{R}\mathbf{U} d\mathbf{X} \quad (3.21.3)$$

Now,  $\mathbf{U}d\mathbf{X}$  describes a pure stretch deformation (Section 3.20) in which there are three mutually perpendicular directions (the eigenvectors of  $\mathbf{U}$ ) along which the material element  $d\mathbf{X}$  stretches (i.e., becomes longer or shorter) but does not rotate. Figure 3.10 depicts the effect of  $\mathbf{U}$  on a spherical volume  $|d\mathbf{X}| = \text{constant}$ ; the spherical volume at  $\mathbf{X}$  becomes an ellipsoid at  $\mathbf{x}$ . (See Example 3.20.2) The effect of  $\mathbf{R}$  in  $\mathbf{R}(\mathbf{U} d\mathbf{X})$  is then simply to rotate this ellipsoid through a rigid body rotation. (See Fig. 3.11)

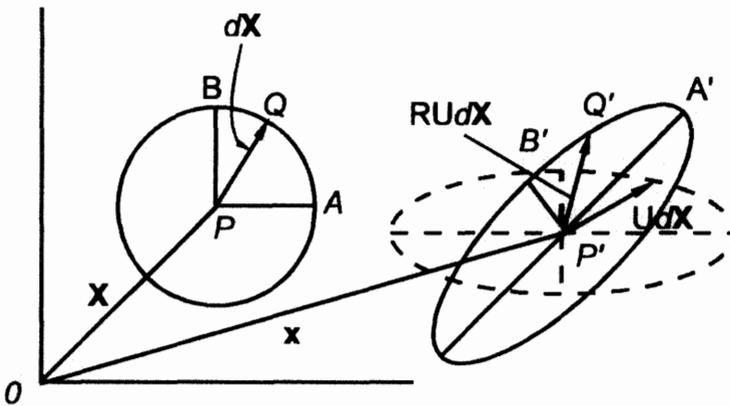


Fig. 3.11

Similarly, the effect of the same deformation gradient can be viewed as a rigid body rotation (described  $\mathbf{R}$ ) of the sphere followed by a pure stretch of the sphere resulting in the same ellipsoid as described in the last paragraph.

From the polar decomposition theorem,  $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$ , it follows immediately that

$$\mathbf{U} = \mathbf{R}^T \mathbf{V} \mathbf{R} \quad (3.21.4)$$

#### Example 3.21.1

Show that if the eigenvector of  $\mathbf{U}$  is  $\mathbf{n}$ , then the eigenvector for  $\mathbf{V}$  is  $\mathbf{R}\mathbf{n}$ ; the eigenvalues for both  $\mathbf{U}$  and  $\mathbf{V}$  are the same

*Solution.* Let  $\mathbf{n}$  be an eigenvector for  $\mathbf{U}$  with eigenvalue  $\lambda$ , then

$$\mathbf{U}\mathbf{n} = \lambda\mathbf{n}, \tag{i}$$

so that

$$\mathbf{R}\mathbf{U}\mathbf{n} = \lambda\mathbf{R}\mathbf{n} \tag{ii}$$

Since  $\mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} = \mathbf{F}$ , therefore, from Eq. (ii), we have

$$\mathbf{V}(\mathbf{R}\mathbf{n}) = \lambda(\mathbf{R}\mathbf{n}) \tag{iii}$$

Thus,  $\mathbf{R}\mathbf{n}$  is an eigenvector of  $\mathbf{V}$  with eigenvalue  $\lambda$ .

### 3.22 Calculation of the Stretch Tensors From the Deformation Gradient

From a given  $\mathbf{F}$ , we have  $\mathbf{F} = \mathbf{R}\mathbf{U}$ , thus,

$$\mathbf{F}^T\mathbf{F} = (\mathbf{R}\mathbf{U})^T(\mathbf{R}\mathbf{U}) = \mathbf{U}^T\mathbf{R}^T\mathbf{R}\mathbf{U} = \mathbf{U}^T\mathbf{U} \tag{i}$$

That is,

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} \tag{3.22.1}$$

From which the positive definite symmetric tensor  $\mathbf{U}$  can be calculated as (See Examples below).

$$\mathbf{U} = (\mathbf{F}^T\mathbf{F})^{1/2} \tag{3.22.2}$$

Once  $\mathbf{U}$  is obtained,  $\mathbf{R}$  can be obtained from the equation

$$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1} \tag{3.22.3}$$

Since

$$\mathbf{U}^{-1}\mathbf{F}^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{U}^{-1}\mathbf{U}^2\mathbf{U}^{-1} = \mathbf{I} \tag{i}$$

therefore, [note that  $\mathbf{U}$  is symmetric],

$$(\mathbf{F}\mathbf{U}^{-1})^T\mathbf{F}\mathbf{U}^{-1} = \mathbf{I} \tag{ii}$$

Thus, from Eq. (3.22.3),

$$\mathbf{R}^T\mathbf{R} = \mathbf{I} \tag{iii}$$

Eq. (iii) states that the tensor  $\mathbf{R}$  obtained from Eq. (3.22.3) is indeed an orthogonal tensor.

The left stretch tensor  $\mathbf{V}$  can be obtained from

$$\mathbf{V} = \mathbf{F}\mathbf{R}^T = \mathbf{R}\mathbf{U}\mathbf{R}^T \tag{3.22.4}$$

## Example 3.22.1

Given

$$x_1 = X_1, \quad x_2 = -3X_3, \quad x_3 = 2X_2 \quad (\text{i})$$

Find (a) the deformation gradient  $\mathbf{F}$ , (b) the right stretch tensor  $\mathbf{U}$ , and (c) the rotation tensor  $\mathbf{R}$  and (d) the left stretch tensor  $\mathbf{V}$ .

*Solution.* (a)

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \quad (\text{ii})$$

(b)

$$[\mathbf{U}^2] = [\mathbf{F}]^T [\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \quad (\text{iii})$$

Thus, the positive definite tensor  $\mathbf{U}$  is given by

$$[\mathbf{U}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (\text{iv})$$

(c)

$$[\mathbf{R}] = [\mathbf{F}][\mathbf{U}^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{v})$$

(d)

$$[\mathbf{V}] = [\mathbf{F}][\mathbf{R}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad (\text{vi})$$

We can also obtain  $\mathbf{V}$  from

$$[\mathbf{V}] = [\mathbf{R}][\mathbf{U}][\mathbf{R}]^T \quad (3.22.2)$$

In this example, the calculation of  $[\mathbf{U}]$  and  $[\mathbf{R}]$  are simple because  $\mathbf{F}^T \mathbf{F}$  happens to be diagonal. If not, one can first diagonalize it to obtain  $[\mathbf{U}]$  and  $[\mathbf{U}]^{-1}$  as diagonal matrices

with respect to the principal axes of  $\mathbf{F}^T\mathbf{F}$ . After that, one then uses the transformation law discussed in Chapter 2 to obtain the matrices with respect to the  $\mathbf{e}_i$  basis. (See Example 3.23.1 below).

Example 3.22.2

(a) Show that if  $\mathbf{F} = \mathbf{R}_1\mathbf{U}_1 = \mathbf{R}_2\mathbf{U}_2$ , then  $\mathbf{R}_1 = \mathbf{R}_2$  and  $\mathbf{U}_1 = \mathbf{U}_2$

(b) Show that if  $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}'$ , then  $\mathbf{R} = \mathbf{R}'$

*Solution.* (a) From  $\mathbf{R}_1\mathbf{U}_1 = \mathbf{R}_2\mathbf{U}_2$ , we have  $(\mathbf{R}_1\mathbf{U}_1)^T = (\mathbf{R}_2\mathbf{U}_2)^T$

Thus,  $\mathbf{U}_1\mathbf{R}_1^T = \mathbf{U}_2\mathbf{R}_2^T$ , so that  $\mathbf{U}_1\mathbf{R}_1^T(\mathbf{R}_1\mathbf{U}_1) = \mathbf{U}_2\mathbf{R}_2^T\mathbf{R}_2\mathbf{U}_2$

In other words,  $\mathbf{U}_1^2 = \mathbf{U}_2^2$ . Since both  $\mathbf{U}_1$  and  $\mathbf{U}_2$  are positive definite, therefore

$$\mathbf{U}_1 = \mathbf{U}_2 = \mathbf{U}$$

and from  $\mathbf{R}_1\mathbf{U} = \mathbf{R}_2\mathbf{U}$ , it follows,

$$\mathbf{R}_1 = \mathbf{R}_2 = \mathbf{R}$$

(b) Since

$$\mathbf{V}\mathbf{R}' = (\mathbf{R}'\mathbf{R}'^{-1})\mathbf{V}\mathbf{R}' = \mathbf{R}'(\mathbf{R}'^{-1}\mathbf{V}\mathbf{R}')$$

thus,

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{R}'(\mathbf{R}'^{-1}\mathbf{V}\mathbf{R}')$$

Noting that  $(\mathbf{R}'^{-1}\mathbf{V}\mathbf{R}')$  is symmetric, from the result of part (a), we have

$$\mathbf{R} = \mathbf{R}'$$

From the decomposition theorem we see that what is responsible for the deformation of a volume of material in a continuum in general motion is the stretch tensor, either  $\mathbf{U}$  (the right stretch tensor) or  $\mathbf{V}$  (the left stretch tensor). Obviously,  $\mathbf{U}^2$  and  $\mathbf{V}^2$  also characterize the deformation, as are many other tensors related to them. In the following sections, we discuss those tensors which have been commonly used to describe finite deformations for a continuum.

### 3.23 Right Cauchy-Green Deformation Tensor

Let

$$\mathbf{C} = \mathbf{U}^2 \tag{3.23.1}$$

where  $\mathbf{U}$  is the right stretch tensor. The tensor  $\mathbf{C}$  is known as the **right Cauchy-Green deformation tensor** (also known as the **Green's deformation tensor**). We note that if there is no deformation,  $\mathbf{U} = \mathbf{C} = \mathbf{I}$ .

Using Eq. (3.22.1), we have

$$\mathbf{C} = \mathbf{F}^T \mathbf{F} \quad (3.23.2)$$

The components of  $\mathbf{C}$  have very simple geometric meanings which are described below.

Consider two material elements  $d\mathbf{x}^{(1)} = \mathbf{F}d\mathbf{X}^{(1)}$  and  $d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)}$ , we have

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(1)} \cdot \mathbf{F}d\mathbf{X}^{(2)} = d\mathbf{X}^{(1)} \cdot \mathbf{F}^T \mathbf{F}d\mathbf{X}^{(2)} \quad (3.23.3)$$

i.e.,

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = d\mathbf{X}^{(1)} \cdot \mathbf{C}d\mathbf{X}^{(2)} \quad (3.23.4)$$

Thus, if  $d\mathbf{x} = ds\mathbf{n}$ , is the deformed vector of the material element  $d\mathbf{X} = dS\mathbf{e}_1$  then Eq. (3.23.4) gives

$$(ds)^2 = (dS)^2 \mathbf{e}_1 \cdot \mathbf{C}\mathbf{e}_1 \quad \text{for } d\mathbf{X}^{(1)} = d\mathbf{X}^{(2)} = dS\mathbf{e}_1$$

That is

$$C_{11} = \left( \frac{ds}{dS} \right)^2 \quad \text{for a material element } d\mathbf{X} = dS\mathbf{e}_1 \quad (3.23.5a)$$

similarly,

$$C_{22} = \left( \frac{ds}{dS} \right)^2 \quad \text{for a material element } d\mathbf{X} = dS\mathbf{e}_2 \quad (3.23.5b)$$

$$C_{33} = \left( \frac{ds}{dS} \right)^2 \quad \text{for a material element } d\mathbf{X} = dS\mathbf{e}_3 \quad (3.23.5c)$$

By considering two material elements  $d\mathbf{X}^{(1)} = dS_1\mathbf{e}_1$  and  $d\mathbf{X}^{(2)} = dS_2\mathbf{e}_2$  which deform into  $d\mathbf{x}^{(1)} = ds_1\mathbf{m}$  and  $d\mathbf{x}^{(2)} = ds_2\mathbf{n}$  where  $\mathbf{m}$  and  $\mathbf{n}$  are unit vectors having an angle of  $\beta$  between them, then Eq. (3.23.4) gives

$$ds_1 ds_2 \cos\beta = dS_1 dS_2 \mathbf{e}_1 \cdot \mathbf{C}\mathbf{e}_2 \quad (3.23.6)$$

That is

$$C_{12} = \frac{ds_1 ds_2}{dS_1 dS_2} \cos(\angle d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}) \quad \text{for } d\mathbf{X}^{(1)} = dS_1\mathbf{e}_1 \text{ and } d\mathbf{X}^{(2)} = dS_2\mathbf{e}_2 \quad (3.23.6a)$$

Similarly

$$C_{13} = \frac{ds_1 ds_3}{dS_1 dS_3} \cos(dx^{(1)}, dx^{(3)}) \text{ for } d\mathbf{X}^{(1)} = dS_1 \mathbf{e}_1 \text{ and } d\mathbf{X}^{(3)} = dS_3 \mathbf{e}_3 \quad (3.23.6b)$$

and

$$C_{23} = \frac{ds_2 ds_3}{dS_2 dS_3} \cos(dx^{(2)}, dx^{(3)}) \text{ for } d\mathbf{X}^{(2)} = dS_2 \mathbf{e}_2 \text{ and } d\mathbf{X}^{(3)} = dS_3 \mathbf{e}_3 \quad (3.23.6c)$$

Example 3.23.1

Given

$$x_1 = X_1 + 2X_2, \quad x_2 = X_2, \quad x_3 = X_3 \quad (i)$$

- (a) Obtain  $\mathbf{C}$
- (b) Obtain the principal values of  $\mathbf{C}$  and the corresponding principal directions
- (c) Obtain the matrix of  $\mathbf{U}$  and  $\mathbf{U}^{-1}$  with respect to the principal directions
- (d) Obtain the matrix of  $\mathbf{U}$  and  $\mathbf{U}^{-1}$  with respect to the  $\mathbf{e}_i$  basis
- (e) Obtain the matrix of  $\mathbf{R}$  with respect to the  $\mathbf{e}_i$  basis

*Solution.* (a) From Eq. (i), we obtain,

$$[\mathbf{F}] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (ii)$$

Thus,

$$[\mathbf{C}] = [\mathbf{F}]^T [\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (iii)$$

The eigenvalues of  $\mathbf{C}$  and their corresponding eigenvectors are easily found to be

$$C_1 = 5.828, \quad \mathbf{n}_1 = \left( \frac{1}{2.613} \right) [\mathbf{e}_1 + 2.414\mathbf{e}_2] = [0.3827\mathbf{e}_1 + 0.9238\mathbf{e}_2]$$

$$C_2 = 0.1716, \quad \mathbf{n}_2 = \left( \frac{1}{1.0824} \right) [\mathbf{e}_1 - 0.4142\mathbf{e}_2] = [0.9238\mathbf{e}_1 - 0.3827\mathbf{e}_2] \quad (iv)$$

$$C_3 = 1, \quad \mathbf{n}_3 = \mathbf{e}_3$$

(b) The matrix of  $\mathbf{C}$  with respect to the principal axis of  $\mathbf{C}$  is

$$[\mathbf{C}] = \begin{bmatrix} 5.828 & 0 & 0 \\ 0 & 0.1716 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (v)$$

(c) The matrix of  $\mathbf{U}$  and  $\mathbf{U}^{-1}$  with respect to the principal axis of  $\mathbf{C}$  are given by

$$[\mathbf{U}]_{\mathbf{n}_i} = \begin{bmatrix} \sqrt{5.828} & 0 & 0 \\ 0 & \sqrt{0.1716} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2.414 & 0 & 0 \\ 0 & 0.4142 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{vi})$$

$$[\mathbf{U}^{-1}]_{\mathbf{n}_i} = \begin{bmatrix} 0.4142 & 0 & 0 \\ 0 & 2.4143 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{vii})$$

(d) The matrix of  $\mathbf{U}$  and  $\mathbf{U}^{-1}$  with respect to  $\mathbf{e}_i$  axes is given by

$$[\mathbf{U}]_{\mathbf{e}_i} = \begin{bmatrix} 0.3827 & 0.9238 & 0 \\ 0.9238 & -0.3827 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2.414 & 0 & 0 \\ 0 & 0.4142 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.3827 & 0.9238 & 0 \\ 0.9238 & -0.3827 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{viii})$$

$$= \begin{bmatrix} 0.7070 & 0.7070 & 0 \\ 0.7070 & 2.121 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\mathbf{U}^{-1}]_{\mathbf{e}_i} = \begin{bmatrix} 0.3827 & 0.9238 & 0 \\ 0.9238 & -0.3827 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.4142 & 0 & 0 \\ 0 & 2.414 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.3827 & 0.9238 & 0 \\ 0.9238 & -0.3827 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.121 & -0.7070 & 0 \\ -0.7070 & 0.7070 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{ix})$$

(e)

$$[\mathbf{R}]_{\mathbf{e}_i} = [\mathbf{F}][\mathbf{U}]^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2.121 & -0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.707 & 0.707 & 0 \\ -0.707 & 0.707 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{x})$$

Using the same procedure as that used in the above example, one can obtain that in general, for

$$[\mathbf{F}] = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3.23.7)$$

$$[\mathbf{R}] = \left(1 + \frac{k^2}{4}\right)^{-1/2} \begin{bmatrix} 1 & \frac{k}{2} & 0 \\ -\frac{k}{2} & 1 & 0 \\ 0 & 0 & \left(1 + \frac{k^2}{4}\right)^{1/2} \end{bmatrix} \quad (3.23.8)$$

Example 3.23.2

Consider the simple shear deformation given by

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3 \quad (i)$$

- (a) What is the stretch for an element which was in the direction of  $\mathbf{e}_1$
- (b) What is the stretch for an element which was in the direction of  $\mathbf{e}_2$
- (c) What is the stretch for an element which was in the direction of  $\mathbf{e}_1 + \mathbf{e}_2$
- (d) In the deformed configuration, what is the angle between the two elements which were in the directions of  $\mathbf{e}_1$  and  $\mathbf{e}_2$

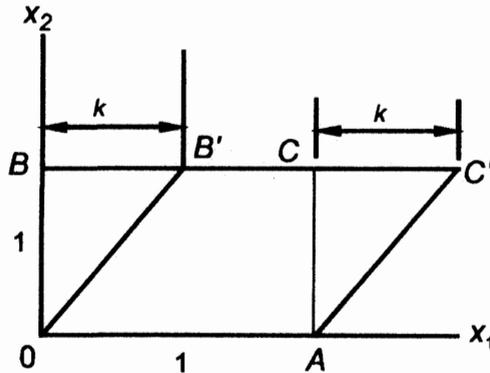


Fig. 3.12

*Solution.*

$$[\mathbf{F}] = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (ii)$$

$$[\mathbf{C}] = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{iii})$$

(a) for  $d\mathbf{X} = dS_1\mathbf{e}_1$ ,  $ds/dS = 1$

(b) for  $d\mathbf{X} = dS_2\mathbf{e}_2$ ,  $ds/dS = \sqrt{1+k^2}$  [e.g.  $OB' = \sqrt{1+k^2}OB$ ]

(c) for  $d\mathbf{X} = (dS/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_2) = dS\mathbf{e}_1'$ ,

$$C'_{11} = \frac{1}{2} [1, 1, 0] \begin{bmatrix} 1 & k & 0 \\ k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 + k + \frac{k^2}{2}$$

thus, for this material element

$$ds/dS = 1 + k + \frac{k^2}{2}$$

(d) For  $d\mathbf{X} = dS_1\mathbf{e}_1$  and  $d\mathbf{X} = dS_2\mathbf{e}_2$

$$\cos(d\mathbf{x}^{(1)}, d\mathbf{x}^{(2)}) = \frac{dS_1}{ds_1} \frac{dS_2}{ds_2} C_{12} = \frac{k}{\sqrt{1+k^2}} \quad [\text{e.g. } \cos AOB' = \frac{k}{\sqrt{1+k^2}}]$$

### Example 3.23.3

Show that (a) the eigenvectors of  $\mathbf{U}$  and  $\mathbf{C}$  are the same and (b) an element which was in the principal direction  $\mathbf{n}$  of  $\mathbf{C}$  becomes, in the deformed state, an element in the direction of  $\mathbf{Rn}$ .

*Solution.* (a) Since  $\mathbf{Un} = \lambda\mathbf{n}$ , therefore  $\mathbf{U}^2\mathbf{n} = \lambda\mathbf{Un} = \lambda^2\mathbf{n}$

i.e.,

$$\mathbf{Cn} = \lambda^2\mathbf{n}$$

Thus,  $\mathbf{n}$  is also an eigenvector of  $\mathbf{C}$  with  $\lambda^2$  as its eigenvalue.

(b) If  $d\mathbf{X} = dS\mathbf{n}$  where  $\mathbf{n}$  is a principal direction of  $\mathbf{U}$  and  $\mathbf{C}$ , then  $\mathbf{U}d\mathbf{X} = dS\mathbf{Un} = dS\lambda\mathbf{n}$  so that

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} = \mathbf{R}\mathbf{U}d\mathbf{X} = \lambda dS(\mathbf{Rn})$$

That is, the deformed vector is in the direction of  $\mathbf{Rn}$ .

### 3.24 Lagrangian Strain Tensor

Let

$$\mathbf{E}^* = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \quad (3.24.1)$$

where  $\mathbf{C}$  is the right Cauchy-Green deformation tensor and  $\mathbf{I}$  is the identity tensor. The tensor  $\mathbf{E}^*$  is known as the **Lagrangian Finite Strain tensor**. We note that if there is no deformation,  $\mathbf{C} = \mathbf{I}$  and  $\mathbf{E}^* = \mathbf{0}$ .

From Eq. (3.23.4), we have

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} - d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} = d\mathbf{X}^{(1)} \cdot (\mathbf{C} - \mathbf{I}) d\mathbf{X}^{(2)}$$

i.e.,

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} - d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} = 2d\mathbf{X}^{(1)} \cdot \mathbf{E}^* d\mathbf{X}^{(2)} \quad (3.24.2)$$

For a material element  $d\mathbf{X} = dS\mathbf{e}_1$ , deforming into  $d\mathbf{x} = ds\mathbf{n}$ , where  $\mathbf{n}$  is a unit vector, Eq. (3.24.2) gives

$$\mathbf{e}_1 \cdot \mathbf{E}^* \mathbf{e}_1 = \frac{ds^2 - dS^2}{2dS^2} \quad (3.24.3)$$

Thus,

$$E_{11}^* = \frac{ds^2 - dS^2}{2dS^2} \quad \text{for } d\mathbf{X} = dS\mathbf{e}_1 \quad (3.24.3a)$$

Similarly,

$$E_{22}^* = \frac{ds^2 - dS^2}{2dS^2} \quad \text{for } d\mathbf{X} = dS\mathbf{e}_2 \quad (3.24.3b)$$

$$E_{33}^* = \frac{ds^2 - dS^2}{2dS^2} \quad \text{for } d\mathbf{X} = dS\mathbf{e}_3 \quad (3.24.3c)$$

We note that for infinitesimal deformations, Eqs.(3.24.3) reduces to Eq. (3.8.1)

By considering two material elements  $d\mathbf{X}^{(1)} = dS_1\mathbf{e}_1$  and  $d\mathbf{X}^{(2)} = dS_2\mathbf{e}_2$ , deforming into  $d\mathbf{x}^{(1)} = ds_1\mathbf{m}$  and  $d\mathbf{x}^{(2)} = ds_2\mathbf{n}$ , where  $\mathbf{m}$  and  $\mathbf{n}$  are unit vectors, then Eq.(3.25.2) gives

$$2E_{12}^* = \frac{ds_1 ds_2}{dS_1 dS_2} \cos(\mathbf{n}, \mathbf{m}) \quad (3.24.4)$$

We note that for infinitesimal deformations, Eq. (3.24.4) reduces to Eq. (3.8.2).

The meanings for  $2E_{13}^*$  and  $2E_{23}^*$  can be established in a similar fashion.

We can also express the components of  $\mathbf{E}^*$  in terms of the displacement components. From Eq. (3.24.1), Eq. (3.23.2) and Eq. (3.18.5), we obtain immediately

$$\mathbf{E}^* = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] + \frac{1}{2}(\nabla\mathbf{u})^T(\nabla\mathbf{u}) \quad (3.24.5a)$$

in component form,

$$E_{ij}^* = \frac{1}{2} \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) + \frac{1}{2} \frac{\partial u_m}{\partial X_i} \frac{\partial u_m}{\partial X_j} \quad (3.24.5b)$$

and in long form,

$$E_{11}^* = \frac{\partial u_1}{\partial X_1} + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial X_1} \right)^2 + \left( \frac{\partial u_2}{\partial X_1} \right)^2 + \left( \frac{\partial u_3}{\partial X_1} \right)^2 \right] \quad (3.24.6a)$$

$$E_{12}^* = \frac{1}{2} \left( \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \right) + \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial X_1} \right) \left( \frac{\partial u_1}{\partial X_2} \right) + \left( \frac{\partial u_2}{\partial X_1} \right) \left( \frac{\partial u_2}{\partial X_2} \right) + \left( \frac{\partial u_3}{\partial X_1} \right) \left( \frac{\partial u_3}{\partial X_2} \right) \right] \quad (3.24.6b)$$

Other components can be similarly written down.

We note that for small values of displacement gradients these equations reduce to those of the infinitesimal deformation tensor.

#### Example 3.24.1

For the simple shear deformation

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3$$

- Compute the Lagrangian Strain tensor  $[\mathbf{E}^*]$
- Referring to Fig. 3.12, by a simple geometrical consideration, find the deformed length for the element  $OB$  in Fig. 3.12.
- Compare the results of (b) with  $E_{22}^*$

*Solution.* (a) Using the  $[\mathbf{C}]$  obtained in Example 3.23.2, we easily obtain from the equation  $2\mathbf{E}^* = \mathbf{C} - \mathbf{I}$  that

$$[\mathbf{E}^*] = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & \frac{k^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(b) From Fig. 3.12, we see from geometry, that  $OB' = OB\sqrt{1+k^2}$

(c)  $E_{22}^* = \frac{k^2}{2}$ . Thus

$$\frac{(\Delta s)^2 - (\Delta S)^2}{2(\Delta S)^2} = \frac{k^2}{2}$$

Thus,  $\Delta s = \Delta S \sqrt{1+k^2}$ , this result is the same as that of (b). We note that if  $k$  is small then  $\Delta s = \Delta S$  to the first order of  $k$ .

Example 3.24.2

Consider the displacement components corresponding to a uniaxial strain field:

$$u_1 = kX_1, \quad u_2 = u_3 = 0 \tag{i}$$

(a) Calculate both the finite Lagrangian strain tensor  $\mathbf{E}^*$  and the infinitesimal strain tensor  $\mathbf{E}$ .

(b) Use the finite strain tensor  $E_{11}^*$  and the infinitesimal strain tensor  $E_{11}$  to calculate  $\frac{\Delta s}{\Delta S}$  for the element  $\Delta \mathbf{X} = \Delta S \mathbf{e}_1$ .

(c) For an element  $\Delta \mathbf{X} = \frac{\Delta S}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$ , calculate  $\frac{\Delta s}{\Delta S}$  from both the finite strain tensor  $\mathbf{E}^*$  and the infinitesimal strain tensor  $\mathbf{E}$ .

*Solution.* (a)

$$[\nabla \mathbf{u}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\nabla \mathbf{u}]^s \tag{ii}$$

Thus, the infinitesimal strain tensor gives

$$[\mathbf{E}] = [\nabla \mathbf{u}]^s = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\nabla \mathbf{u}] \tag{iii}$$

and

$$[\mathbf{E}^*] = [\nabla \mathbf{u}]^s + \frac{1}{2} [\nabla \mathbf{u}]^T [\nabla \mathbf{u}] = \begin{bmatrix} k(1 + \frac{k}{2}) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{iv})$$

(b) Based on  $E_{11}^* = k + \frac{k^2}{2}$ ,  $\frac{(\Delta s)^2 - (\Delta S)^2}{2(\Delta S)^2} = k + \frac{k^2}{2}$ , therefore

$$(\Delta s)^2 = [1 + 2k + k^2] (\Delta S)^2, \quad \Delta s = (1 + k) \Delta S. \quad (\text{v})$$

Based on  $E_{11} = k$ ,  $\frac{\Delta s - \Delta S}{\Delta S} = k$ , therefore

$$\Delta s = (1 + k) \Delta S \quad (\text{vi})$$

We see both the finite and the infinitesimal strain tensor components lead to the same answer whether  $k$  is large or small.

(c) Let  $\mathbf{e}_1' = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$  then,

$$E_{11}' = \frac{1}{2} [1, 1, 0] \begin{bmatrix} k + \frac{k^2}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{k}{2} + \frac{k^2}{4} \quad (\text{vii})$$

Thus,  $\frac{(\Delta s)^2 - (\Delta S)^2}{2(\Delta S)^2} = \frac{k}{2} + \frac{k^2}{4}$ , from which we find  $\Delta s = \sqrt{1 + k + k^2/2} \Delta S$ . This result is easily confirmed by the geometry in Fig. 3.12 for any value of  $k$ . On the other hand, the infinitesimal strain component

$$E_{11}' = \frac{1}{2} [1, 1, 0] \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{k}{2}$$

Thus,  $\frac{\Delta s - \Delta S}{\Delta S} = \frac{k}{2}$ , from which we find  $\Delta s = (1 + \frac{k}{2}) \Delta S$ . From Fig 3.12, we can easily conclude that this result is acceptable only if  $k$  is very small.

This example demonstrates clearly that in the case of finite deformations, the concept of unit elongation (i.e., change of length per unit length) is inadequate.

### 3.25 Left Cauchy-Green Deformation Tensor

Let

$$\mathbf{B} = \mathbf{V}^2 \quad (3.25.1)$$

where  $\mathbf{V}$  is the left stretch tensor. The tensor  $\mathbf{B}$  is known as the **left Cauchy-Green deformation tensor** (also known as the **Finger deformation tensor**). We note that if there is no deformation,  $\mathbf{V} = \mathbf{B} = \mathbf{I}$ .

Since  $\mathbf{F} = \mathbf{VR}$ , and  $\mathbf{R}^T\mathbf{R} = \mathbf{I}$ , it is easily verified that

$$\mathbf{B} = \mathbf{FF}^T \quad (3.25.2)$$

Thus, one can calculate  $\mathbf{B}$  directly from the deformation gradient  $\mathbf{F}$ .

Substituting  $\mathbf{F} = \mathbf{RU}$  in Eq. (3.25.2), we obtain the relation between  $\mathbf{B}$  and  $\mathbf{C}$  as follows:

$$\mathbf{B} = \mathbf{RCR}^T \text{ and } \mathbf{C} = \mathbf{R}^T\mathbf{BR} \quad (3.25.3)$$

We also note that if  $\mathbf{n}$  is an eigenvector of  $\mathbf{C}$  with eigenvalue  $\lambda$ , then  $\mathbf{Rn}$  is an eigenvector of  $\mathbf{B}$  with the same eigenvalue  $\lambda$ .

The components of  $\mathbf{B}$  have simple geometric meanings which are described below:

Consider a material element  $d\mathbf{X} = dS\mathbf{n}$ , where  $\mathbf{n} = \mathbf{R}^T\mathbf{e}_1$ ,  $\mathbf{R}$  being the rotation tensor associated with the deformation gradient  $\mathbf{F}$ . Then from Eq. (3.23.4), we have

$$ds^2 = dS^2\mathbf{n} \cdot \mathbf{C}\mathbf{n} = dS^2\mathbf{R}^T\mathbf{e}_1 \cdot \mathbf{C}\mathbf{R}^T\mathbf{e}_1 = dS^2\mathbf{e}_1 \cdot (\mathbf{C}\mathbf{R}^T)^T\mathbf{R}^T\mathbf{e}_1 = dS^2\mathbf{e}_1 \cdot \mathbf{RCR}^T\mathbf{e}_1 \quad (3.25.4)$$

That is

$$ds^2 = dS^2\mathbf{e}_1 \cdot \mathbf{B}\mathbf{e}_1 \text{ for } d\mathbf{X} = dS(\mathbf{R}^T\mathbf{e}_1), \quad (3.25.5)$$

That is

$$B_{11} = \left(\frac{ds}{dS}\right)^2 \text{ for a material element } d\mathbf{X} = dS(\mathbf{R}^T\mathbf{e}_1) \quad (3.25.5a)$$

similarly,

$$B_{22} = \left(\frac{ds}{dS}\right)^2 \text{ for a material element } d\mathbf{X} = dS(\mathbf{R}^T\mathbf{e}_2) \quad (3.25.5b)$$

$$B_{33} = \left(\frac{ds}{dS}\right)^2 \text{ for a material element } d\mathbf{X} = dS(\mathbf{R}^T\mathbf{e}_3) \quad (3.25.5c)$$

By considering two material elements  $d\mathbf{X}^{(1)} = dS_1(\mathbf{R}^T \mathbf{e}_1)$  and  $d\mathbf{X}^{(2)} = dS_2(\mathbf{R}^T \mathbf{e}_2)$  which deform into  $d\mathbf{x}^{(1)} = ds_1 \mathbf{m}$  and  $d\mathbf{x}^{(2)} = ds_2 \mathbf{n}$  where  $\mathbf{m}$  and  $\mathbf{n}$  are unit vectors having an angle of  $\beta$  between them, then Eq. (3.23.4) gives

$$ds_1 ds_2 \cos \beta = dS_1 dS_2 (\mathbf{R}^T \mathbf{e}_1) \cdot \mathbf{C} (\mathbf{R}^T \mathbf{e}_2) = dS_1 dS_2 \mathbf{e}_1 \cdot \mathbf{B} \mathbf{e}_2 \quad (3.25.6)$$

That is

$$B_{12} = \frac{ds_1 ds_2}{dS_1 dS_2} \cos(dx_1, dx_2) \quad \text{for } d\mathbf{X}^{(1)} = dS_1(\mathbf{R}^T \mathbf{e}_1) \text{ and } d\mathbf{X}^{(2)} = dS_2(\mathbf{R}^T \mathbf{e}_2) \quad (3.25.6a)$$

Similarly

$$B_{13} = \frac{ds_1 ds_3}{dS_1 dS_3} \cos(dx_1, dx_3) \quad \text{for } d\mathbf{X}^{(1)} = dS_1(\mathbf{R}^T \mathbf{e}_1) \text{ and } d\mathbf{X}^{(3)} = dS_3(\mathbf{R}^T \mathbf{e}_3) \quad (3.25.6b)$$

and

$$B_{23} = \frac{ds_2 ds_3}{dS_2 dS_3} \cos(dx_2, dx_3) \quad \text{for } d\mathbf{X}^{(2)} = dS_2(\mathbf{R}^T \mathbf{e}_2) \text{ and } d\mathbf{X}^{(3)} = dS_3(\mathbf{R}^T \mathbf{e}_3) \quad (3.25.6c)$$

We can also express the components of  $\mathbf{B}$  in terms of the displacement components.

Using Eq. (3.18.5), we have

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = (\mathbf{I} + \nabla \mathbf{u})(\mathbf{I} + \nabla \mathbf{u})^T = \mathbf{I} + [\nabla \mathbf{u} + (\nabla \mathbf{u})^T] + (\nabla \mathbf{u})(\nabla \mathbf{u})^T \quad (3.25.7a)$$

and in component form,

$$B_{ij} = \delta_{ij} + \left( \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} \right) + \frac{\partial u_i}{\partial X_m} \frac{\partial u_j}{\partial X_m} \quad (3.25.7b)$$

We note that for small displacement gradients,  $\frac{1}{2}(B_{ij} - \delta_{ij})$  reduces to  $2E_{ij}$  of Eq. (3.7.10a).

### Example 3.25.1

For the simple shear deformation

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3 \quad (i)$$

(a) Obtain the left Cauchy-Green deformation tensor.

(b) Calculate  $\mathbf{R}^T \mathbf{e}_1$  and  $\mathbf{R}^T \mathbf{e}_2$



### 3.26 Eulerian Strain Tensor

Let

$$\mathbf{e}^* \equiv \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1}) \quad (3.26.1)$$

where  $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ , then the tensor  $\mathbf{e}^*$  is known as the **Eulerian Strain Tensor**. We note that if there is no deformation, then  $\mathbf{B}^{-1} = \mathbf{I}$  and  $\mathbf{e}^* = 0$ .

The geometric meaning of the components of  $\mathbf{e}^*$  and  $\mathbf{B}^{-1}$  are described below:

From

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \quad (3.26.2)$$

we have

$$d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x} \quad (3.26.3)$$

where  $\mathbf{F}^{-1}$  is the inverse of  $\mathbf{F}$ . In rectangular Cartesian coordinates, Eq. (3.26.3) reads

$$dX_i = F_{ij}^{-1}dx_j \quad (3.26.4)$$

Thus,

$$F_{ij}^{-1} = \frac{\partial X_i}{\partial x_j} \quad (3.26.5)$$

where  $X_i = X_i(x_1, x_2, x_3, t)$  is the inverse function of  $x_i = x_i(X_1, X_2, X_3, t)$ .

In other words, when rectangular Cartesian coordinates are used for both the reference and the current configuration,

$$[\mathbf{F}^{-1}] = \begin{bmatrix} \frac{\partial X_1}{\partial x_1} & \frac{\partial X_1}{\partial x_2} & \frac{\partial X_1}{\partial x_3} \\ \frac{\partial X_2}{\partial x_1} & \frac{\partial X_2}{\partial x_2} & \frac{\partial X_2}{\partial x_3} \\ \frac{\partial X_3}{\partial x_1} & \frac{\partial X_3}{\partial x_2} & \frac{\partial X_3}{\partial x_3} \end{bmatrix} \quad (3.26.6)$$

Now,

$$d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} = \mathbf{F}^{-1}d\mathbf{x}^{(1)} \cdot \mathbf{F}^{-1}d\mathbf{x}^{(2)} = d\mathbf{x}^{(1)} \cdot (\mathbf{F}^{-1})^T \mathbf{F}^{-1}d\mathbf{x}^{(2)} = d\mathbf{x}^{(1)} \cdot (\mathbf{F}\mathbf{F}^T)^{-1}d\mathbf{x}^{(2)}$$

i.e.,

$$d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} = d\mathbf{x}^{(1)} \cdot \mathbf{B}^{-1}d\mathbf{x}^{(2)} \quad (3.26.7)$$

and

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} - d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} = d\mathbf{x}^{(1)} \cdot (\mathbf{I} - \mathbf{B}^{-1})d\mathbf{x}^{(2)}$$

Or,

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} - d\mathbf{X}^{(1)} \cdot d\mathbf{X}^{(2)} = 2d\mathbf{x}^{(1)} \cdot \mathbf{e}^* d\mathbf{x}^{(2)} \tag{3.26.8}$$

Thus, if we consider a material element, which at time  $t$  is in the direction of  $\mathbf{e}_1$ , i.e.,  $d\mathbf{x} = ds\mathbf{e}_1$  and which at the reference time is  $d\mathbf{X} = dS\mathbf{n}$ , where  $\mathbf{n}$  is a unit vector, then Eq. (3.26.7) and Eq. (3.26.8) give:

For  $d\mathbf{x} = ds\mathbf{e}_1$

$$B_{11}^{-1} = \frac{dS^2}{ds^2}$$

and

$$e_{11}^* = \frac{(ds^2 - dS^2)}{dS^2}$$

Similar meanings hold for the other diagonal elements of  $\mathbf{B}^{-1}$  and  $\mathbf{e}^*$ .

By considering two material elements  $d\mathbf{x}^{(1)} = ds_1\mathbf{e}_1$  and  $d\mathbf{x}^{(2)} = ds_2\mathbf{e}_2$  at time  $t$  corresponding to  $d\mathbf{X}^{(1)} = dS_1\mathbf{n}$  and  $d\mathbf{X}^{(2)} = dS_2\mathbf{m}$  at the reference time,  $\mathbf{n}$  and  $\mathbf{m}$  are unit vectors, Eq. (3.26.7) and Eq. (3.26.8) give

$$B_{12}^{-1} = \frac{dS_1 dS_2}{ds_1 ds_2} \cos(\mathbf{n}, \mathbf{m})$$

$$2e_{12}^* = 1 - \frac{dS_1 dS_2}{ds_1 ds_2} \cos(\mathbf{n}, \mathbf{m})$$

Similar meanings hold for the other off-diagonal elements of  $\mathbf{B}^{-1}$  and  $\mathbf{e}^*$ .

We can also express  $\mathbf{B}^{-1}$  and  $\mathbf{e}^*$  in terms of the displacement components:

From  $\mathbf{u} = \mathbf{x} - \mathbf{X}$ , we can write

$$\mathbf{X} = \mathbf{x} - \mathbf{u}(x_1, x_2, x_3, t) \tag{3.26.9a}$$

or,

$$X_i = x_i - u_i(x_1, x_2, x_3, t) \tag{3.26.9b}$$

where we have used the spatial description of the displacement field because we intend to differentiate this equation with respect to the spatial coordinates  $x_i$ . Thus,

$$\frac{\partial X_i}{\partial x_j} = \delta_{ij} - \frac{\partial u_i}{\partial x_j} \quad (3.26.10a)$$

or,

$$\mathbf{F}^{-1} = \mathbf{I} - \nabla_{\mathbf{x}} \mathbf{u} \quad (3.26.10b)$$

Thus, (dropping the subscript  $\mathbf{x}$  from  $\nabla_{\mathbf{x}} \mathbf{u}$ )

$$\mathbf{B}^{-1} = (\mathbf{F}^{-1})^T \mathbf{F}^{-1} = (\mathbf{I} - \nabla_{\mathbf{x}} \mathbf{u})^T (\mathbf{I} - \nabla_{\mathbf{x}} \mathbf{u}) = \mathbf{I} - [\nabla_{\mathbf{x}} \mathbf{u} + (\nabla_{\mathbf{x}} \mathbf{u})^T] + (\nabla_{\mathbf{x}} \mathbf{u})^T (\nabla_{\mathbf{x}} \mathbf{u}) \quad (3.26.11)$$

and

$$\mathbf{e}^* = \frac{[\nabla_{\mathbf{x}} \mathbf{u} + (\nabla_{\mathbf{x}} \mathbf{u})^T]}{2} - \frac{(\nabla_{\mathbf{x}} \mathbf{u})^T (\nabla_{\mathbf{x}} \mathbf{u})}{2} \quad (3.26.12a)$$

In component form, Eq. (3.26.12a) is

$$e_{ij}^* = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{2} \frac{\partial u_m}{\partial x_i} \frac{\partial u_m}{\partial x_j} \quad (3.26.12b)$$

and in long form,

$$e_{11}^* = \frac{\partial u_1}{\partial x_1} - \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \left( \frac{\partial u_3}{\partial x_1} \right)^2 \right] \quad (3.26.13a)$$

$$e_{12}^* = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) - \frac{1}{2} \left[ \left( \frac{\partial u_1}{\partial x_1} \right) \left( \frac{\partial u_1}{\partial x_2} \right) + \left( \frac{\partial u_2}{\partial x_1} \right) \left( \frac{\partial u_2}{\partial x_2} \right) + \left( \frac{\partial u_3}{\partial x_1} \right) \left( \frac{\partial u_3}{\partial x_2} \right) \right] \quad (3.26.13b)$$

The other components can be similarly written down. We note that for infinitesimal deformation,  $\frac{\partial u_i}{\partial X_j} \approx \frac{\partial u_i}{\partial x_j}$ , Eq. (3.26.12) reduces to Eq. (3.7.10a).

### Example 3.26.1

For the simple shear deformation

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3 \quad (i)$$

(a) Find  $\mathbf{B}^{-1}$  and  $\mathbf{e}^*$ .

(b) Use the geometry in Fig. 3.13 to discuss the meaning of  $e_{11}$  and  $e_{22}$ .

*Solution.* (a)

$$[\mathbf{F}] = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [\mathbf{F}^{-1}] = \begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(ii)}$$

$$[\mathbf{B}^{-1}] = [\mathbf{F}^{-1}]^T [\mathbf{F}^{-1}] = \begin{bmatrix} 1 & 0 & 0 \\ -k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -k & 0 \\ -k & 1+k^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(iii)}$$

$$[\mathbf{e}^*] = \frac{1}{2}[\mathbf{I} - \mathbf{B}^{-1}] = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & \frac{-k^2}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{(iv)}$$

(b) Since  $e_{11}^* = 0$ , an element which is in  $\mathbf{e}_1$  direction in the deformed state (such as  $B'C'$ ) had the same length in the undeformed state ( $BC$  in Fig. 3.13). Also since  $e_{22}^* = -\frac{k^2}{2}$ , an element which is in the  $\mathbf{e}_2$  direction in the deformed state (such as  $AH'$ ) had a length  $AH$  given by the equation

$$(AH')^2 - (AH)^2 = 2(AH')^2 e_{22}^* \quad \text{(v)}$$

from which one obtains

$$AH = (AH')\sqrt{1+k^2} \quad \text{(vi)}$$

This result checks with the geometry in Fig. 3.13.

### 3.27 Compatibility Conditions for Components of Finite Deformation Tensor

Whenever the three pathline equations (or equivalently, the three displacement functions) are given, one can always obtain the six components of  $\mathbf{e}^*$  or  $\mathbf{C}$  or  $\mathbf{B}$  or  $\mathbf{E}^*$  etc. by differentiation. On the other hand, if the six components of  $\mathbf{e}^*$  etc. are given, there exist three displacement functions corresponding to the given strain components only when compatibility conditions for the components are satisfied. This is because in general, it is not possible to solve for three unknown functions from six differential equations. The compatibility conditions can in principle be obtained by the elimination of the three displacement components  $u_i$  from the six equations relating strain components with the displacement components such as Eqs. (3.26.12b) by partial differentiation and elimination as was done for the infinitesimal components (Section 3.16) The procedure is very lengthy and will be omitted. Only the conditions for  $e_{ij}^*$  are given below with the super \* dropped for convenience:

$$\begin{aligned}
 & \frac{\partial^2 e_{kn}}{\partial x_l \partial x_m} + \frac{\partial^2 e_{lm}}{\partial x_k \partial x_n} - \frac{\partial^2 e_{km}}{\partial x_l \partial x_n} - \frac{\partial^2 e_{ln}}{\partial x_k \partial x_m} \\
 & - \left( \frac{\partial e_{ks}}{\partial x_n} + \frac{\partial e_{ns}}{\partial x_k} - \frac{\partial e_{kn}}{\partial x_s} \right) \left( \frac{\partial e_{ls}}{\partial x_m} + \frac{\partial e_{ms}}{\partial x_l} - \frac{\partial e_{lm}}{\partial x_s} \right) + \left( \frac{\partial e_{ks}}{\partial x_m} + \frac{\partial e_{ms}}{\partial x_k} - \frac{\partial e_{km}}{\partial x_s} \right) \left( \frac{\partial e_{ls}}{\partial x_n} + \frac{\partial e_{ns}}{\partial x_l} - \frac{\partial e_{ln}}{\partial x_s} \right) \\
 & + 2e_{rs} \left[ -2 \left( \frac{\partial e_{kr}}{\partial x_n} + \frac{\partial e_{nr}}{\partial x_k} - \frac{\partial e_{kn}}{\partial x_r} \right) \left( \frac{\partial e_{ls}}{\partial x_m} + \frac{\partial e_{ms}}{\partial x_l} - \frac{\partial e_{lm}}{\partial x_s} \right) - 4 \left( \frac{\partial e_{kr}}{\partial x_m} + \frac{\partial e_{mr}}{\partial x_k} - \frac{\partial e_{km}}{\partial x_r} \right) \left( \frac{\partial e_{ls}}{\partial x_n} + \frac{\partial e_{ns}}{\partial x_l} - \frac{\partial e_{ln}}{\partial x_s} \right) \right. \\
 & \left. + 4 \left( \frac{\partial e_{kr}}{\partial x_n} + \frac{\partial e_{nr}}{\partial x_k} - \frac{\partial e_{kn}}{\partial x_r} \right) \left( \frac{\partial e_{ls}}{\partial x_m} + \frac{\partial e_{ms}}{\partial x_l} - \frac{\partial e_{lm}}{\partial x_s} \right) \right] = 0
 \end{aligned} \tag{3.27.1}$$

We note that for infinitesimal deformation, Eq. (3.27.1) reduces to

$$\frac{\partial^2 e_{kn}}{\partial x_l \partial x_m} + \frac{\partial^2 e_{lm}}{\partial x_k \partial x_n} - \frac{\partial^2 e_{km}}{\partial x_l \partial x_n} - \frac{\partial^2 e_{ln}}{\partial x_k \partial x_m} = 0 \tag{3.27.2}$$

which are the same as those given in Sect. 3.16.

### 3. 28 Change of Area due to Deformation

Consider two material elements  $d\mathbf{X}^{(1)} = dS_1 \mathbf{e}_1$  and  $d\mathbf{X}^{(2)} = dS_2 \mathbf{e}_2$  emanating from  $\mathbf{X}$ . The rectangular area formed by  $d\mathbf{X}^{(1)}$  and  $d\mathbf{X}^{(2)}$  at the reference time  $t_0$  is given by

$$d\mathbf{A}_0 = d\mathbf{X}^{(1)} \times d\mathbf{X}^{(2)} = dS_1 dS_2 \mathbf{e}_3 = dA_0 \mathbf{e}_3 \tag{3.28.1}$$

where  $dA_0$  is the magnitude of the undeformed area and  $\mathbf{e}_3$  is normal to the area. At time  $t$ ,  $d\mathbf{X}^{(1)}$  deforms into  $d\mathbf{x}^{(1)} = \mathbf{F}d\mathbf{X}^{(1)}$  and  $d\mathbf{X}^{(2)}$  deforms into  $d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)}$  and the area is

$$d\mathbf{A} = \mathbf{F}d\mathbf{X}^{(1)} \times \mathbf{F}d\mathbf{X}^{(2)} = dS_1 dS_2 \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 = dA_0 \mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2 \tag{3.28.2}$$

Thus, the orientation of the deformed area is normal to  $\mathbf{F}\mathbf{e}_1$  and  $\mathbf{F}\mathbf{e}_2$ . Let this direction be denoted by the unit vector  $\mathbf{n}$ , i.e.,

$$d\mathbf{A} = dA \mathbf{n} \tag{3.28.3}$$

then,

$$dA \mathbf{n} = dA_0 (\mathbf{F}\mathbf{e}_1 \times \mathbf{F}\mathbf{e}_2) \tag{i}$$

From the above equation, it is clear that

$$\mathbf{F}\mathbf{e}_1 \cdot dA \mathbf{n} = \mathbf{F}\mathbf{e}_2 \cdot dA \mathbf{n} = 0 \tag{ii}$$

and

$$\mathbf{Fe}_3 \cdot d\mathbf{A} = dA_o (\mathbf{Fe}_3 \cdot \mathbf{Fe}_1 \times \mathbf{Fe}_2) \quad (\text{iii})$$

Recall that for any vectors  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ ,

$\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$  = determinant whose rows are components of  $\mathbf{a}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$ . Therefore

$$\mathbf{Fe}_3 \cdot \mathbf{Fe}_1 \times \mathbf{Fe}_2 = \det \mathbf{F} \quad (\text{iv})$$

Eq. (iii) becomes

$$\mathbf{Fe}_3 \cdot d\mathbf{A}\mathbf{n} = dA_o \det \mathbf{F} \quad (\text{v})$$

Using the definition of transpose of a tensor, Eqs. (ii) become

$$\mathbf{e}_1 \cdot \mathbf{F}^T \mathbf{n} = \mathbf{e}_2 \cdot \mathbf{F}^T \mathbf{n} = 0 \quad (\text{vi})$$

and Eq. (v) becomes

$$\mathbf{e}_3 \cdot \mathbf{F}^T \mathbf{n} = \left( \frac{dA_o}{dA} \right) \det \mathbf{F} \quad (\text{vii})$$

Thus,  $\mathbf{F}^T \mathbf{n}$  is in the direction of  $\mathbf{e}_3$ , so that

$$\mathbf{F}^T \mathbf{n} = \frac{dA_o}{dA} (\det \mathbf{F}) \mathbf{e}_3 \quad (\text{vii})$$

Therefore,

$$d\mathbf{A}\mathbf{n} = dA_o (\det \mathbf{F}) (\mathbf{F}^{-1})^T \mathbf{e}_3 \quad (3.28.4)$$

Equation (3.28.4) states that the deformed area has a normal in the direction of  $(\mathbf{F}^{-1})^T \mathbf{e}_3$  and with a magnitude given by

$$dA = dA_o (\det \mathbf{F}) |(\mathbf{F}^{-1})^T \mathbf{e}_3| \quad (3.28.5)$$

In deriving Eq. (3.28.4), we have chosen the initial area to be the rectangular area formed by the Cartesian base vectors  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , it can be shown that the formula remains valid for any material area except that  $\mathbf{e}_3$  be replaced by the normal vector of the undeformed area  $\mathbf{n}_o$ . That is in general,

$$d\mathbf{A}\mathbf{n} = dA_o (\det \mathbf{F}) (\mathbf{F}^{-1})^T \mathbf{n}_o \quad (3.28.6)$$

### 3.29 Change of Volume due to Deformation

Consider three material elements

$$d\mathbf{X}^{(1)} = dS_1 \mathbf{e}_1, \quad d\mathbf{X}^{(2)} = dS_2 \mathbf{e}_2 \quad \text{and} \quad d\mathbf{X}^{(3)} = dS_3 \mathbf{e}_3,$$

emanating from  $\mathbf{X}$ . The rectangular volume formed by  $d\mathbf{X}^{(1)}$ ,  $d\mathbf{X}^{(2)}$  and  $d\mathbf{X}^{(3)}$  at the reference time  $t_o$  is given by

$$dV_o = dS_1 dS_2 dS_3 \quad (3.29.1)$$

At time  $t$ ,  $d\mathbf{X}^{(1)}$  deforms into  $d\mathbf{x}^{(1)} = \mathbf{F}d\mathbf{X}^{(1)}$ ,  $d\mathbf{X}^{(2)}$  deforms into  $d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(2)}$  and  $d\mathbf{X}^{(3)}$  deforms into  $d\mathbf{x}^{(3)} = \mathbf{F}d\mathbf{X}^{(3)}$  and the volume is

$$\begin{aligned} dV &= \mathbf{F}d\mathbf{X}^{(1)} \cdot \mathbf{F}d\mathbf{X}^{(2)} \times \mathbf{F}d\mathbf{X}^{(3)} = dS_1 dS_2 dS_3 (\mathbf{F}\mathbf{e}_1 \cdot \mathbf{F}\mathbf{e}_2 \times \mathbf{F}\mathbf{e}_3) \\ &= dV_o (\mathbf{F}\mathbf{e}_1 \cdot \mathbf{F}\mathbf{e}_2 \times \mathbf{F}\mathbf{e}_3) \end{aligned} \quad (3.29.2)$$

That is,

$$dV = (\det \mathbf{F})dV_o \quad (3.29.3)$$

Since  $\mathbf{C} = \mathbf{F}^T \mathbf{F}$  and  $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ , therefore

$$\det \mathbf{C} = \det \mathbf{B} = (\det \mathbf{F})^2 \quad (3.29.4)$$

Thus, Eq. (3.29.3) can also be written as

$$dV = \sqrt{\det \mathbf{C}} dV_o = \sqrt{\det \mathbf{B}} dV_o \quad (3.29.5)$$

We note that for an incompressible material,  $dV = dV_o$ , so that

$$\det \mathbf{F} = \det \mathbf{C} = \det \mathbf{B} = 1 \quad (3.29.6)$$

We note also that due to Eq. (3.29.3), the conservation of mass equation can be written as:

$$\rho = \frac{\rho_o}{\det \mathbf{F}} \quad (3.29.7)$$

#### Example 3.29.1

Consider the deformation given by

$$x_1 = \lambda_1 X_1, \quad x_2 = -\lambda_3 X_3, \quad x_3 = \lambda_2 X_2 \quad (i)$$

(a) Find the deformed volume of the unit cube shown in Fig. 3.14.

(b) Find the deformed area of  $OABC$ .

(c) Find the rotation tensor and the axial vector of the antisymmetric part of the rotation tensor.

*Solution.* (a) From Eq. (i),

$$[\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix}$$

Thus,

$$\det \mathbf{F} = \lambda_1 \lambda_2 \lambda_3 \quad (\text{ii})$$

Since  $\det \mathbf{F}$  is a constant, from the equation

$$dV = (\det \mathbf{F}) dV_0$$

we have, with  $\Delta V_0 = 1$ ,

$$\Delta V = \lambda_1 \lambda_2 \lambda_3 \quad (\text{iii})$$

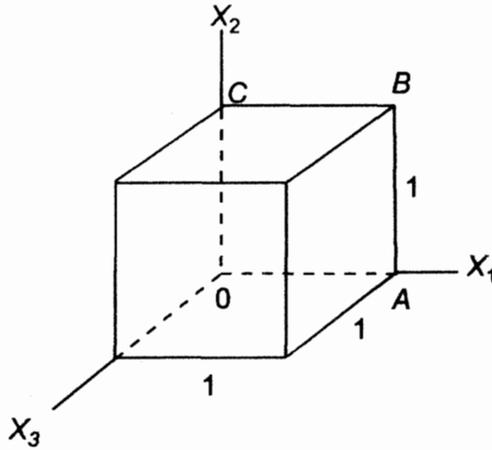


Fig. 3.14

(b) Using Eq. (3.28.6), with  $\Delta A_0 = 1$ , and  $\mathbf{n}_0 = -\mathbf{e}_3$ , we have

$$\Delta A \mathbf{n} = (1) (\det \mathbf{F}) (\mathbf{F}^{-1})^T \mathbf{n}_0 =$$

$$(\lambda_1 \lambda_2 \lambda_3) \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & 0 & -\frac{1}{\lambda_3} \\ 0 & \frac{1}{\lambda_2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_1 \lambda_2 \\ 0 \end{bmatrix} \quad (\text{iv})$$

i.e.,

$$\Delta A \mathbf{n} = \lambda_1 \lambda_2 \mathbf{e}_2$$

Thus, the area  $OABC$ , which was of unit area and having a normal in the direction of  $-\mathbf{e}_3$  becomes an area whose normal is in the direction of  $\mathbf{e}_2$  and with a magnitude of  $\lambda_1 \lambda_2$ .

(c)

$$[\mathbf{U}]^2 = [\mathbf{F}]^T [\mathbf{F}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 \\ 0 & -\lambda_3 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}$$

$$[\mathbf{U}] = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad (\text{v})$$

$$[\mathbf{R}] = [\mathbf{F}][\mathbf{U}]^{-1} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & 0 & -\lambda_3 \\ 0 & \lambda_2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda_1} & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & 0 \\ 0 & 0 & \frac{1}{\lambda_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad (\text{vi})$$

It is easily verified that  $\mathbf{R}$  corresponds to a  $90^\circ$  rotation about the  $\mathbf{e}_1$ , which is the axial vector of the antisymmetric part of  $\mathbf{R}$

### 3.30 Components of Deformation Tensors in other Coordinates

The deformation gradient  $\mathbf{F}$  transforms a differential material element  $d\mathbf{X}$  in the reference configuration into a material element  $d\mathbf{x}$  in the current configuration in accordance with the equation

$$d\mathbf{x} = \mathbf{F}d\mathbf{X} \quad (3.30.1)$$

where

$$\mathbf{x} = \mathbf{x}(X_1, X_2, X_3, t)$$

describes the motion. If the same rectangular coordinate system is used for both the reference and the current configurations, then since the set of base vectors  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$  is the same at every point, we have

$$\mathbf{e}_i \cdot d\mathbf{x} = \mathbf{e}_i \cdot \mathbf{F}(dX_m \mathbf{e}_m) = dX_m (\mathbf{e}_i \cdot \mathbf{F}\mathbf{e}_m) = F_{im} dX_m \quad (\text{i})$$

That is

$$dx_i = F_{im} dX_m \quad (\text{ii})$$

Thus

$$F_{im} = \frac{\partial x_i}{\partial X_m} \quad (3.30.2)$$

i.e.,

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \quad (3.30.3)$$

We have already used this matrix for computing the components of  $\mathbf{F}$  in a few examples above. The situation is more complicated if the base vectors at the reference configuration are different from those at the current configuration. Such situations arise not only in the case where different coordinate systems are used for the two configuration ( for example, a rectangular coordinate system for the reference and a cylindrical coordinates for the current configuration , see (B) below) , but also in cases where the same curvilinear coordinates are used for the two configurations. The following are examples.

(A) Cylindrical coordinate system for both the reference and the current configuration

Let

$$r = r(r_o, \theta_o, z_o, t), \quad \theta = \theta(r_o, \theta_o, z_o, t), \quad z = z(r_o, \theta_o, z_o, t) \quad (iii)$$

be the pathline equations. We shall show in the following that

$$\mathbf{F}\mathbf{e}_{or} = \left( \frac{\partial r}{\partial r_o} \right) \mathbf{e}_r + \left( \frac{r \partial \theta}{\partial r_o} \right) \mathbf{e}_\theta + \left( \frac{\partial z}{\partial r_o} \right) \mathbf{e}_z \quad (3.30.4a)$$

$$\mathbf{F}\mathbf{e}_{o\theta} = \left( \frac{\partial r}{r_o \partial \theta_o} \right) \mathbf{e}_r + \left( \frac{r \partial \theta}{r_o \partial \theta_o} \right) \mathbf{e}_\theta + \left( \frac{\partial z}{r_o \partial \theta_o} \right) \mathbf{e}_z \quad (3.30.4b)$$

$$\mathbf{F}\mathbf{e}_{oz} = \left( \frac{\partial r}{\partial z_o} \right) \mathbf{e}_r + \left( \frac{r \partial \theta}{\partial z_o} \right) \mathbf{e}_\theta + \left( \frac{\partial z}{\partial z_o} \right) \mathbf{e}_z \quad (3.30.4c)$$

and

$$\mathbf{F}^T \mathbf{e}_r = \left( \frac{\partial r}{\partial r_o} \right) \mathbf{e}_{or} + \left( \frac{\partial r}{r_o \partial \theta_o} \right) \mathbf{e}_{o\theta} + \left( \frac{\partial r}{\partial z_o} \right) \mathbf{e}_{oz} \quad (3.30.5a)$$

$$\mathbf{F}^T \mathbf{e}_\theta = \left( \frac{r \partial \theta}{\partial r_o} \right) \mathbf{e}_{or} + \left( \frac{r \partial \theta}{r_o \partial \theta_o} \right) \mathbf{e}_{o\theta} + \left( \frac{r \partial \theta}{\partial z_o} \right) \mathbf{e}_{oz} \quad (3.30.5b)$$

$$\mathbf{F}^T \mathbf{e}_z = \left( \frac{\partial z}{\partial r_o} \right) \mathbf{e}_{or} + \left( \frac{\partial z}{r_o \partial \theta_o} \right) \mathbf{e}_{o\theta} + \left( \frac{\partial z}{\partial z_o} \right) \mathbf{e}_{oz} \quad (3.30.5c)$$

where  $\mathbf{e}_{oi}$  denotes base vectors at the reference position and  $\mathbf{e}_i$  those at the current position.

Substituting

$$d\mathbf{x} = dr\mathbf{e}_r + r d\theta\mathbf{e}_\theta + dz\mathbf{e}_z \quad \text{and} \quad d\mathbf{X} = dr_o\mathbf{e}_{or} + r_o d\theta_o\mathbf{e}_{o\theta} + dz_o\mathbf{e}_{oz} \quad (\text{iv})$$

in the equation  $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ , we obtain

$$dr = dr_o(\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_{or}) + r_o d\theta_o(\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_{o\theta}) + dz_o(\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_{oz}) \quad (\text{v})$$

$$r d\theta = dr_o(\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_{or}) + r_o d\theta_o(\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_{o\theta}) + dz_o(\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_{oz}) \quad (\text{vi})$$

etc. Thus,

$$\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_{or} = \frac{\partial r}{\partial r_o}, \quad \mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_{o\theta} = \frac{\partial r}{r_o \partial \theta_o}, \quad \mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_{oz} = \frac{\partial r}{\partial z_o} \quad (3.30.6a)$$

$$\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_{or} = \frac{r \partial \theta}{\partial r_o}, \quad \mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_{o\theta} = \frac{r \partial \theta}{r_o \partial \theta_o}, \quad \mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_{oz} = \frac{r \partial \theta}{\partial z_o} \quad (3.30.6b)$$

$$\mathbf{e}_z \cdot \mathbf{F}\mathbf{e}_{or} = \frac{\partial z}{\partial r_o}, \quad \mathbf{e}_z \cdot \mathbf{F}\mathbf{e}_{o\theta} = \frac{\partial z}{r_o \partial \theta_o}, \quad \mathbf{e}_z \cdot \mathbf{F}\mathbf{e}_{oz} = \frac{\partial z}{\partial z_o} \quad (3.30.6c)$$

These equations are equivalent to Eqs. (3.30.4).

The matrix

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial r}{\partial r_o} & \frac{\partial r}{r_o \partial \theta_o} & \frac{\partial r}{\partial z_o} \\ \frac{r \partial \theta}{\partial r_o} & \frac{r \partial \theta}{r_o \partial \theta_o} & \frac{r \partial \theta}{\partial z_o} \\ \frac{\partial z}{\partial r_o} & \frac{\partial z}{r_o \partial \theta_o} & \frac{\partial z}{\partial z_o} \end{bmatrix} \{ \mathbf{e}_i \}, \{ \mathbf{e}_{o_j} \} \quad (3.30.7)$$

is based on two sets of bases, one at the reference configuration ( $\mathbf{e}_{or}, \mathbf{e}_{o\theta}, \mathbf{e}_{oz}$ ) and the other the current configuration ( $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ ). The components in this matrix is called **the two point components of the tensor F** with respect to ( $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z$ ) and ( $\mathbf{e}_{or}, \mathbf{e}_{o\theta}, \mathbf{e}_{oz}$ ).

By using the definition of transpose of a tensor, one can easily establish Eqs. (3.30.5) from Eq. (3.30.4). [see Prob. 3.73]

The components of the left Cauchy-Green tensor, with respect to the basis at the spatial position  $\mathbf{x}$  can be obtained as follows. From the definition  $\mathbf{B} = \mathbf{F}\mathbf{F}^T$ , and by using Eqs. (3.30.4) and (3.30.5) we obtain

$$\begin{aligned}
 B_{rr} &= \mathbf{e}_r \cdot \mathbf{B} \mathbf{e}_r = \mathbf{e}_r \cdot \mathbf{F} \mathbf{F}^T \mathbf{e}_r \\
 &= \left( \frac{\partial r}{\partial r_o} \right) \mathbf{e}_r \cdot \mathbf{F} \mathbf{e}_{or} + \left( \frac{\partial r}{\partial r_o \partial \theta_o} \right) \mathbf{e}_r \cdot \mathbf{F} \mathbf{e}_{o\theta} + \left( \frac{\partial r}{\partial z_o} \right) \mathbf{e}_r \cdot \mathbf{F} \mathbf{e}_{oz} \\
 &= \left( \frac{\partial r}{\partial r_o} \right)^2 + \left( \frac{\partial r}{\partial r_o \partial \theta_o} \right)^2 + \left( \frac{\partial r}{\partial z_o} \right)^2
 \end{aligned} \tag{vii}$$

Similarly,

$$\begin{aligned}
 B_{r\theta} &= \mathbf{e}_r \cdot \mathbf{B} \mathbf{e}_\theta = \mathbf{e}_r \cdot \mathbf{F} \mathbf{F}^T \mathbf{e}_\theta \\
 &= \left( \frac{r \partial \theta}{\partial r_o} \right) \left( \frac{\partial r}{\partial r_o} \right) + \left( \frac{r \partial \theta}{\partial r_o \partial \theta_o} \right) \left( \frac{\partial r}{\partial r_o \partial \theta} \right) + \left( \frac{r \partial \theta}{\partial z_o} \right) \left( \frac{\partial r}{\partial z_o} \right)
 \end{aligned} \tag{viii}$$

Other components can be obtained in the same way [see Prob. 3.74]. We list all the components below:

$$B_{rr} = \left( \frac{\partial r}{\partial r_o} \right)^2 + \left( \frac{\partial r}{\partial r_o \partial \theta_o} \right)^2 + \left( \frac{\partial r}{\partial z_o} \right)^2 \tag{3.30.8a}$$

$$B_{\theta\theta} = \left( \frac{r \partial \theta}{\partial r_o} \right)^2 + \left( \frac{r \partial \theta}{\partial r_o \partial \theta_o} \right)^2 + \left( \frac{r \partial \theta}{\partial z_o} \right)^2 \tag{3.30.8b}$$

$$B_{zz} = \left( \frac{\partial z}{\partial r_o} \right)^2 + \left( \frac{\partial z}{\partial r_o \partial \theta_o} \right)^2 + \left( \frac{\partial z}{\partial z_o} \right)^2 \tag{3.30.8c}$$

$$B_{r\theta} = \left( \frac{\partial r}{\partial r_o} \right) \left( \frac{r \partial \theta}{\partial r_o} \right) + \left( \frac{\partial r}{\partial r_o \partial \theta_o} \right) \left( \frac{r \partial \theta}{\partial r_o \partial \theta_o} \right) + \left( \frac{\partial r}{\partial z_o} \right) \left( \frac{r \partial \theta}{\partial z_o} \right) \tag{3.30.8d}$$

$$B_{rz} = \left( \frac{\partial r}{\partial r_o} \right) \left( \frac{\partial z}{\partial r_o} \right) + \left( \frac{\partial r}{\partial r_o \partial \theta_o} \right) \left( \frac{\partial z}{\partial r_o \partial \theta_o} \right) + \left( \frac{\partial r}{\partial z_o} \right) \left( \frac{\partial z}{\partial z_o} \right) \tag{3.30.8e}$$

$$B_{z\theta} = \left( \frac{\partial z}{\partial r_o} \right) \left( \frac{r \partial \theta}{\partial r_o} \right) + \left( \frac{\partial z}{\partial r_o \partial \theta_o} \right) \left( \frac{r \partial \theta}{\partial r_o \partial \theta_o} \right) + \left( \frac{\partial z}{\partial z_o} \right) \left( \frac{r \partial \theta}{\partial z_o} \right) \tag{3.30.8f}$$

The components of  $\mathbf{B}^{-1}$  can be obtained either by inverting the tensor  $\mathbf{B}$  or by inverting the pathline equations. Let

$$r_o = r_o(r, \theta, z, t), \quad \theta_o = \theta_o(r, \theta, z, t), \quad z_o = z_o(r, \theta, z, t) \tag{ix}$$

be the inverse of Eq. (iii). Then from the equation  $d\mathbf{X} = \mathbf{F}^{-1} d\mathbf{x}$ , one can obtain

$$B_{rr}^{-1} = \left( \frac{\partial r_o}{\partial r} \right)^2 + \left( \frac{r_o \partial \theta_o}{\partial r} \right)^2 + \left( \frac{\partial z_o}{\partial r} \right)^2 \quad (3.30.8g)$$

$$B_{\theta\theta}^{-1} = \left( \frac{\partial r_o}{r \partial \theta} \right)^2 + \left( \frac{r_o \partial \theta_o}{r \partial \theta} \right)^2 + \left( \frac{\partial z_o}{r \partial \theta} \right)^2 \quad (3.30.8h)$$

$$B_{zz}^{-1} = \left( \frac{\partial r_o}{\partial z} \right)^2 + \left( \frac{r_o \partial \theta_o}{\partial z} \right)^2 + \left( \frac{\partial z_o}{\partial z} \right)^2 \quad (3.30.8i)$$

$$B_{r\theta}^{-1} = \left( \frac{\partial r_o}{\partial r} \right) \left( \frac{\partial r_o}{r \partial \theta} \right) + \left( \frac{r_o \partial \theta_o}{\partial r} \right) \left( \frac{r_o \partial \theta_o}{r \partial \theta} \right) + \left( \frac{\partial z_o}{\partial r} \right) \left( \frac{\partial z_o}{r \partial \theta} \right) \quad (3.30.8j)$$

$$B_{rz}^{-1} = \left( \frac{\partial r_o}{\partial r} \right) \left( \frac{\partial r_o}{\partial z} \right) + \left( \frac{r_o \partial \theta_o}{\partial r} \right) \left( \frac{r_o \partial \theta_o}{\partial z} \right) + \left( \frac{\partial z_o}{\partial r} \right) \left( \frac{\partial z_o}{\partial z} \right) \quad (3.30.8k)$$

$$B_{z\theta}^{-1} = \left( \frac{\partial r_o}{\partial z} \right) \left( \frac{\partial r_o}{r \partial \theta} \right) + \left( \frac{r_o \partial \theta_o}{\partial z} \right) \left( \frac{r_o \partial \theta_o}{r \partial \theta} \right) + \left( \frac{\partial z_o}{\partial z} \right) \left( \frac{\partial z_o}{r \partial \theta} \right) \quad (3.30.8l)$$

The components of the right Cauchy-Green tensor  $\mathbf{C}$ , with respect to the basis at the reference position  $\mathbf{X}$  can be obtained as follows. From the definition  $\mathbf{C} = \mathbf{F}^T \mathbf{F}$ , and by using Eqs. (3.30.4) and (3.30.5) we obtain

$$\begin{aligned} C_{r_o r_o} &= \mathbf{e}_{or} \cdot \mathbf{C} \mathbf{e}_{or} = \mathbf{e}_{or} \cdot \mathbf{F}^T \mathbf{F} \mathbf{e}_{or} \\ &= \left( \frac{\partial r}{\partial r_o} \right) \mathbf{e}_{or} \cdot \mathbf{F}^T \mathbf{e}_r + \left( \frac{r \partial \theta}{\partial r_o} \right) \mathbf{e}_{or} \cdot \mathbf{F}^T \mathbf{e}_\theta + \left( \frac{\partial z}{\partial r_o} \right) \mathbf{e}_{or} \cdot \mathbf{F}^T \mathbf{e}_z \\ &= \left( \frac{\partial r}{\partial r_o} \right)^2 + \left( \frac{r \partial \theta}{\partial r_o} \right)^2 + \left( \frac{\partial z}{\partial r_o} \right)^2 \end{aligned} \quad (x)$$

Similarly,

$$\begin{aligned} C_{r_o \theta_o} &= \mathbf{e}_{or} \cdot \mathbf{C} \mathbf{e}_{o\theta} = \mathbf{e}_{or} \cdot \mathbf{F}^T \mathbf{F} \mathbf{e}_{o\theta} \\ &= \left( \frac{\partial r}{r_o \partial \theta_o} \right) \left( \frac{\partial r}{\partial r_o} \right) + \left( \frac{r \partial \theta}{r_o \partial \theta_o} \right) \left( \frac{r \partial \theta}{\partial r_o} \right) + \left( \frac{\partial z}{r_o \partial \theta_o} \right) \left( \frac{\partial z}{\partial r_o} \right) \end{aligned} \quad (xi)$$

Other components can be obtained in the same way [see Prob. 3.75]. We list all the components below:

$$C_{r_o r_o} = \left( \frac{\partial r}{\partial r_o} \right)^2 + \left( \frac{r \partial \theta}{\partial r_o} \right)^2 + \left( \frac{\partial z}{\partial r_o} \right)^2 \quad (3.30.9a)$$

$$C_{\theta_o \theta_o} = \left( \frac{\partial r}{r_o \partial \theta_o} \right)^2 + \left( \frac{r \partial \theta}{r_o \partial \theta_o} \right)^2 + \left( \frac{\partial z}{r_o \partial \theta_o} \right)^2 \quad (3.30.9b)$$

$$C_{z_0 z_0} = \left( \frac{\partial r}{\partial z_0} \right)^2 + \left( \frac{r \partial \theta}{\partial z_0} \right)^2 + \left( \frac{\partial z}{\partial z_0} \right)^2 \quad (3.30.9c)$$

$$C_{r_0 \theta_0} = \left( \frac{\partial r}{\partial r_0} \right) \left( \frac{\partial r}{r_0 \partial \theta_0} \right) + \left( \frac{r \partial \theta}{\partial r_0} \right) \left( \frac{r \partial \theta}{r_0 \partial \theta_0} \right) + \left( \frac{\partial z}{\partial r_0} \right) \left( \frac{\partial z}{r_0 \partial \theta_0} \right) \quad (3.30.9d)$$

$$C_{r_0 z_0} = \left( \frac{\partial r}{\partial r_0} \right) \left( \frac{\partial r}{\partial z_0} \right) + \left( \frac{r \partial \theta}{\partial r_0} \right) \left( \frac{r \partial \theta}{\partial z_0} \right) + \left( \frac{\partial z}{\partial r_0} \right) \left( \frac{\partial z}{\partial z_0} \right) \quad (3.30.9e)$$

$$C_{z_0 \theta_0} = \left( \frac{\partial r}{\partial z_0} \right) \left( \frac{\partial r}{r_0 \partial \theta_0} \right) + \left( \frac{r \partial \theta}{\partial z_0} \right) \left( \frac{r \partial \theta}{r_0 \partial \theta_0} \right) + \left( \frac{\partial z}{\partial z_0} \right) \left( \frac{\partial z}{r_0 \partial \theta_0} \right) \quad (3.30.9f)$$

Again, the components of  $C^{-1}$  can be obtained by using the equation  $d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$  and Eq. (ix). We list here two of the six components. The other four components can be easily written down following the patterns of these two equations.

$$C_{r_0 r_0}^{-1} = \left( \frac{\partial r_0}{\partial r} \right)^2 + \left( \frac{\partial r_0}{r \partial \theta} \right)^2 + \left( \frac{\partial r_0}{\partial z} \right)^2 \quad (3.30.9g)$$

$$C_{r_0 \theta_0}^{-1} = \left( \frac{\partial r_0}{\partial r} \right) \left( \frac{r_0 \partial \theta_0}{\partial r} \right) + \left( \frac{\partial r_0}{r \partial \theta} \right) \left( \frac{r_0 \partial \theta_0}{r \partial \theta} \right) + \left( \frac{\partial r_0}{\partial z} \right) \left( \frac{r_0 \partial \theta_0}{\partial z} \right) \quad (3.30.9h)$$

(B) Cylindrical coordinates  $(r, \theta, z)$  for the current configuration and rectangular Cartesian coordinates  $(X, Y, Z)$  for the reference configuration.

Let

$$r = r(X, Y, Z, t) \quad \theta = \theta(X, Y, Z, t) \quad z = z(X, Y, Z, t) \quad (\text{xii})$$

describe the motions. Then using the same procedure as described for the case where one single cylindrical coordinate is used, it can be derived that [see Prob.3.76].

$$\mathbf{F}\mathbf{e}_X = \frac{\partial r}{\partial X}\mathbf{e}_r + \frac{r \partial \theta}{\partial X}\mathbf{e}_\theta + \frac{\partial z}{\partial X}\mathbf{e}_z \quad (3.30.10a)$$

$$\mathbf{F}\mathbf{e}_Y = \frac{\partial r}{\partial Y}\mathbf{e}_r + \frac{r \partial \theta}{\partial Y}\mathbf{e}_\theta + \frac{\partial z}{\partial Y}\mathbf{e}_z \quad (3.30.10b)$$

$$\mathbf{F}\mathbf{e}_Z = \frac{\partial r}{\partial Z}\mathbf{e}_r + \frac{r \partial \theta}{\partial Z}\mathbf{e}_\theta + \frac{\partial z}{\partial Z}\mathbf{e}_z \quad (3.30.10c)$$

The matrix

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial r}{\partial X} & \frac{\partial r}{\partial Y} & \frac{\partial r}{\partial Z} \\ r \frac{\partial \theta}{\partial X} & r \frac{\partial \theta}{\partial Y} & r \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} (\mathbf{e}_{r \dots}), (\mathbf{e}_X \dots) \quad (3.30.11)$$

gives the two point components of  $\mathbf{F}$  with respect to the two sets of bases, one at the reference configuration, the other at the current configuration.

The components of the left Cauchy-Green deformation tensor  $\mathbf{B}$  with respect to the basis at the current configuration are given by [see Prob.3.77]

$$B_{rr} = \left( \frac{\partial r}{\partial X} \right)^2 + \left( \frac{\partial r}{\partial Y} \right)^2 + \left( \frac{\partial r}{\partial Z} \right)^2 \quad (3.30.12a)$$

$$B_{\theta\theta} = \left( r \frac{\partial \theta}{\partial X} \right)^2 + \left( r \frac{\partial \theta}{\partial Y} \right)^2 + \left( r \frac{\partial \theta}{\partial Z} \right)^2 \quad (3.30.12b)$$

$$B_{zz} = \left( \frac{\partial z}{\partial X} \right)^2 + \left( \frac{\partial z}{\partial Y} \right)^2 + \left( \frac{\partial z}{\partial Z} \right)^2 \quad (3.30.12c)$$

$$B_{r\theta} = \left( \frac{\partial r}{\partial X} \right) \left( r \frac{\partial \theta}{\partial X} \right) + \left( \frac{\partial r}{\partial Y} \right) \left( r \frac{\partial \theta}{\partial Y} \right) + \left( \frac{\partial r}{\partial Z} \right) \left( r \frac{\partial \theta}{\partial Z} \right) \quad (3.30.12d)$$

$$B_{rz} = \left( \frac{\partial r}{\partial X} \right) \left( \frac{\partial z}{\partial X} \right) + \left( \frac{\partial r}{\partial Y} \right) \left( \frac{\partial z}{\partial Y} \right) + \left( \frac{\partial r}{\partial Z} \right) \left( \frac{\partial z}{\partial Z} \right) \quad (3.30.12e)$$

$$B_{\theta z} = \left( r \frac{\partial \theta}{\partial X} \right) \left( \frac{\partial z}{\partial X} \right) + \left( r \frac{\partial \theta}{\partial Y} \right) \left( \frac{\partial z}{\partial Y} \right) + \left( r \frac{\partial \theta}{\partial Z} \right) \left( \frac{\partial z}{\partial Z} \right) \quad (3.30.12f)$$

Again, the components of  $\mathbf{B}^{-1}$  can be obtained by using the equation  $d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$  and the inverse of Eq. (xii). We list here two of the six components. The other four components can be easily written down following the patterns of these two equations.

$$B_{rr}^{-1} = \left( \frac{\partial X}{\partial r} \right)^2 + \left( \frac{\partial Y}{\partial r} \right)^2 + \left( \frac{\partial Z}{\partial r} \right)^2 \quad (3.30.12g)$$

$$B_{r\theta}^{-1} = \left( \frac{\partial X}{\partial r} \right) \left( \frac{\partial X}{r \partial \theta} \right) + \left( \frac{\partial Y}{\partial r} \right) \left( \frac{\partial Y}{r \partial \theta} \right) + \left( \frac{\partial Z}{\partial r} \right) \left( \frac{\partial Z}{r \partial \theta} \right) \quad (3.30.12h)$$

The components of the right Cauchy-Green deformation tensor  $\mathbf{C}$  with respect to the basis at the reference configuration are given by : [see Prob. 3.78]

$$C_{XX} = \left( \frac{\partial r}{\partial X} \right)^2 + \left( r \frac{\partial \theta}{\partial X} \right)^2 + \left( \frac{\partial z}{\partial X} \right)^2 \quad (3.30.13a)$$

$$C_{YY} = \left( \frac{\partial r}{\partial Y} \right)^2 + \left( \frac{r \partial \theta}{\partial Y} \right)^2 + \left( \frac{\partial z}{\partial Y} \right)^2 \quad (3.30.13b)$$

$$C_{ZZ} = \left( \frac{\partial r}{\partial Z} \right)^2 + \left( \frac{r \partial \theta}{\partial Z} \right)^2 + \left( \frac{\partial z}{\partial Z} \right)^2 \quad (3.30.13c)$$

$$C_{XY} = \left( \frac{\partial r}{\partial X} \right) \left( \frac{\partial r}{\partial Y} \right) + \left( \frac{r \partial \theta}{\partial X} \right) \left( \frac{r \partial \theta}{\partial Y} \right) + \left( \frac{\partial z}{\partial X} \right) \left( \frac{\partial z}{\partial Y} \right) \quad (3.30.13d)$$

$$C_{XZ} = \left( \frac{\partial r}{\partial X} \right) \left( \frac{\partial r}{\partial Z} \right) + \left( \frac{r \partial \theta}{\partial X} \right) \left( \frac{r \partial \theta}{\partial Z} \right) + \left( \frac{\partial z}{\partial X} \right) \left( \frac{\partial z}{\partial Z} \right) \quad (3.30.13e)$$

$$C_{YZ} = \left( \frac{\partial r}{\partial Y} \right) \left( \frac{\partial r}{\partial Z} \right) + \left( \frac{r \partial \theta}{\partial Y} \right) \left( \frac{r \partial \theta}{\partial Z} \right) + \left( \frac{\partial z}{\partial Y} \right) \left( \frac{\partial z}{\partial Z} \right) \quad (3.30.13f)$$

The components of  $C^{-1}$  can be obtained as:

$$C_{XX}^{-1} = \left( \frac{\partial X}{\partial r} \right)^2 + \left( \frac{\partial X}{r \partial \theta} \right)^2 + \left( \frac{\partial X}{\partial z} \right)^2 \quad (3.30.13g)$$

$$C_{XY}^{-1} = \left( \frac{\partial X}{\partial r} \right) \left( \frac{\partial Y}{\partial r} \right) + \left( \frac{\partial X}{r \partial \theta} \right) \left( \frac{\partial Y}{r \partial \theta} \right) + \left( \frac{\partial X}{\partial z} \right) \left( \frac{\partial Y}{\partial z} \right) \quad (3.30.13h)$$

and the other four components can be easily written down following the patterns of these two equation.

(C) Spherical coordinate system for both the reference and the current configuration

Let

$$r = r(r_o, \theta_o, \phi_o, t), \quad \theta = \theta(r_o, \theta_o, \phi_o, t), \quad \phi = \phi(r_o, \theta_o, \phi_o, t) \quad (xii)$$

be the pathline equations. Then using the same procedure as described for the cylindrical coordinate case, it can derive that the two point components for  $\mathbf{F}$  with respect to  $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$  at  $\mathbf{x}$  and  $(\mathbf{e}_{or}, \mathbf{e}_{o\theta}, \mathbf{e}_{o\phi})$  at  $\mathbf{X}$  are given by the matrix

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial r}{\partial r_o} & \frac{\partial r}{r_o \partial \theta_o} & \frac{\partial r}{r_o \sin \theta_o \partial \phi_o} \\ \frac{r \partial \theta}{\partial r_o} & \frac{r \partial \theta}{r_o \partial \theta_o} & \frac{r \partial \theta}{r_o \sin \theta_o \partial \phi_o} \\ \frac{r \sin \theta \partial \phi}{\partial r_o} & \frac{r \sin \theta \partial \phi}{r_o \partial \theta_o} & \frac{r \sin \theta \partial \phi}{r_o \sin \theta_o \partial \phi_o} \end{bmatrix} (\mathbf{e}_i), (\mathbf{e}_{oj}) \quad (3.30.14)$$

The components of the left Cauchy-Green tensor are:

$$B_{rr} = \left( \frac{\partial r}{\partial r_o} \right)^2 + \left( \frac{\partial r}{r_o \partial \theta_o} \right)^2 + \left( \frac{\partial r}{r_o \sin \theta_o \partial \phi_o} \right)^2 \quad (3.30.15a)$$

$$B_{\theta\theta} = \left( \frac{r \partial \theta}{\partial r_o} \right)^2 + \left( \frac{r \partial \theta}{r_o \partial \theta_o} \right)^2 + \left( \frac{r \partial \theta}{r_o \sin \theta_o \partial \phi_o} \right)^2 \quad (3.30.15b)$$

$$B_{\phi\phi} = \left( \frac{r \sin \theta \partial \phi}{\partial r_o} \right)^2 + \left( \frac{r \sin \theta \partial \phi}{r_o \partial \theta_o} \right)^2 + \left( \frac{r \sin \theta \partial \phi}{r_o \sin \theta_o \partial \phi_o} \right)^2 \quad (3.30.15c)$$

$$B_{r\theta} = \left( \frac{\partial r}{\partial r_o} \right) \left( \frac{r \partial \theta}{\partial r_o} \right) + \left( \frac{\partial r}{r_o \partial \theta_o} \right) \left( \frac{r \partial \theta}{r_o \partial \theta_o} \right) + \left( \frac{\partial r}{r_o \sin \theta_o \partial \phi_o} \right) \left( \frac{r \partial \theta}{r_o \sin \theta_o \partial \phi_o} \right) \quad (3.30.15d)$$

$$B_{r\phi} = \left( \frac{\partial r}{\partial r_o} \right) \left( \frac{r \sin \theta \partial \phi}{\partial r_o} \right) + \left( \frac{\partial r}{r_o \partial \theta_o} \right) \left( \frac{r \sin \theta \partial \phi}{r_o \partial \theta_o} \right) + \left( \frac{\partial r}{r_o \sin \theta_o \partial \phi_o} \right) \left( \frac{r \sin \theta \partial \phi}{r_o \sin \theta_o \partial \phi_o} \right) \quad (3.30.15e)$$

$$B_{\phi\theta} = \left( \frac{r \sin \theta \partial \phi}{\partial r_o} \right) \left( \frac{r \partial \theta}{\partial r_o} \right) + \left( \frac{r \sin \theta \partial \phi}{r_o \partial \theta_o} \right) \left( \frac{r \partial \theta}{r_o \partial \theta_o} \right) + \left( \frac{r \sin \theta \partial \phi}{r_o \sin \theta_o \partial \phi_o} \right) \left( \frac{r \partial \theta}{r_o \sin \theta_o \partial \phi_o} \right) \quad (3.30.15f)$$

The components of  $\mathbf{B}^{-1}$  can be written:

$$B_{rr}^{-1} = \left( \frac{\partial r_o}{\partial r} \right)^2 + \left( \frac{r_o \partial \theta_o}{\partial r} \right)^2 + \left( \frac{r_o \sin \theta_o \partial \phi_o}{\partial r} \right)^2 \quad (3.30.15g)$$

$$B_{rz}^{-1} = \left( \frac{\partial r_o}{\partial r} \right) \left( \frac{\partial r_o}{r \sin \theta \partial \phi} \right) + \left( \frac{r_o \partial \theta_o}{\partial r} \right) \left( \frac{r_o \partial \theta_o}{r \sin \theta \partial \phi} \right) + \left( \frac{r_o \sin \theta_o \partial \phi_o}{\partial r} \right) \left( \frac{r_o \sin \theta_o \partial \phi_o}{r \sin \theta \partial \phi} \right) \quad (3.30.15h)$$

etc.

The components of  $\mathbf{C}$  are:

$$C_{r\mathcal{J}_o} = \left( \frac{\partial r}{\partial r_o} \right)^2 + \left( \frac{r \partial \theta}{\partial r_o} \right)^2 + \left( \frac{r \sin \theta \partial \phi}{\partial r_o} \right)^2 \quad (3.30.16a)$$

$$C_{\theta\mathcal{J}_o} = \left( \frac{\partial r}{r_o \partial \theta_o} \right)^2 + \left( \frac{r \partial \theta}{r_o \partial \theta_o} \right)^2 + \left( \frac{r \sin \theta \partial \phi}{r_o \partial \theta_o} \right)^2 \quad (3.30.16b)$$

$$C_{z\mathcal{J}_o} = \left( \frac{\partial r}{r_o \sin \theta_o \partial \phi_o} \right)^2 + \left( \frac{r \partial \theta}{r_o \sin \theta_o \partial \phi_o} \right)^2 + \left( \frac{r \sin \theta \partial \phi}{r_o \sin \theta_o \partial \phi_o} \right)^2 \quad (3.30.16c)$$

$$C_{r\theta\mathcal{J}_o} = \left( \frac{\partial r}{\partial r_o} \right) \left( \frac{\partial r}{r_o \partial \theta_o} \right) + \left( \frac{r \partial \theta}{\partial r_o} \right) \left( \frac{r \partial \theta}{r_o \partial \theta_o} \right) + \left( \frac{r \sin \theta \partial \phi}{\partial r_o} \right) \left( \frac{r \sin \theta \partial \phi}{r_o \partial \theta_o} \right) \quad (3.30.16d)$$

$$C_{r_z\phi_o} = \left(\frac{\partial r}{\partial r_o}\right) \left(\frac{\partial r}{r_o \sin\theta_o \partial \phi_o}\right) + \left(\frac{r\partial\theta}{\partial r_o}\right) \left(\frac{r\partial\theta}{r_o \sin\theta_o \partial \phi_o}\right) + \left(\frac{r\sin\theta\partial\phi}{\partial r_o}\right) \left(\frac{r\sin\theta\partial\phi}{r_o \sin\theta_o \partial \phi_o}\right) \quad (3.30.16e)$$

$$C_{z_o\theta_o} = \left(\frac{\partial r}{r_o \sin\theta_o \partial \phi_o}\right) \left(\frac{\partial r}{r_o \partial \theta_o}\right) + \left(\frac{r\partial\theta}{r_o \sin\theta_o \partial \phi_o}\right) \left(\frac{r\partial\theta}{r_o \partial \theta_o}\right) + \left(\frac{r\sin\theta\partial\phi}{r_o \sin\theta_o \partial \phi_o}\right) \left(\frac{\partial z}{r_o \partial \theta_o}\right) \quad (3.30.16f)$$

The components of  $C^{-1}$  can be written:

$$C_{r\phi_o}^{-1} = \left(\frac{\partial r_o}{\partial r}\right)^2 + \left(\frac{\partial r_o}{r\partial\theta}\right)^2 + \left(\frac{\partial r_o}{r\sin\theta\partial\phi}\right)^2 \quad (3.30.16g)$$

$$C_{r_o\theta_o}^{-1} = \left(\frac{\partial r_o}{\partial r}\right) \left(\frac{r_o\partial\theta_o}{\partial r}\right) + \left(\frac{\partial r_o}{r\partial\theta}\right) \left(\frac{r_o\partial\theta_o}{r\partial\theta}\right) + \left(\frac{\partial r_o}{r\sin\theta\partial\phi}\right) \left(\frac{r_o\partial\theta_o}{r\sin\theta\partial\phi}\right) \quad (3.30.16h)$$

etc.

### 3.31 Current Configuration as the Reference Configuration

Let  $\mathbf{x}'$  be the position at time  $\tau$  of a material point which is at the spatial position  $\mathbf{x}$  at time  $t$ , then the kinematic equations of motion ( the pathline equations) take the form of

$$\mathbf{x}' = \mathbf{x}'_t(\mathbf{x}, \tau), \quad \text{with } \mathbf{x} = \mathbf{x}_t(\mathbf{x}, t) \quad (3.31.1)$$

Equations (3.31.1) describe the motion using the current configuration as the reference configuration. The subscript  $t$  in  $\mathbf{x}'_t$  indicates that the current time  $t$  is the reference time and as such in addition to  $\mathbf{x}$  and  $\tau$ , it is also a function of  $t$ .

#### Example 3.31.1

Given the velocity field

$$v_1 = kx_2, \quad v_2 = v_3 = 0 \quad (i)$$

Find the pathline equations using the current configuration as the reference configuration.

*Solution.* Let  $x'_1(\mathbf{x}, \tau)$ ,  $x'_2(\mathbf{x}, \tau)$ ,  $x'_3(\mathbf{x}, \tau)$  be the position at time  $\tau$  then

$$\frac{dx'_1}{d\tau} = kx'_2, \quad \frac{dx'_2}{d\tau} = \frac{dx'_3}{d\tau} = 0 \quad (ii)$$

The second and the third equation state that both  $x'_2$  and  $x'_3$  are constants. Since they must be  $x_2$  and  $x_3$  at time  $t$ , therefore,

$$x'_2 = x_2, \quad x'_3 = x_3$$

Now from the first equation, since  $x'_2 = x_2$ , we have

$$x_1' = kx_2 \tau + C$$

so that

$$x_1' = x_1 + kx_2(\tau - t) \quad (\text{ii})$$

When the current configuration is used as the reference, it is customary also to denote tensors based on such a reference with a subscript  $t$ , e.g.,

$$\mathbf{F}_t \equiv \nabla_{\mathbf{x}} \mathbf{x}' \quad (\text{relative deformation gradient})$$

$$\mathbf{C}_t \equiv \mathbf{F}_t^T \mathbf{F}_t \quad (\text{relative right Cauchy-Green Tensor})$$

$$\mathbf{B}_t \equiv \mathbf{F}_t \mathbf{F}_t^T \quad (\text{relative left Cauchy-Green Tensor})$$

etc. All the formulas derived earlier, based on a fixed reference configuration, can be easily rewritten for the case where the current configuration is used as the reference. For example, let  $(r', \theta', z', \tau)$  denote the cylindrical coordinates for the position  $\mathbf{x}'$  at time  $\tau$  for a material point which is at  $(r, \theta, z)$  at time  $t$  i.e.,

$$r' = r'(r, \theta, z, \tau), \quad \theta' = \theta'(r, \theta, z, \tau), \quad z' = z'(r, \theta, z, \tau)$$

then, with respect to the current bases  $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$

$$(C_t)_{rr} = \left( \frac{\partial r'}{\partial r} \right)^2 + \left( \frac{r' \partial \theta'}{\partial r} \right)^2 + \left( \frac{\partial z'}{\partial r} \right)^2 \quad (3.31.1a)$$

$$(C_t)_{\theta\theta} = \left( \frac{\partial r'}{r \partial \theta} \right)^2 + \left( \frac{r' \partial \theta'}{r \partial \theta} \right)^2 + \left( \frac{\partial z'}{r \partial \theta} \right)^2 \quad (3.31.1b)$$

$$(C_t)_{zz} = \left( \frac{\partial r'}{\partial z} \right)^2 + \left( \frac{r' \partial \theta'}{\partial z} \right)^2 + \left( \frac{\partial z'}{\partial z} \right)^2 \quad (3.31.1c)$$

$$(C_t)_{r\theta} = \left( \frac{\partial r'}{\partial r} \right) \left( \frac{\partial r'}{r \partial \theta} \right) + \left( \frac{r' \partial \theta'}{\partial r} \right) \left( \frac{r' \partial \theta'}{r \partial \theta} \right) + \left( \frac{\partial z'}{\partial r} \right) \left( \frac{\partial z'}{r \partial \theta} \right) \quad (3.31.1d)$$

$$(C_t)_{rz} = \left( \frac{\partial r'}{\partial r} \right) \left( \frac{\partial r'}{\partial z} \right) + \left( \frac{r' \partial \theta'}{\partial r} \right) \left( \frac{r' \partial \theta'}{\partial z} \right) + \left( \frac{\partial z'}{\partial r} \right) \left( \frac{\partial z'}{\partial z} \right) \quad (3.31.1e)$$

$$(C_t)_{z\theta} = \left( \frac{\partial r'}{\partial z} \right) \left( \frac{\partial r'}{r \partial \theta} \right) + \left( \frac{r' \partial \theta'}{\partial z} \right) \left( \frac{r' \partial \theta'}{r \partial \theta} \right) + \left( \frac{\partial z'}{\partial z} \right) \left( \frac{\partial z'}{r \partial \theta} \right) \quad (3.31.1f)$$

We will have more to say about relative deformation tensors in Chapter 8 where we shall discuss the constitutive equations for Non-Newtonian fluids.

## PROBLEMS

3.1. Consider the motion

$$x_1 = kt + X_1$$

$$x_2 = X_2$$

$$x_3 = X_3$$

where the material coordinates  $X_i$  designate the position of a particle at  $t = 0$ .

(a) Determine the velocity and acceleration of a particle in both a material and spatial description.

(b) If in a spatial description, there is a temperature field  $\theta = Ax_1$ , find the material derivative  $D\theta/Dt$ .

(c) Do part (b) if the temperature field is given by  $\theta = Bx_2$ .

3.2. Consider the motion

$$x_1 = X_1$$

$$x_2 = kX_1^2 t^2 + X_2$$

$$x_3 = X_3$$

where  $X_i$  are the material coordinates.

(a) At  $t = 0$  the corners of a unit square are at  $A(0,0,0)$ ,  $B(0,1,0)$ ,  $C(1,1,0)$  and  $D(1,0,0)$ . Determine the position of  $A, B, C, D$  at  $t = 1$ , and sketch the new shape of the square.

(b) Find the velocity  $\mathbf{v}$  and the acceleration  $D\mathbf{v}/Dt$  in a material description.

(c) Show that the spatial velocity field is given by

$$v_1 = v_3 = 0, \quad v_2 = 2kx_1^2 t.$$

3.3. Consider the motion

$$x_1 = kX_2^2 t^2 + X_1$$

$$x_2 = kX_2 t + X_2$$

$$x_3 = X_3$$

(a) At  $t = 0$ , the corners of a unit square are at  $A(0,0,0)$ ,  $B(0,1,0)$ ,  $C(1,1,0)$ , and  $D(1,0,0)$ . Sketch the deformed shape of the square at  $t = 2$ .

(b) Obtain the spatial description of the velocity field.

(c) Obtain the spatial description of the acceleration field.

3.4. Consider the motion

$$x_1 = (k + X_1)t + X_1$$

$$x_2 = X_2$$

$$x_3 = X_3$$

- (a) For this motion, repeat part (a) of the previous problem.
- (b) Find the velocity and acceleration as a function of time of a particle that is initially at the origin.
- (c) Find the velocity and acceleration as a function of time of the particles that are passing through the origin.

3.5. The position at time  $t$  of a particle initially at  $(X_1, X_2, X_3)$  is given by

$$x_1 = X_1 - 2X_2^2 t^2, \quad x_2 = X_2 - X_3 t, \quad x_3 = X_3$$

- (a) Sketch the deformed shape, at time  $t = 1$  of the material line  $OA$  which was a straight line at  $t = 0$  with  $O$  at  $(0,0,0)$  and  $A$  at  $(0,1,0)$ .
- (b) Find the velocity at  $t = 2$ , of the particle which is at  $(1,3,1)$  at  $t = 0$ .
- (c) Find the velocity of a particle which is at  $(1,3,1)$  at  $t = 2$ .

3.6. The position at time  $t$  of a particle initially at  $(X_1, X_2, X_3)$ , is given by

$$x_1 = X_1 + (X_1 + X_2)t, \quad x_2 = X_2 + (X_1 + X_2)t, \quad x_3 = X_3$$

- (a) Find the velocity at  $t = 2$  for the particle which was at  $(1,1,0)$  at the reference time.
- (b) Find the velocity at  $t = 2$  for the particle which is at the position  $(1,1,0)$  at  $t = 2$ .

3.7. Consider the motion

$$x_1 = \frac{1+t}{1+t_0} X_1, \quad x_2 = X_2, \quad x_3 = X_3$$

- (a) Show that reference time is  $t = t_0$ .
- (b) Find the velocity field in spatial coordinates.
- (c) Show that the velocity field is identical to that of the following motion

$$x_1 = (1+t)X_1, \quad x_2 = X_2, \quad x_3 = X_3.$$

3.8. The position at time  $t$  of a particle initially at  $(X_1, X_2, X_3)$  is given by

$$x_1 = X_1 + X_2^2 t^2, \quad x_2 = X_2 + X_2 t, \quad x_3 = X_3$$

- (a) For the particle which was initially at  $(1,1,0)$ , what are its positions in the following instants of time:  $t = 0, t = 1, t = 2$ .
- (b) Find the initial position for a particle which is at  $(1,3,2)$  at  $t = 2$ .
- (c) Find the acceleration at  $t = 2$  of the particle which was initially at  $(1,3,2)$ .
- (d) Find the acceleration of a particle which is at  $(1,3,2)$  at  $t = 2$ .

3.9. (a) Show that the velocity field

$$v_i = \frac{x_i}{1+t}$$

corresponds to the motion

$$x_i = X_i(1+t)$$

(b) Find the acceleration of this motion in the material description.

3.10. Given the two-dimensional velocity field

$$v_x = -2y, \quad v_y = 2x$$

(a) Obtain the acceleration field.

(b) Obtain the pathline equations.

3.11. Given the two-dimensional velocity field

$$v_x = kx, \quad v_y = -ky$$

(a) Obtain the acceleration field.

(b) Obtain the pathline equations.

3.12. Given the two-dimensional velocity field,

$$v_x = x^2 - y^2, \quad v_y = -2xy$$

Obtain the acceleration field.

3.13. In a spatial description the equation to evaluate the acceleration

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\mathbf{v}$$

is nonlinear. That is, if we consider two velocity fields  $\mathbf{v}^A$  and  $\mathbf{v}^B$ , then

$$\mathbf{a}^A + \mathbf{a}^B \neq \mathbf{a}^{A+B}$$

where  $\mathbf{a}^A$  and  $\mathbf{a}^B$  denote respectively the acceleration fields corresponding to the velocity fields  $\mathbf{v}^A$  and  $\mathbf{v}^B$  each existing alone,  $\mathbf{a}^{A+B}$  denotes the acceleration field corresponding to the combined velocity field  $\mathbf{v}^A + \mathbf{v}^B$ . Verify this inequality for the velocity fields

$$\mathbf{v}^A = -2x_2\mathbf{e}_1 + 2x_1\mathbf{e}_2$$

$$\mathbf{v}^B = 2x_2\mathbf{e}_1 - 2x_1\mathbf{e}_2$$

3.14. Consider the motion

$$x_1 = X_1$$

$$x_2 = X_2 + (\sin \pi t)(\sin \pi X_1)$$

$$x_3 = X_3$$

(a) At  $t = 0$  a material filament coincides with the straight line that extends from  $(0,0,0)$  to  $(1,0,0)$ . Sketch the deformed shape of this filament at  $t = 1/2$ ,  $t = 1$ , and  $t = 3/2$ .

(b) Find the velocity and acceleration in a material and a spatial description.

3.15. Consider the following velocity and temperature fields:

$$\mathbf{v} = \frac{x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2}{x_1^2 + x_2^2}, \quad \theta = k(x_1^2 + x_2^2)$$

(a) Determine the velocity at several positions and indicate the general nature of this velocity field. What do the isotherms look like?

(b) At the point  $A(1,1,0)$ , determine the acceleration and the material derivative of the temperature field.

3.16. Do the previous problem for the temperature and velocity fields:

$$\mathbf{v} = \frac{-x_2 \mathbf{e}_1 + x_1 \mathbf{e}_2}{x_1^2 + x_2^2}, \quad \theta = k(x_1^2 + x_2^2).$$

3.17. Consider the motion  $\mathbf{x} = \mathbf{X} + X_1 k \mathbf{e}_1$  and let  $d\mathbf{X}^{(1)} = (dS_1 / \sqrt{2})(\mathbf{e}_1 + \mathbf{e}_2)$  and  $d\mathbf{X}^{(2)} = (dS_2 / \sqrt{2})(-\mathbf{e}_1 + \mathbf{e}_2)$  be differential material elements in the undeformed configuration.

(a) Find the deformed elements  $d\mathbf{x}^{(1)}$  and  $d\mathbf{x}^{(2)}$ .

(b) Evaluate the stretches of these elements,  $ds_1 / dS_1$  and  $ds_2 / dS_2$ , and the change in the angle between them.

(c) Do part (b) for  $k = 1$  and  $k = 10^{-2}$ .

(d) Compare the results of part(c) to that predicted by the small strain tensor  $\mathbf{E}$ .

3.18. The motion of a continuum from initial position  $\mathbf{X}$  to current position  $\mathbf{x}$  is given by

$$\mathbf{x} = (\mathbf{I} + \mathbf{B})\mathbf{X}$$

where  $\mathbf{I}$  is the identity tensor and  $\mathbf{B}$  is a tensor whose components  $B_{ij}$  are constants and small compared to unity. If the components of  $\mathbf{x}$  are  $x_i$  and those of  $\mathbf{X}$  are  $X_i$ , find

(a) the components of the displacement vector  $\mathbf{u}$ , and

(b) the small strain tensor  $\mathbf{E}$ .

3.19. At time  $t$ , the position of a particle initially at  $(X_1, X_2, X_3)$  is defined by

$$x_1 = X_1 + kX_3$$

$$x_2 = X_2 + kX_2$$

$$x_3 = X_3$$

where  $k = 10^{-5}$ .

- (a) Find the components of the strain tensor.  
 (b) Find the unit elongation of an element initially in the direction of  $\mathbf{e}_1 + \mathbf{e}_2$ .

3.20. Consider the displacement field

$$u_1 = k(2X_1^2 + X_1 X_2), \quad u_2 = kX_2^2, \quad u_3 = 0,$$

where  $k = 10^{-4}$ .

- (a) Find the unit elongations and the change of angle for two material elements  $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$  and  $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$  that emanate from a particle designated by  $\mathbf{X} = \mathbf{e}_1 + \mathbf{e}_2$ .  
 (b) Find the deformed shape of these two elements.

3.21. For the displacement field of Example 3.8.3, determine the increase in length for the diagonal element of the cube in the direction of  $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$  (a) by using the strain tensor and (b) by geometry.

3.22. With reference to a rectangular Cartesian coordinate system, the state of strain at a point is given by the matrix

$$[\mathbf{E}] = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \times 10^{-4}$$

- (a) What is the unit elongation in the direction  $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$ ?  
 (b) What is the change of angle between two perpendicular lines (in the undeformed state) emanating from the point and in the directions of  $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$  and  $3\mathbf{e}_1 - 6\mathbf{e}_3$ ?

3.23. Do the previous problem for (a) the unit elongation in the direction  $3\mathbf{e}_1 - 4\mathbf{e}_2$ , (b) the change in angle between two elements in the direction  $3\mathbf{e}_1 - 4\mathbf{e}_3$  and  $4\mathbf{e}_1 + 3\mathbf{e}_3$ .

3.24. (a) For Prob. 3.22, determine the principal scalar invariants of the strain tensor.

(b) Show that the following matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \times 10^{-4}$$

cannot represent the same state of strain of Prob. 3.22.

3.25. For the displacement field

$$u_1 = kX_1^2, \quad u_2 = kX_2X_3, \quad u_3 = k(2X_1X_3 + X_1^2), \quad k = 10^{-6}$$

find the maximum unit elongation for an element that is initially at  $(1, 0, 0)$ .

3.26. Given the matrix of an infinitesimal strain field

$$[\mathbf{E}] = \begin{bmatrix} k_1 X_2 & 0 & 0 \\ 0 & -k_2 X_2 & 0 \\ 0 & 0 & -k_2 X_2 \end{bmatrix}$$

- (a) Find the location of the particle that does not undergo any volume change.  
 (b) What should be the relation between  $k_1$  and  $k_2$  be such that no element changes volume?

**3.27.** The displacement components for a body are

$$u_1 = k(X_1^2 + X_2), \quad u_2 = k(4X_3^2 - X_1), \quad u_3 = 0, \quad k = 10^{-4}$$

- (a) Find the strain tensor.  
 (b) Find the change of length per unit length for an element which was at (1,2,1) and in the direction of  $\mathbf{e}_1 + \mathbf{e}_2$ .  
 (c) What is the maximum unit elongation at the same point (1,2,1)?  
 (d) What is the change of volume for the unit cube with a corner at the origin and with three of its edges along the positive coordinate axes.

**3.28.** For any motion the mass of a particle (material volume) remains constant. Consider the mass to be a product of its volume times its mass density and show that (a) for infinitesimal deformation  $\rho(1 + E_{kk}) = \rho_0$ , where  $\rho_0$  denotes the initial density and  $\rho$  the current density.

- (b) Use the smallness of  $E_{kk}$  to show that the current density is given by

$$\rho = \rho_0(1 - E_{kk})$$

**3.29.** True or false: At any point in a body, there always exist two mutually perpendicular material elements which do not suffer any change of angle in an arbitrary small deformation of the body. Give reasons.

**3.30.** Given the following strain components at a point in a continuum:

$$E_{11} = E_{12} = E_{22} = k, \quad E_{33} = 3k, \quad E_{13} = E_{23} = 0 \quad k > 0$$

Does there exist a material element at the point which decreases in length under the deformation? Explain your answer.

**3.31.** The unit elongations at a certain point on the surface of a body are measured experimentally by means of strain gages that are arranged  $45^\circ$  apart (called the  $45^\circ$  strain rosette) in the directions  $\mathbf{e}_1$ ,  $(\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2)$  and  $\mathbf{e}_2$ . If these unit elongations are designated by  $a, b, c$  respectively, what are the strain components  $E_{11}, E_{22}, E_{12}$ .

**3.32.** (a) Do Problem 3.31 if the measured strains are  $200 \times 10^{-6}$ ,  $50 \times 10^{-6}$ ,  $100 \times 10^{-6}$ , respectively.

- (b) If  $E_{33} = E_{32} = E_{31} = 0$ , find the principal strains and directions of part (a).  
 (c) How will the result of part (b) be altered if  $E_{33} \neq 0$ ?

3.33. Repeat Problem 3.32 except that  $a = b = c = 1000 \times 10^{-6}$ .

3.34. The unit elongations at a certain point on the surface of a body are measured experimentally by means of strain gages that are arranged  $60^\circ$  apart (called the  $60^\circ$  strain rosette) in the directions  $\mathbf{e}_1$ ,  $\frac{1}{2}(\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2)$ , and  $\frac{1}{2}(-\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2)$ . If these elongations are designated by  $a, b, c$  respectively, what are the strain components  $E_{11}, E_{22}, E_{12}$ ?

3.35. Do Problem 3.34 if the strain rosette measurements give  $a = 2 \times 10^{-6}$ ,  $b = 1 \times 10^{-6}$ ,  $c = 1.5 \times 10^{-6}$ .

3.36. Do Problem 3.35 except that  $a = b = c = 2000 \times 10^{-6}$ .

3.37. For the velocity field,  $\mathbf{v} = (kx_2^2)\mathbf{e}_1$

(a) Find the rate of deformation and spin tensors.

(b) Find the rate of extensions of a material element  $d\mathbf{x} = (ds)\mathbf{n}$  where

$$\mathbf{n} = (\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2) \text{ at } \mathbf{x} = 5\mathbf{e}_1 + 3\mathbf{e}_2.$$

3.38. For the velocity field

$$\mathbf{v} = \left( \frac{t+k}{1+x_1} \right) \mathbf{e}_1$$

find the rates of extension for the following material elements:  $d\mathbf{x}^{(1)} = ds_1\mathbf{e}_1$  and  $d\mathbf{x}^{(2)} = (ds_2/\sqrt{2})(\mathbf{e}_1 + \mathbf{e}_2)$  at the origin at time  $t = 1$ .

3.39. (a) Find the rate of deformation and spin tensors for the velocity field  $\mathbf{v} = (\cos t)(\sin \pi x_1)\mathbf{e}_2$ .

(b) For the velocity field of part (a), find the rates of extension of the elements  $d\mathbf{x}^{(1)} = (ds_1)\mathbf{e}_1$ ,  $d\mathbf{x}^{(2)} = (ds_2)\mathbf{e}_2$ ,  $d\mathbf{x}^{(3)} = ds_3/\sqrt{2}(\mathbf{e}_1 + \mathbf{e}_2)$  at the origin at  $t = 0$ .

3.40. Show that the following velocity components correspond to a rigid body motion:

$$v_1 = x_2 - x_3, \quad v_2 = -x_1 + x_3, \quad v_3 = x_1 - x_2$$

3.41. For the velocity field of Prob.3.15

(a) Find the rate of deformation and spin tensors.

(b) Find the rate of extension of a radial material line element.

3.42. Given the two-dimensional velocity field in cylindrical coordinates

$$v_r = 0, \quad v_\theta = 2r + \frac{4}{r}$$

(a) Find the acceleration at  $r = 2$ .

(b) Find the rate of deformation tensor at  $r = 2$ .

3.43. Given the velocity field in spherical coordinates

$$v_r = 0, \quad v_\theta = 0, \quad v_\phi = \left( Ar + \frac{B}{r^2} \right) \sin\theta$$

- (a) Find the acceleration field.
- (b) Find the rate of deformation field.

3.44. A motion is said to be irrotational if the spin tensor vanishes. Show that the velocity field of Prob.3.16 describes an irrotational motion.

3.45. (a) Let  $d\mathbf{x}^{(1)} = (ds_1)\mathbf{n}$ , and  $d\mathbf{x}^{(2)} = (ds_2)\mathbf{m}$  be two material elements that emanate from a particle  $P$  which at present has a rate of deformation  $\mathbf{D}$ . Consider  $(D/Dt)(d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)})$  and show that

$$\left( \frac{1}{ds_1} \frac{D(ds_1)}{Dt} + \frac{1}{ds_2} \frac{D(ds_2)}{Dt} \right) \cos\theta - (\sin\theta) \frac{D\theta}{Dt} = 2\mathbf{m} \cdot \mathbf{D}\mathbf{n}.$$

where  $\theta$  is the angle between  $\mathbf{m}$  and  $\mathbf{n}$ .

(b) Consider the special cases (i)  $d\mathbf{x}^{(1)} = d\mathbf{x}^{(2)}$  and (ii)  $\theta = \pi/2$ . Show that the above expression reduces to the results of Section 3.13.

3.46. Let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  and  $D_1, D_2, D_3$  be the principal directions and values of the rate of deformation tensor  $\mathbf{D}$ . Further, let

$$d\mathbf{x}^{(1)} = (ds_1)\mathbf{e}_1, \quad d\mathbf{x}^{(2)} = (ds_2)\mathbf{e}_2, \quad d\mathbf{x}^{(3)} = (ds_3)\mathbf{e}_3$$

be three material line elements. Consider the material derivative  $(D/Dt)[d\mathbf{x}^{(1)} \cdot (d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)})]$  and show that

$$\frac{1}{dV} \frac{D(dV)}{Dt} = D_1 + D_2 + D_3$$

where the infinitesimal volume  $dV = (ds_1)(ds_2)(ds_3)$ .

3.47. Consider a material element  $d\mathbf{x} = ds\mathbf{n}$

(a) Show that

$$(D/Dt)(\mathbf{n}) = \mathbf{D}\mathbf{n} + \mathbf{W}\mathbf{n} - (\mathbf{n} \cdot \mathbf{D}\mathbf{n})\mathbf{n}$$

(b) Show that if  $\mathbf{n}$  is an eigenvector of  $\mathbf{D}$  then

$$\frac{D\mathbf{n}}{Dt} = \mathbf{W}\mathbf{n} = \boldsymbol{\omega} \times \mathbf{n}$$

where  $\boldsymbol{\omega}$  is the axial vector of  $\mathbf{W}$ .

3.48. Given the following velocity field

$$v_1 = k(x_2 - 2)^2 x_3, \\ v_2 = -x_1 x_2 \quad ,$$

$$v_3 = kx_1x_3 \quad ,$$

for an incompressible fluid, determine  $k$  such that the equation of mass conservation is satisfied.

3.49. Given the velocity field in cylindrical coordinates

$$v_r = f(r, \theta), \quad v_\theta = 0, \quad v_z = 0$$

For an incompressible material, from the conservation of mass principle, obtain the most general form of the function  $f(r, \theta)$ .

3.50. An incompressible fluid undergoes a two-dimensional motion with

$$v_r = \frac{k \cos \theta}{\sqrt{r}}$$

find  $v_\theta$  if  $v_\theta = 0$  at  $\theta = 0$ .

3.51. Are the fluid motions described in (a) Prob.3.15 and (b) Prob.3.16 incompressible?

3.52. In a spatial description, the density of an incompressible fluid is given by  $\rho = kx_2$ . Find the permissible form for the velocity field with  $v_3 = 0$ , so that the conservation of mass equation is satisfied.

3.53. Consider the velocity field

$$\mathbf{v} = \left( \frac{x_1}{1+t} \right) \mathbf{e}_1$$

(a) Find the density if it is independent of spatial position, i.e.,  $\rho = \rho(t)$ .

(b) Find the density if it is a function  $x_1$  alone.

3.54. Given the velocity field

$$\mathbf{v} = x_1 t \mathbf{e}_1 + x_2 t \mathbf{e}_2,$$

determine how the fluid density varies with time, if in a spatial description it is a function of time only.

3.55. Check whether or not the following distribution of the state of strain satisfies the compatibility conditions:

$$[\mathbf{E}] = k \begin{bmatrix} X_1 + X_2 & X_1 & X_2 \\ X_1 & X_2 + X_3 & X_3 \\ X_2 & X_3 & X_1 + X_3 \end{bmatrix}$$

where  $k = 10^{-4}$ .

3.56. Check whether or not the following distribution of the state of strain satisfies the compatibility conditions:

$$[\mathbf{E}] = k \begin{bmatrix} X_1^2 & X_2^2 + X_3^2 & X_1X_3 \\ X_2^2 + X_3^2 & 0 & X_1 \\ X_1X_3 & X_1 & X_2^2 \end{bmatrix}$$

where  $k = 10^{-4}$ .

3.57. Does the displacement field

$$u_1 = \sin X_1, \quad u_2 = X_1^3 X_2, \quad u_3 = \cos X_3$$

correspond to a compatible strain field?

3.58. Given the strain field

$$E_{12} = E_{21} = k X_1 X_2$$

where  $k = 10^{-4}$  and all other  $E_{ij} = 0$ .

(a) Check the equations of compatibility for this strain field.

(b) By attempting to integrate the strain field, show that there does not exist a continuous displacement field for this strain field.

3.59. The strain components are given by

$$\begin{aligned} E_{11} &= \frac{1}{\alpha} f(X_2, X_3) \\ E_{22} = E_{33} &= -\frac{\nu}{\alpha} f(X_2, X_3) \\ E_{12} = E_{13} = E_{23} &= 0 \end{aligned}$$

Show that for the strains to be compatible  $f(X_2, X_3)$  must be linear.

3.60. In cylindrical coordinates  $(r, \theta, z)$ , consider a differential volume bounded by the three pairs of faces  $r = r_o, r = r_o + dr; \theta = \theta_o, \theta = \theta_o + d\theta; z = z_o, z = z_o + dz$ . The rate at which mass is flowing into the volume across the face  $r = r_o$  is given by  $(\rho v_r)(r_o d\theta)(dz)$  and similar expressions for other faces. By demanding that the net rate of inflow of mass must be equal to the rate of increase of mass inside the volume, obtain the equation of conservation of mass in cylindrical coordinates as that given in Eq.(3.15.5).

3.61. Given the following deformation in rectangular Cartesian coordinates

$$\begin{aligned} x_1 &= 3X_3 \\ x_2 &= -X_1 \\ x_3 &= -2X_2 \end{aligned}$$

Determine (a) the deformation gradient  $\mathbf{F}$ , (b) the right Cauchy-Green deformation tensor  $\mathbf{C}$ , (c) the left Cauchy-Green deformation tensor  $\mathbf{B}$ , (d) the rotation tensor  $\mathbf{R}$ , (e) the Lagrangian strain tensor, (f) the Euler strain tensor, (g) ratio of deformed volume to the initial

volume, (h) the deformed area (magnitude and its normal) for the area whose normal was in the direction of  $\mathbf{e}_2$  and whose magnitude was unity in the undeformed state.

3.62. Do Prob. 3.61 for the following deformation

$$x_1 = 2X_2$$

$$x_2 = 3X_3$$

$$x_3 = X_1$$

3.63. Do Prob. 3.61 for the following deformation

$$x_1 = X_1$$

$$x_2 = 3X_3$$

$$x_3 = -2X_2$$

3.64. Do Prob. 3.61 for the following deformation

$$x_1 = 2X_2$$

$$x_2 = -X_1$$

$$x_3 = 3X_3$$

3.65. Given

$$x_1 = X_1 + 3X_2, \quad x_2 = X_2, \quad x_3 = X_3$$

Obtain

(a)  $\mathbf{F}$ ,  $\mathbf{C}$ .

(b) the eigenvalues and eigenvectors of  $\mathbf{C}$ .

(c) the matrix of  $\mathbf{U}$  and  $\mathbf{U}^{-1}$  using the eigenvectors of  $\mathbf{C}$  as basis.

(d) the matrix of  $\mathbf{U}$  and  $\mathbf{U}^{-1}$  with respect to the  $\mathbf{e}_i$  basis.

(d) the rotation tensor  $\mathbf{R}$  with respect to the  $\mathbf{e}_i$  basis.

You may check your results with the formulas given in the next problem.

3.66. Verify that with respect to rectangular Cartesian base vectors, the right stretch tensor  $\mathbf{U}$  and the rotation tensor  $\mathbf{R}$  for the simple shear deformation

$$x_1 = X_1 + kX_2, \quad x_2 = X_2, \quad x_3 = X_3$$

are given by

$$[\mathbf{U}] = \begin{bmatrix} f & \frac{k_f}{2} & 0 \\ \frac{k_f}{2} & (1 + \frac{k^2}{2})f & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad [\mathbf{R}] = \begin{bmatrix} f & \frac{k_f}{2} & 0 \\ -\frac{k_f}{2} & f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where  $f = (1 + \frac{k^2}{4})^{-\frac{1}{2}}$ .

3.67. Let  $d\mathbf{X}^{(1)} = dS_1 \mathbf{N}^{(1)}$  and  $d\mathbf{X}^{(2)} = dS_2 \mathbf{N}^{(2)}$  be two material elements at a point P. Show that if  $\theta$  denotes the angle between their respective deformed elements  $dx^{(1)}$  and  $dx^{(2)}$ , then

$$\cos\theta = \frac{C_{\alpha\beta} N_{\alpha}^{(1)} N_{\beta}^{(1)}}{\lambda_1 \lambda_2}$$

where  $\mathbf{N}^{(1)} = N_{\alpha}^{(1)} \mathbf{e}_{\alpha}$ ,  $\mathbf{N}^{(2)} = N_{\alpha}^{(2)} \mathbf{e}_{\alpha}$ ,  $\lambda_1 = \frac{ds_1}{dS_1}$  and  $\lambda_2 = \frac{ds_2}{dS_2}$ .

3.68. Given the following right Cauchy-Green deformation tensor at a point

$$[\mathbf{C}] = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0.36 \end{bmatrix}$$

- Find the stretch for the material elements which were in the direction of  $\mathbf{e}_1$ ,  $\mathbf{e}_2$  and  $\mathbf{e}_3$  directions.
- Find the stretch for the material element which was in the direction of  $\mathbf{e}_1 + \mathbf{e}_2$
- Find  $\cos\theta$ , where  $\theta$  is the angle between  $dx^{(1)}$  and  $dx^{(2)}$ .

3.69. Show that for any tensor  $\mathbf{A}(X_1, X_2, X_3)$

$$\frac{\partial}{\partial X_m} \det \mathbf{A} = (\det \mathbf{A}) (A^{-1})_{nj} \frac{\partial A_{jn}}{\partial X_m}$$

3.70. Given

$$r = r_o, \quad \theta = \theta_o + kz_o, \quad z = z_o$$

where  $(r, \theta, z)$  and  $(r_o, \theta_o, z_o)$  are cylindrical coordinates for the current and reference configuration respectively.

- Obtain the components of the left Cauchy-Green tensor  $\mathbf{B}$  with respect to the basis at the current configuration.
- Obtain the components of the right Cauchy-Green tensor  $\mathbf{C}$  with respect to the basis at the reference configuration.

3.71. Given

$$r = (2aX + b)^{1/2}, \quad \theta = \frac{Y}{a}, \quad z = Z$$

where  $(r, \theta, z)$  is cylindrical coordinates for the current configuration and  $(X, Y, Z)$  are rectangular Cartesian coordinates for the reference configuration.

(a) Calculate the change of volume.

(b) Obtain the components of the left Cauchy-Green tensor  $\mathbf{B}$  with respect to the basis at the current configuration.

3.72. Given

$$r = f(X), \quad \theta = g(Y), \quad z = h(Z)$$

where  $(r, \theta, z)$  and  $(X, Y, Z)$  are cylindrical coordinates and rectangular Cartesian coordinates for the current and reference configuration, respectively. Obtain the components of the right Cauchy-Green tensor  $\mathbf{C}$  with respect to the basis at the reference configuration

3.73. From Eqs.(3.30.4a), obtain Eqs.(3.30.5).

3.74. Verify Eq.(3.30.8b) and (3.30.8d).

3.75. Verify Eq.(3.30.9b) and (3.30.9d).

3.76. Derive Eqs.(3.30.10).

3.77. Using Eqs.(3.30.10) derive Eqs.(3.30.12a) and (3.30.12d).

3.78. Verify Eqs. (3.30.13 a) and (3.30.13d).

# 4

## Stress

In the previous chapter, we considered the purely kinematical description of the motion of a continuum without any consideration of the forces that cause the motion and deformation. In this chapter, we shall consider a means of describing the forces in the interior of a body idealized as a continuum. It is generally accepted that matter is formed of molecules which in turn consists of atoms and subatomic particles. Therefore, the internal forces in real matter are those between the above particles. In the classical continuum theory the internal forces are introduced through the concept of body forces and surface forces. Body forces are those that act throughout a volume (e.g., gravity, electrostatic force) by a long-range interaction with matter or charge at a distance. Surface forces are those that act on a surface (real or imagined) separating parts of the body. We shall assume that it is adequate to describe the surface force at a point of a surface through the definition of a **stress vector**, discussed in Section 4.1, which pays no attention to the curvature of the surface at the point. Such an assumption is known as **Cauchy's stress principle** which is one of the basic axioms of classical continuum mechanics.

### 4.1 Stress Vector

Let us consider a body depicted in Fig. 4.1. Imagine a plane such as  $S$ , which passes through an arbitrary internal point  $P$  and which has a unit normal vector  $\mathbf{n}$ . The plane cuts the body into two portions. One portion lies on the side of the arrow of  $\mathbf{n}$  (designated by II in the figure) and the other portion on the tail of  $\mathbf{n}$  (designated by I). Considering portion I as a free body, there will be on plane  $S$  a resultant force  $\Delta\mathbf{F}$  acting on a small area  $\Delta A$  containing  $P$ . We define the stress vector (from II to I) at the point  $P$  on the plane  $S$  as the limit of the ratio  $\Delta\mathbf{F}/\Delta A$  as  $\Delta A \rightarrow 0$ . That is, with  $\mathbf{t}_n$  denoting the stress vector,

$$\mathbf{t}_n = \lim_{\Delta A \rightarrow 0} \frac{\Delta\mathbf{F}}{\Delta A} \quad (4.1.1)$$

If portion II is considered as a free body, then by Newton's law of action and reaction, we shall have a stress vector (from I to II),  $\mathbf{t}_{-n}$  at the same point on the same plane equal and opposite to that given by Eq. (4.1.1). That is,

$$\mathbf{t}_n = -\mathbf{t}_{-n} \quad (4.1.2)$$

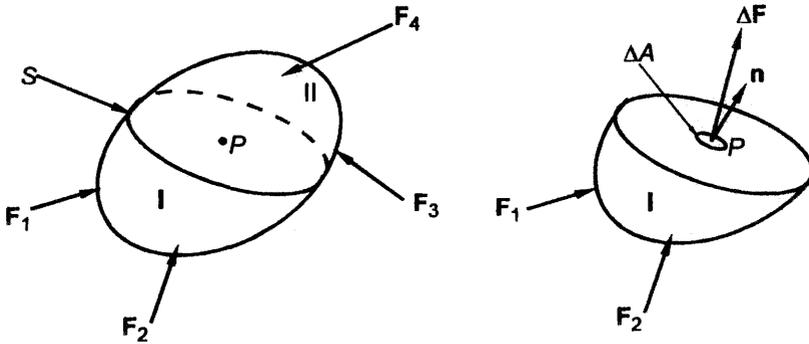


Fig. 4.1

Next, let  $S$  be a surface (instead of a plane) passing the point  $P$ . Let  $\Delta F$  be the resultant force on a small area  $\Delta S$  on the surface  $S$ . The **Cauchy stress vector** at  $P$  on  $S$  is defined as

$$t = \lim_{\Delta S \rightarrow 0} \frac{\Delta F}{\Delta S} \tag{4.1.3}$$

We now state the following principle, known as the **Cauchy's stress principle**: The stress vector at any given place and time has a common value on all parts of material having a common tangent plane at  $P$  and lying on the same side of it. In other words, if  $n$  is the unit outward normal (i.e., a vector of unit length pointing outward away from the material) to the tangent plane, then

$$t = t(x, t, n) \tag{4.1.4}$$

where the scalar  $t$  denotes time.

In the following section, we shall show from Newton's second law that this dependence on  $n$  can be expressed as

$$t(x, t, n) = T(x, t)n \tag{4.1.5}$$

where  $T$  is a linear transformation.

## 4.2 Stress Tensor

According to Eq. (4.1.4) of the previous section, the stress vector on a plane passing through a given spatial point  $x$  at a given time  $t$  depends only on the unit normal vector  $n$  to the plane. Thus, let  $T$  be the transformation such that

$$t_n = Tn \tag{4.2.1}$$

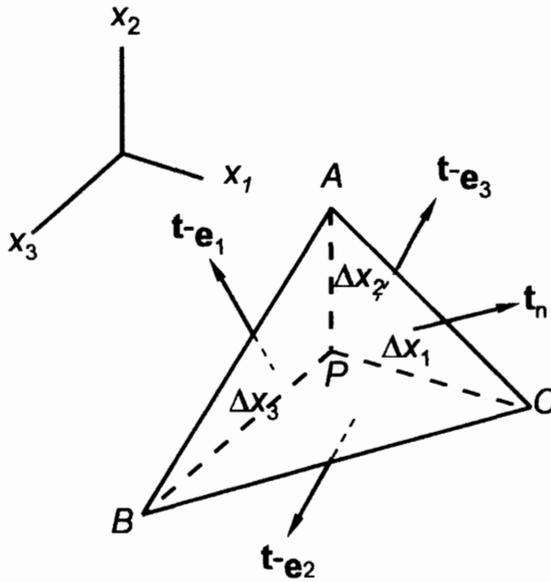


Fig. 4.2

Let a small tetrahedron be isolated from the body with the point  $P$  as one of its vertices (see Fig. 4.2). The size of the tetrahedron will ultimately be made to approach zero volume so that, in the limit, the inclined plane will pass through the point  $P$ . The outward normal to the face  $PAB$  is  $-\mathbf{e}_1$ . Thus, the stress vector on this face is denoted by  $\mathbf{t}_{-\mathbf{e}_1}$  and the force on the face is

$$\mathbf{t}_{-\mathbf{e}_1} \Delta A_1$$

where  $\Delta A_1$  is the area of  $PAB$ . Similarly, the forces acting on  $PBC$ ,  $PAC$  and the inclined face  $ABC$  are

$$\mathbf{t}_{-\mathbf{e}_2} \Delta A_2, \quad \mathbf{t}_{-\mathbf{e}_3} \Delta A_3$$

and

$$\mathbf{t}_n \Delta A_n$$

respectively. Thus, from Newton's second law written for the tetrahedron, we have

$$\Sigma \mathbf{F} = \mathbf{t}_{-\mathbf{e}_1}(\Delta A_1) + \mathbf{t}_{-\mathbf{e}_2}(\Delta A_2) + \mathbf{t}_{-\mathbf{e}_3}(\Delta A_3) + \mathbf{t}_n(\Delta A_n) = m\mathbf{a} \tag{i}$$

Since the mass  $m = (\text{density})(\text{volume})$ , and the volume of the tetrahedron is proportional to the product of three infinitesimal lengths, (in fact, the volume equals to  $(1/6)\Delta x_1 \Delta x_2 \Delta x_3$ ), when the size of the tetrahedron approaches zero, the right hand side of Eq. (i) will approach zero

faster than the terms on the left where the stress vectors are multiplied by areas, the product of two infinitesimal lengths. Thus, in the limit, the acceleration term drops out exactly from Eq. (i) (We note also that any body force e.g. weight that is acting will be of the same order of magnitude as that of the acceleration term and will also drop out). Thus,

$$\mathbf{t}_{-e_1}(\Delta A_1) + \mathbf{t}_{-e_2}(\Delta A_2) + \mathbf{t}_{-e_3}(\Delta A_3) + \mathbf{t}_n(\Delta A_n) = \mathbf{0} \quad (\text{ii})$$

Let the unit normal vector of the inclined plane  $ABC$  be

$$\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3 \quad (4.2.2)$$

The areas  $\Delta A_1, \Delta A_2$ , and  $\Delta A_3$ , being the projections of  $\Delta A_n$  are related to  $\Delta A_n$  by

$$\Delta A_1 = n_1\Delta A_n, \quad \Delta A_2 = n_2\Delta A_n, \quad \Delta A_3 = n_3\Delta A_n \quad (4.2.3)$$

Using Eq. (4.2.3), Eq. (ii) becomes

$$\mathbf{t}_{-e_1}n_1 + \mathbf{t}_{-e_2}n_2 + \mathbf{t}_{-e_3}n_3 + \mathbf{t}_n = \mathbf{0} \quad (4.2.4)$$

But from the law of action and reaction,

$$\mathbf{t}_{-e_1} = -\mathbf{t}_{e_1}, \quad \mathbf{t}_{-e_2} = -\mathbf{t}_{e_2}, \quad \mathbf{t}_{-e_3} = -\mathbf{t}_{e_3} \quad (\text{iii})$$

Thus, Eq. (4.2.4) becomes

$$\mathbf{t}_n = n_1\mathbf{t}_{e_1} + n_2\mathbf{t}_{e_2} + n_3\mathbf{t}_{e_3} \quad (4.2.5)$$

Now, using Eq. (4.2.2) and Eq. (4.2.5), Eq. (4.2.1) becomes

$$\mathbf{T}(n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3) = n_1\mathbf{T}\mathbf{e}_1 + n_2\mathbf{T}\mathbf{e}_2 + n_3\mathbf{T}\mathbf{e}_3 \quad (4.2.6)$$

That is, the transformation  $\mathbf{T}$  defined by

$$\mathbf{t}_n = \mathbf{T}\mathbf{n} \quad (4.2.7)$$

is a linear transformation [see Eq. (2B1.2)]. It is called the **stress tensor**, or **Cauchy stress tensor**.

### 4.3 Components of Stress Tensor

According to Eq. (4.2.7) of the previous section, the stress vectors  $\mathbf{t}_{e_i}$  on the three coordinate planes (the  $e_i$ -planes) are related to the stress tensor  $\mathbf{T}$  by

$$\mathbf{t}_{e_1} = \mathbf{T}\mathbf{e}_1, \quad \mathbf{t}_{e_2} = \mathbf{T}\mathbf{e}_2, \quad \mathbf{t}_{e_3} = \mathbf{T}\mathbf{e}_3 \quad (4.3.1)$$

By the definition of the components of a tensor, Eq. (2B2.1b), we have

$$\mathbf{T}\mathbf{e}_i = T_{mi}\mathbf{e}_m \quad (4.3.2)$$

Thus,

$$\mathbf{t}_{\mathbf{e}_1} = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3 \quad (4.3.3a)$$

$$\mathbf{t}_{\mathbf{e}_2} = T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3 \quad (4.3.3b)$$

$$\mathbf{t}_{\mathbf{e}_3} = T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3 \quad (4.3.3c)$$

Since  $\mathbf{t}_{\mathbf{e}_1}$  is the stress vector acting on the plane whose outward normal is  $\mathbf{e}_1$ , it is clear from Eq. (4.3.3a) that  $T_{11}$  is its normal component and  $T_{21}$  and  $T_{31}$  are its tangential components. Similarly,  $T_{22}$  is the normal component on the  $\mathbf{e}_2$ -plane and  $T_{12}$ ,  $T_{32}$  are the tangential components on the same plane, etc.

We note that for each stress component  $T_{ij}$ , the second index  $j$  indicates the plane on which the stress component acts and the first index indicates the direction of the component; e.g.,  $T_{12}$  is the stress component in the direction of  $\mathbf{e}_1$  acting on the plane whose outward normal is in the direction of  $\mathbf{e}_2$ . We also note that positive **normal stresses** are also known as **tensile stresses** and negative normal stresses as **compressive stresses**. Tangential stresses are also known as shearing stresses. Both  $T_{21}$  and  $T_{31}$  are shearing stress components acting on the *same* plane (the  $\mathbf{e}_1$ -plane), thus the resultant shearing stress on this plane is given by

$$\boldsymbol{\tau}_1 = T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3 \quad (4.3.4a)$$

the magnitude of this shearing stress is given by

$$|\boldsymbol{\tau}_1| = \sqrt{T_{21}^2 + T_{31}^2}$$

Similarly, on  $\mathbf{e}_2$ -plane

$$\boldsymbol{\tau}_2 = T_{12}\mathbf{e}_1 + T_{32}\mathbf{e}_3 \quad (4.3.4b)$$

and on  $\mathbf{e}_3$ -plane

$$\boldsymbol{\tau}_3 = T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 \quad (4.3.4c)$$

From  $\mathbf{t} = \mathbf{T}\mathbf{n}$ , the components of  $\mathbf{t}$  are related to those of  $\mathbf{T}$  and  $\mathbf{n}$  by the equation

$$t_i = T_{ij}n_j \quad (4.3.5a)$$

Or, in a form more convenient for computations,

$$[\mathbf{t}] = [\mathbf{T}][\mathbf{n}] \quad (4.3.5b)$$

Thus, it is clear that if the matrix of  $\mathbf{T}$  is known, the stress vector  $\mathbf{t}$  on any inclined plane is uniquely determined from Eq. (4.3.5b). In other words, the state of stress at a point is completely characterized by the stress tensor  $\mathbf{T}$ . Also since  $\mathbf{T}$  is a second-order tensor, any one matrix of  $\mathbf{T}$  determines the other matrices of  $\mathbf{T}$ , see Section 2B13 of Chapter 2.

We should also note that some authors use the convention  $\mathbf{t} = \mathbf{T}^T \mathbf{n}$  so that  $\mathbf{t}_e = T_{ij} \mathbf{e}_j$ . Under that convention, for example,  $T_{21}$  and  $T_{23}$  are tangential components of the stress vector on the plane whose normal is  $\mathbf{e}_2$  etc. These differences in meaning regarding the nondiagonal elements of  $\mathbf{T}$  disappear if the stress tensor  $\mathbf{T}$  is symmetric.

**4.4 Symmetry of Stress Tensor- Principle of Moment of Momentum**

By the use of moment of momentum equation for a differential element, we shall now show that the stress tensor is generally a symmetric tensor<sup>†</sup>.

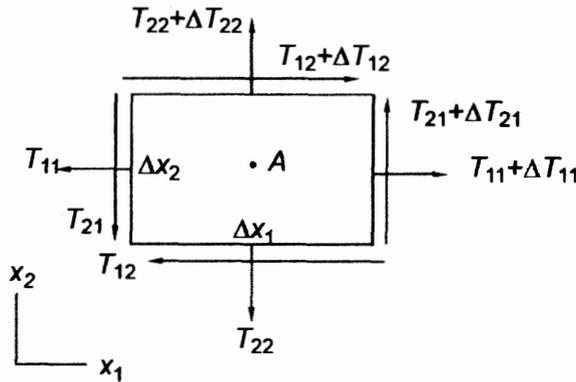


Fig. 4.3

Consider the free-body diagram of a differential parallelepiped isolated from a body as shown in Fig. 4.3. Let us find the moment of all the forces about an axis passing through the center point  $A$  and parallel to the  $x_3$ -axis:

$$\begin{aligned} \Sigma(M_A)_3 = & T_{21}(\Delta x_2)(\Delta x_3) \left( \frac{\Delta x_1}{2} \right) + (T_{21} + \Delta T_{21})(\Delta x_2)(\Delta x_3) \left( \frac{\Delta x_1}{2} \right) \\ & - T_{12}(\Delta x_1)(\Delta x_3) \left( \frac{\Delta x_2}{2} \right) - (T_{12} + \Delta T_{12})(\Delta x_1)(\Delta x_3) \left( \frac{\Delta x_2}{2} \right) \end{aligned} \tag{i}$$

In writing Eq. (i) we have assumed the absence of body moments.

Dropping the terms containing small quantities of higher order, we obtain

<sup>†</sup> See Prob. 4.27 for a case where the stress tensor is not symmetric

$$\Sigma(M_A)_3 = (T_{21} - T_{12})\Delta x_1 \Delta x_2 \Delta x_3 \quad (\text{ii})$$

Now, whether the elements is in static equilibrium or not,  $\Sigma(M_A)_3$  is equal to zero because the angular acceleration term is proportional to the moment of inertia which is given by  $(1/12)$  (density)  $(\Delta x_1 \Delta x_2 \Delta x_3)[(\Delta x_1)^2 + (\Delta x_2)^2]$  and is therefore a small quantity of higher order than the right side of Eq. (ii). Thus,

$$T_{12} = T_{21} \quad (4.4.1a)$$

Similarly, one can obtain

$$T_{13} = T_{31} \quad (4.4.1b)$$

and

$$T_{23} = T_{32} \quad (4.4.1c)$$

These equations state that the stress tensor is symmetric, i.e.,  $\mathbf{T} = \mathbf{T}^T$ . Therefore, there are only six independent stress components.

#### Example 4.4.1

The state of stress at a certain point is  $\mathbf{T} = -p\mathbf{I}$ , where  $p$  is a scalar. Show that there is no shearing stress on any plane containing this point.

*Solution.* The stress vector on any plane passing through the point with normal  $\mathbf{n}$  is

$$\mathbf{t} = \mathbf{T}\mathbf{n} = -p\mathbf{I}\mathbf{n} = -p\mathbf{n}$$

Therefore, it is normal to the plane. This simple stress state is called a **hydrostatic state of stress**.

#### Example 4.4.2

With reference to an  $xyz$  coordinate system, the matrix of a state of stress at a certain point of a body is given by:

$$[\mathbf{T}] = \begin{bmatrix} 2 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix} \text{ MPa} \quad (\text{i})$$

(a) Find the stress vector and the magnitude of the normal stress on a plane that passes through the point and is parallel to the plane

$$x + 2y + 2z - 6 = 0$$

(b) If

$$\mathbf{e}_1' = \frac{1}{3}(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)$$

and

$$\mathbf{e}_2' = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2)$$

find  $T'_{12}$ .

*Solution.* (a) The plane  $x+2y+2z-6=0$  has a unit normal  $\mathbf{n}$  given by

$$\mathbf{n} = \frac{1}{3}(\mathbf{e}_1 + 2\mathbf{e}_2 + 2\mathbf{e}_3) \quad (\text{ii})$$

The stress vector is obtained from Eq. (4.2.7) as

$$[\mathbf{t}] = [\mathbf{T}][\mathbf{n}] = \frac{1}{3} \begin{bmatrix} 2 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 16 \\ 4 \\ 1 \end{bmatrix} \quad (\text{iii})$$

or,

$$\mathbf{t} = \frac{1}{3}(16\mathbf{e}_1 + 4\mathbf{e}_2 + \mathbf{e}_3) \text{ MPa} \quad (\text{iv})$$

The magnitude of the normal stress is simply, with  $T_n \equiv T_{(n)(n)}$ ,

$$T_n = \mathbf{t} \cdot \mathbf{n} = \frac{1}{9}(16+8+2) = 2.89 \text{ MPa} \quad (\text{v})$$

(b) To find the primed components of the stress, we have,

$$T'_{12} = \mathbf{e}_1' \cdot \mathbf{T}\mathbf{e}_2' = \frac{1}{3\sqrt{2}}[2, 2, 1] \begin{bmatrix} 2 & 4 & 3 \\ 4 & 0 & 0 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad (\text{vi})$$

Therefore,

$$T'_{12} = \frac{7}{3\sqrt{2}} = 1.65 \text{ MPa} \quad (\text{vii})$$

### Example 4.4.3

The distribution of stress inside a body is given by the matrix

$$\begin{bmatrix} -p+\rho gy & 0 & 0 \\ 0 & -p+\rho gy & 0 \\ 0 & 0 & -p+\rho gy \end{bmatrix} \quad (\text{i})$$

where  $p$ ,  $\rho$  and  $g$  are constants. Figure 4.4 shows a rectangular block inside the body.

(a) What is the distribution of the stress vector on the six faces of the block?

(b) Find the total resultant force acting on the faces  $y = 0$  and  $x = 0$ .

*Solution.* (a) We have, from  $\mathbf{t} = \mathbf{T}\mathbf{n}$ ,

$$\text{on } x = 0, [\mathbf{n}] = [-1, 0, 0], \quad [\mathbf{t}] = [p - \rho gy, 0, 0]$$

$$\text{on } x = a, [\mathbf{n}] = [+1, 0, 0], \quad [\mathbf{t}] = [-p + \rho gy, 0, 0]$$

$$\text{on } y = 0, [\mathbf{n}] = [0, -1, 0], \quad [\mathbf{t}] = [0, p, 0]$$

$$\text{on } y = b, [\mathbf{n}] = [0, +1, 0], \quad [\mathbf{t}] = [0, -p + \rho gb, 0] \tag{ii}$$

$$\text{on } z = 0, [\mathbf{n}] = [0, 0, -1], \quad [\mathbf{t}] = [0, 0, p - \rho gy]$$

$$\text{on } z = c, [\mathbf{n}] = [0, 0, +1], \quad [\mathbf{t}] = [0, 0, -p + \rho gy]$$

A section of the distribution of the stress vector is shown in Fig. 4.5.

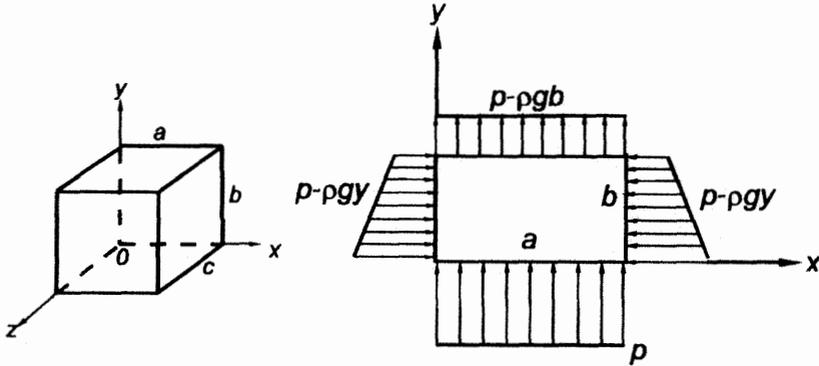


Fig.4.4

Fig.4.5

(b) On the face  $y = 0$ , the total force is

$$\mathbf{F}_1 = \int \mathbf{t}dA = (p \int dA)\mathbf{e}_2 = p a c \mathbf{e}_2 \tag{iii}$$

On the face  $x = 0$ , the total force is

$$\mathbf{F}_2 = \left[ \int (p - \rho gy)dA \right] \mathbf{e}_1 = \left[ p \int dA - \rho g \int ydA \right] \mathbf{e}_1 \tag{iv}$$

The second integral can be evaluated directly by replacing  $(dA)$  by  $(c dy)$  and integrating from  $y = 0$  to  $y = b$ . Or since  $\int ydA$  is the first moment of the face area about the  $z$ -axis, it is therefore equal to the product of the centroidal distance and the total area. Thus,

$$\mathbf{F}_2 = \left[ pbc - \frac{\rho gb^2 c}{2} \right] \mathbf{e}_1 \tag{v}$$

#### 4.5 Principal Stresses

From Sect. 2B18, we know that for any symmetric stress tensor  $\mathbf{T}$ , there exists at least three mutually perpendicular principal directions ( the eigenvectors of  $\mathbf{T}$  ). The planes having these directions as their normals are known as the **principal planes**. On these planes, the stress vector is normal to the plane (i.e., no shearing stresses ) and the normal stresses are known as the **principal stresses**. Thus, the principal stresses (eigenvalues of  $\mathbf{T}$  ) include the maximum and the minimum values of normal stresses among all planes passing through a given point.

The principal stresses are to be obtained from the characteristic equation of  $\mathbf{T}$ , which may be written:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \quad (4.5.1)$$

where

$$I_1 = T_{11} + T_{22} + T_{33} \quad (4.5.2a)$$

$$I_2 = \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} \quad (4.5.2b)$$

$$I_3 = \det[\mathbf{T}] \quad (4.5.2c)$$

are the three principal scalar invariants of the stress tensor. For the computations of the principal directions, we refer the reader to Sect. 2B17.

#### 4.6 Maximum Shearing Stress

In this section, we show that the maximum shearing stress is equal to one-half the difference between the maximum and the minimum principal stresses and acts on the plane that bisects the right angle between the directions of the maximum and minimum principal stresses.

Let  $\mathbf{e}_1$ ,  $\mathbf{e}_2$  and  $\mathbf{e}_3$  be the principal directions of  $\mathbf{T}$  and let  $T_1$ ,  $T_2$ ,  $T_3$  be the principal stresses. If  $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$  is the unit normal to a plane, the components of the stress vector on the plane is given by

$$\begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} n_1 T_1 \\ n_2 T_2 \\ n_3 T_3 \end{bmatrix} \quad (4.6.1a)$$

i.e.,

$$\mathbf{t} = n_1 T_1 \mathbf{e}_1 + n_2 T_2 \mathbf{e}_2 + n_3 T_3 \mathbf{e}_3 \quad (4.6.1b)$$

and the normal stress on the same plane is given by

$$T_n = \mathbf{n} \cdot \mathbf{t} = n_1^2 T_1 + n_2^2 T_2 + n_3^2 T_3 \quad (4.6.2)$$

Thus, if  $T_s$  denotes the magnitude of the total shearing stress on the plane, we have (see Fig. 4.6)

$$T_s^2 = |t|^2 - T_n^2 \tag{4.6.3}$$

i.e.,

$$T_s^2 = T_1^2 n_1^2 + T_2^2 n_2^2 + T_3^2 n_3^2 - (T_1 n_1^2 + T_2 n_2^2 + T_3 n_3^2)^2 \tag{4.6.4}$$

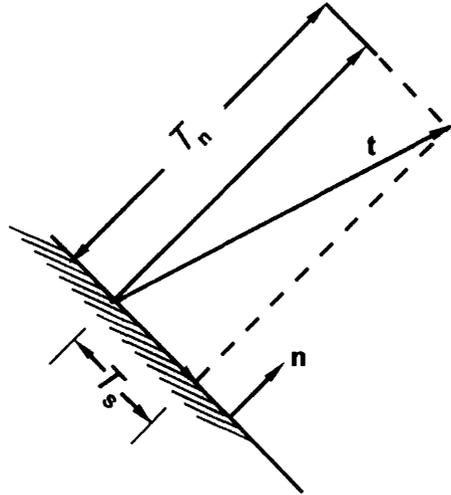


Fig. 4.6

For known values of  $T_1$ ,  $T_2$ , and  $T_3$ , Eq. (4.6.4) states that  $T_s^2$  is a function of  $n_1$ ,  $n_2$ , and  $n_3$ , i.e.,

$$T_s^2 = f(n_1, n_2, n_3) \tag{4.6.5}$$

We wish to find the triple  $(n_1, n_2, n_3)$  for which  $f$  attains a maximum. However,

$$n_1^2 + n_2^2 + n_3^2 = 1 \tag{4.6.6}$$

thus, we are looking for a maximum for the value of the function  $f(n_1, n_2, n_3)$  subjected to the constraint that  $n_1^2 + n_2^2 + n_3^2 = 1$ .

Taking the total derivative of Eq. (4.6.5), we obtain

$$d(T_s^2) = \frac{\partial T_s^2}{\partial n_1} dn_1 + \frac{\partial T_s^2}{\partial n_2} dn_2 + \frac{\partial T_s^2}{\partial n_3} dn_3 = 0 \tag{i}$$

If  $dn_1, dn_2,$  and  $dn_3$  can vary independently of one another, then Eq. (i) gives the familiar condition for the determination of the triple  $(n_1, n_2, n_3)$  for the stationary value of  $T_s^2$

$$\frac{\partial T_s^2}{\partial n_1} = 0, \quad \frac{\partial T_s^2}{\partial n_2} = 0, \quad \text{and} \quad \frac{\partial T_s^2}{\partial n_3} = 0$$

But the  $dn_1, dn_2$  and  $dn_3$  can not vary independently. Indeed, taking the total derivative of Eq. (4.6.6), i.e.,  $n_1^2 + n_2^2 + n_3^2 = 1$ . we obtain

$$n_1 dn_1 + n_2 dn_2 + n_3 dn_3 = 0 \tag{ii}$$

If we let

$$\frac{\partial T_s^2}{\partial n_1} = \lambda n_1 \tag{iii}$$

$$\frac{\partial T_s^2}{\partial n_2} = \lambda n_2 \tag{iv}$$

and

$$\frac{\partial T_s^2}{\partial n_3} = \lambda n_3 \tag{v}$$

then by substituting Eqs. (iii) (iv) and (v) into Eq. (i), we see clearly that Eq. (i) is satisfied if Eq. (4.6.6) is enforced. Thus, Eqs. (iii), (iv), (v) and (4.6.6) are four equations for the determination of the four unknown values of  $n_1, n_2, n_3$  and  $\lambda$  which correspond to stationary values of  $T_s^2$ . This is the Lagrange multiplier method and the parameter  $\lambda$  is known as the Lagrange multiplier (whose value is however of little interest).

Computing the partial derivatives from Eq. (4.6.4), Eqs. (iii), (iv), and (v) become

$$2n_1[T_1^2 - 2(T_1n_1^2 + T_2n_2^2 + T_3n_3^2)T_1] = n_1 \lambda \tag{vi}$$

$$2n_2[T_2^2 - 2(T_1n_1^2 + T_2n_2^2 + T_3n_3^2)T_2] = n_2 \lambda \tag{vii}$$

$$2n_3[T_3^2 - 2(T_1n_1^2 + T_2n_2^2 + T_3n_3^2)T_3] = n_3 \lambda \tag{viii}$$

From Eqs. (vi), (vii), (viii) and (4.6.6), the following stationary points  $(n_1, n_2, n_3)$  can be obtained (The procedure is straight forward, but the detail is somewhat tedious, we leave it as an exercise.):

$$(1,0,0), (0,1,0), (0,0,1) \tag{ix}$$

$$\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{2}}, 0, \pm\frac{1}{\sqrt{2}}\right), \left(0, \frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) \tag{x}$$

The planes determined by the solutions given by Eq. (ix) are nothing but the principal planes, on which  $T_s = 0$ . Thus, on these planes the values of  $T_s^2$  is a minimum (in fact, zero).

The values of  $T_s^2$  on the planes given by the solutions (x) are easily obtained from Eq. (4.6.4) to be the following:

$$\text{for } \mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{e}_1 \pm \frac{1}{\sqrt{2}}\mathbf{e}_2, \quad T_s^2 = \frac{(T_1 - T_2)^2}{4} \quad (4.6.5)$$

$$\text{for } \mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{e}_1 \pm \frac{1}{\sqrt{2}}\mathbf{e}_3, \quad T_s^2 = \frac{(T_1 - T_3)^2}{4} \quad (4.6.6)$$

and

$$\text{for } \mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{e}_2 \pm \frac{1}{\sqrt{2}}\mathbf{e}_3, \quad T_s^2 = \frac{(T_2 - T_3)^2}{4} \quad (4.6.7)$$

Thus, the maximum magnitude of the shearing stress is given by the largest of the three values

$$\frac{|T_1 - T_2|}{2}, \quad \frac{|T_1 - T_3|}{2}, \quad \text{and} \quad \frac{|T_2 - T_3|}{2}$$

In other words,

$$(T_s)_{\max} = \frac{(T_n)_{\max} - (T_n)_{\min}}{2} \quad (4.6.8)$$

where  $(T_n)_{\max}$  and  $(T_n)_{\min}$  are the largest and the smallest normal stress respectively. It can also be shown that on the plane of maximum shearing stress, the normal stress is

$$T_n = \frac{[(T_n)_{\max} + (T_n)_{\min}]}{2} \quad (4.6.9)$$

#### Example 4.6.1

If the state of stress is such that the components  $T_{13}, T_{23}, T_{33}$  are equal to zero, then it is called a state of plane stress.

- (a) For plane stress, find the principal values and the corresponding principal directions.
- (b) Determine the maximum shearing stress.

*Solution.* (a) For the stress matrix

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (i)$$

the characteristic equation has the form

$$\lambda[\lambda^2 - (T_{11} + T_{22})\lambda + (T_{11}T_{22} - T_{12}^2)] = 0 \quad (\text{ii})$$

Therefore  $\lambda = 0$  is an eigenvalue and its direction is obviously  $\mathbf{n} = \mathbf{e}_3$ . The remaining eigenvalues are

$$\begin{cases} T_1 \\ T_2 \end{cases} = \frac{T_{11} + T_{22} \pm \sqrt{(T_{11} - T_{22})^2 + 4T_{12}^2}}{2} \quad (4.6.10)$$

To find the corresponding eigenvectors, we set  $(T_{ij} - \lambda\delta_{ij})n_j = 0$  and obtain for either  $\lambda = T_1$  or  $T_2$ ,

$$(T_{11} - \lambda)n_1 + T_{12}n_2 = 0 \quad (\text{iii})$$

$$T_{12}n_1 + (T_{22} - \lambda)n_2 = 0 \quad (\text{iv})$$

$$-\lambda n_3 = 0 \quad (\text{v})$$

The third equation gives  $n_3 = 0$ . Let the eigenvector  $\mathbf{n} = \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$  (see Fig. 4.7). Then, from the first equation

$$\tan\theta = \frac{n_2}{n_1} = -\frac{T_{11} - \lambda}{T_{12}} \quad (4.6.11)$$

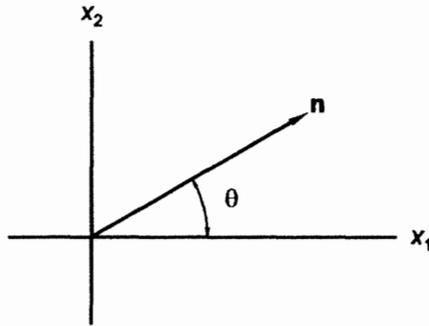


Fig. 4.7

(b) Since the third eigenvalue is always zero, the maximum shearing stress will be the greatest of the value

$$\frac{|T_1|}{2}, \quad \frac{|T_2|}{2}$$

and

$$\left| \frac{T_1 - T_2}{2} \right| = \frac{\sqrt{(T_{11} - T_{22})^2 + 4T_{12}^2}}{2} \quad (4.6.12)$$

### Example 4.6.2

Do the previous example for the following state of stress:  $T_{12} = T_{21} = 1000$  MPa, all other  $T_{ij}$  are zero.

*Solution.* From Eq. (4.6.10), we have

$$\begin{cases} T_1 \\ T_2 \end{cases} = \pm \frac{\sqrt{(4)(1000)^2}}{2} = \pm 1000 \text{ MPa} \quad (i)$$

Corresponding to the maximum normal stress  $T_1 = 1000$  MPa, Eq. (4.6.11) gives

$$\tan \theta_1 = \frac{0 - 1000}{1000} = -1, \text{ i.e., } \theta_1 = 45^\circ$$

and corresponding to the minimum normal stress  $T_2 = -1000$  MPa, (i.e., maximum compressive stress),

$$\tan \theta_2 = \frac{0 - (-1000)}{1000} = 1, \text{ i.e., } \theta_2 = -45^\circ$$

The maximum shearing stress is given by

$$(T_s)_{\max} = \frac{1000 - (-1000)}{2} = 1000 \text{ MPa}$$

which acts on the planes bisecting the planes of maximum and minimum normal stresses, i.e., the  $e_1$ -plane and the  $e_2$ -plane in this problem.

## 4.7 Equations of Motion - Principle of Linear Momentum

In this section, we derive the differential equations of motion for any continuum in motion. The basic postulate is that each particle of the continuum must satisfy Newton's law of motion.

Fig. 4.8 shows the stress vectors that are acting on the six faces of a small rectangular element that is isolated from the continuum in the neighborhood of the position designated by  $x_i$ .

Let  $\mathbf{B} = B_i \mathbf{e}_i$  be the body force (such as weight) per unit mass,  $\rho$  be the mass density at  $x_i$  and  $\mathbf{a}$  the acceleration of a particle currently at the position  $x_i$ ; then Newton's law of motion takes the form, valid in rectangular Cartesian coordinate systems

$$\left[ \left( \frac{\mathbf{t}_{e_1}(x_1 + \Delta x_1, x_2, x_3) - \mathbf{t}_{e_1}(x_1, x_2, x_3)}{\Delta x_1} \right) \right] + \left( \frac{\mathbf{t}_{e_2}(x_1, x_2 + \Delta x_2, x_3) - \mathbf{t}_{e_2}(x_1, x_2, x_3)}{\Delta x_1} \right)$$

$$\left[ \left( \frac{\mathbf{t}_{e_3}(x_1, x_2, x_3 + \Delta x_3) - \mathbf{t}_{e_3}(x_1, x_2, x_3)}{\Delta x_1} \right) \right] \Delta x_1 \Delta x_2 \Delta x_3 + \rho \mathbf{B} \Delta x_1 \Delta x_2 \Delta x_3 = (\rho \mathbf{a}) \Delta x_1 \Delta x_2 \Delta x_3 \tag{i}$$

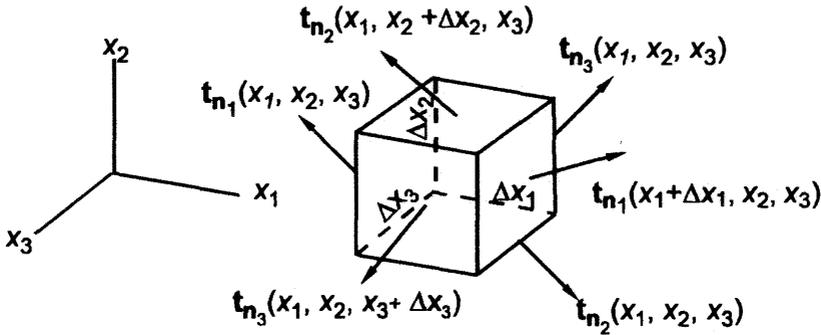


Fig. 4.8

Dividing by  $\Delta x_1, \Delta x_2, \Delta x_3$  and letting  $\Delta x_i \rightarrow 0$ , we have

$$\frac{\partial \mathbf{t}_{e_1}}{\partial x_1} + \frac{\partial \mathbf{t}_{e_2}}{\partial x_2} + \frac{\partial \mathbf{t}_{e_3}}{\partial x_3} + \rho \mathbf{B} = \rho \mathbf{a} \tag{4.7.1}$$

Since  $\mathbf{t}_{e_i} = \mathbf{T} \mathbf{e}_i = T_{ji} \mathbf{e}_j$ , therefore we have (noting that all  $\mathbf{e}_i$  are of fixed directions in Cartesian coordinates)

$$\frac{\partial T_{ij}}{\partial x_j} \mathbf{e}_i + \rho B_i \mathbf{e}_i = \rho a_i \mathbf{e}_i \tag{ii}$$

In invariant form, the above equation can be written

$$\text{div} \mathbf{T} + \rho \mathbf{B} = \rho \mathbf{a} \tag{4.7.2a}$$

and in Cartesian component form

$$\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = \rho a_i \tag{4.7.2b}$$

These are the equations that must be satisfied for any continuum in motion, whether it be solid or fluid. They are called **Cauchy's equations of motion**. If the acceleration vanishes, then Eq. (4.7.2) reduces to the **equilibrium equations**

$$\operatorname{div} \mathbf{T} + \rho \mathbf{B} = 0 \quad (4.7.3a)$$

or,

$$\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = 0 \quad (4.7.3b)$$

#### Example 4.7.1

In the absence of body forces, does the stress distribution

$$\begin{aligned} T_{11} &= x_2^2 + \nu(x_1^2 - x_2^2), & T_{12} &= -2\nu x_1 x_2 \\ T_{22} &= x_1^2 + \nu(x_2^2 - x_1^2), & T_{23} &= T_{13} = 0 \\ T_{33} &= \nu(x_1^2 + x_2^2) \end{aligned} \quad (i)$$

where  $\nu$  is a constant, satisfy the equations of equilibrium?

*Solution.* Writing the first ( $i = 1$ ) equilibrium equation, we have

$$\frac{\partial T_{1j}}{\partial x_j} = \frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = 2\nu x_1 - 2\nu x_1 + 0 = 0 \quad (ii)$$

Similarly, for  $i = 2$ , we have

$$\frac{\partial T_{2j}}{\partial x_j} = \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = -2\nu x_2 + 2\nu x_2 + 0 = 0 \quad (iii)$$

and for  $i = 3$

$$\frac{\partial T_{3j}}{\partial x_j} = \frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = 0 + 0 + 0 = 0 \quad (iv)$$

Therefore, the given stress distribution does satisfy the equilibrium equations

#### Example 4.7.2

Write the equations of motion if the stress components have the form  $T_{ij} = -p\delta_{ij}$  where  $p = p(x_1, x_2, x_3, t)$

*Solution.* Substituting the given stress distribution in the first term on the left-hand side of Eq. (4.7.3b), we obtain

$$\frac{\partial T_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j} \delta_{ij} = -\frac{\partial p}{\partial x_i} \quad (i)$$

Therefore,

$$-\frac{\partial p}{\partial x_i} + \rho B_i = \rho a_i \quad (4.7.4a)$$

or,

$$-\nabla p + \rho \mathbf{B} = \rho \mathbf{a} \quad (4.7.4b)$$

#### 4.8 Equations of Motion in Cylindrical and Spherical Coordinates

In Chapter 2, we presented the components of  $\text{div} \mathbf{T}$  in cylindrical and in spherical coordinates. Using those formulas, we have the following equations of motion: [See also Prob. 4.34]

*Cylindrical coordinates*

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} + \rho B_r = \rho a_r \quad (4.8.1a)$$

$$\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{2T_{\theta r}}{r} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{r\theta} - T_{\theta r}}{r} + \rho B_\theta = \rho a_\theta \quad (4.8.1b)$$

$$\frac{\partial T_{zr}}{\partial r} + \frac{1}{r} \frac{\partial T_{z\theta}}{\partial \theta} + \frac{T_{zr}}{r} + \frac{\partial T_{zz}}{\partial z} + \rho B_\theta = \rho a_z \quad (4.8.1c)$$

We note that for symmetric stress tensor,  $T_{r\theta} - T_{\theta r} = 0$ .

##### Example 4.8.1

The stress field for the Kelvin's problem (an infinite elastic space loaded by a concentrated load at the origin) is given by the following stress components in cylindrical coordinates

$$\begin{aligned} T_{rr} &= \frac{Az}{R^3} - \frac{3r^2z}{R^5}, & T_{\theta\theta} &= \frac{Az}{R^3}, & T_{zz} &= -\left[ \frac{Az}{R^3} + \frac{3z^3}{R^5} \right] \\ T_{rz} &= -\left[ \frac{Ar}{R^3} + \frac{3rz^2}{R^5} \right], & T_{z\theta} &= T_{\theta z} = 0 \end{aligned} \quad (i)$$

where

$$R^2 = r^2 + z^2 \quad (ii)$$

and  $A$  is a constant. Verify that the given state of stress is in equilibrium in the absence of body forces.

*Solution.* From  $R^2 = r^2 + z^2$ , we obtain

$$\frac{\partial R}{\partial r} = \frac{r}{R}, \quad \frac{\partial R}{\partial z} = \frac{z}{R} \quad (\text{iii})$$

Thus,

$$\begin{aligned} \frac{\partial T_{rr}}{\partial r} &= -\frac{3Azr}{R^5} - \frac{6rz}{R^5} + \frac{15r^3z}{R^7} \\ \frac{T_{rr} - T_{\theta\theta}}{r} &= -\frac{3rz}{R^5} \end{aligned} \quad (\text{iv})$$

$$\frac{\partial T_{rz}}{\partial z} = \frac{3Azr}{R^5} - \frac{6rz}{R^5} + \frac{15rz^3}{R^7}$$

Thus, the left hand side of Eq. (4.8.1a) becomes, with  $B_r = 0$

$$-\frac{15rz}{R^5} + \frac{15rz}{R^7}(r^2 + z^2) = -\frac{15rz}{R^5} + \frac{15rz}{R^5} = 0$$

In other words, the  $r$ -equation of equilibrium is satisfied.

Since  $T_{r\theta} = T_{\theta z} = 0$  and  $T_{\theta\theta}$  is independent of  $\theta$ , therefore, with  $B_\theta = a_\theta = 0$ , the second equation of equilibrium is also satisfied.

The third equation of equilibrium Eq. (4.8.1c) with  $B_z = a_z = 0$  can be similarly verified. [see Prob. 4.35].

### Spherical coordinates

$$\frac{1}{r^2} \frac{\partial(r^2 T_{rr})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(T_{r\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{r\phi}}{\partial \phi} - \frac{T_{\theta\theta} + T_{\phi\phi}}{r} + \rho B_r = \rho a_r \quad (4.8.2a)$$

$$\frac{1}{r^3} \frac{\partial(r^3 T_{\theta r})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(T_{\theta\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{\theta\phi}}{\partial \phi} - \frac{T_{\phi\phi} \cot \theta}{r} + \frac{T_{r\theta} - T_{\theta r}}{r} + \rho B_\theta = \rho a_\theta \quad (4.8.2b)$$

$$\frac{1}{r^3} \frac{\partial(r^3 T_{\phi r})}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial(T_{\phi\theta} \sin \theta)}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{T_{\theta\phi} \cot \theta}{r} + \frac{T_{r\phi} - T_{\phi r}}{r} + \rho B_\phi = \rho a_\phi \quad (4.8.2c)$$

Again, we note that for symmetric stress tensor,  $T_{r\theta} - T_{\theta r} = 0$  and  $T_{r\phi} - T_{\phi r} = 0$ .

### 4.9 Boundary Condition for the Stress Tensor

If on the boundary of some body there are applied distributive forces, we call them **surface tractions**. We wish to find the relation between the surface tractions and the stress field that is defined within the body.

If we consider an infinitesimal tetrahedron cut from the boundary of a body with its inclined face coinciding with the boundary surface (Fig. 4.9), then as in Section 4.1, we obtain

$$\mathbf{t} = \mathbf{T}\mathbf{n} \tag{4.9.1}$$

where  $\mathbf{n}$  is the unit outward normal vector to the boundary,  $\mathbf{T}$  is the stress tensor evaluated at the boundary and  $\mathbf{t}$  is the force vector per unit area on the boundary. Equation (4.9.1) is called the stress boundary condition.

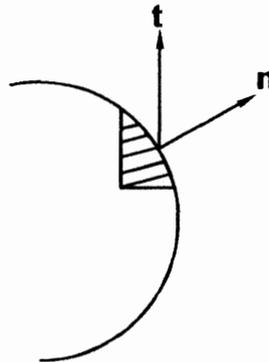


Fig. 4.9

#### Example 4.9.1

Given that the stress field in a thick wall elastic cylinder is

$$\begin{aligned} T_{rr} &= A + \frac{B}{r^2}, \quad T_{\theta\theta} = A - \frac{B}{r^2}, \quad T_{zz} = \text{constant} \\ T_{r\theta} &= T_{rz} = T_{\theta z} = 0 \end{aligned} \tag{i}$$

where  $A$  and  $B$  are constants.

(a) Verify that the given state of stress satisfies the equations of equilibrium in the absence of body forces.

(b) Find the stress vector on a cylindrical surface  $r = a$ .

(c) If the surface traction on the inner surface  $r = r_i$  is a uniform pressure  $p_i$  and the outer surface  $r = r_o$  is free of surface traction, find the constants  $A$  and  $B$ .

*Solution.*

$$(a) \text{ We have, } \frac{\partial T_{rr}}{\partial r} = -\frac{2B}{r^3}, \quad \frac{T_{rr} - T_{\theta\theta}}{r} = \frac{2B}{r^3} \quad (ii)$$

The above results, together with  $T_{r\theta} = T_{rz} = 0$ , give a value of zero for the left hand side of Eq. (4.8.1a) in the absence of a body force component. Thus, the r-equation of equilibrium is satisfied. Also, by inspection, one easily sees that Eq. (4.8.1b) and Eq. (4.8.1c) are satisfied when  $B_\theta = B_z = a_\theta = a_z = 0$ .

(b) The unit normal vector to the cylindrical surface is  $\mathbf{n} = \mathbf{e}_r$ , thus the stress vector is given by

$$\begin{bmatrix} t_r \\ t_\theta \\ t_z \end{bmatrix} = \begin{bmatrix} T_{rr} & T_{r\theta} & T_{rz} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta z} \\ T_{zr} & T_{z\theta} & T_{zz} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{rr} \\ T_{\theta r} \\ T_{zr} \end{bmatrix} \quad (iii)$$

i.e.,

$$\mathbf{t} = T_{rr}\mathbf{e}_r + T_{\theta r}\mathbf{e}_\theta + T_{zr}\mathbf{e}_z = \left(A + \frac{B}{a^2}\right)\mathbf{e}_r \quad (iv)$$

The boundary conditions are

$$\text{At } r = r_i, \quad \mathbf{t} = -p_i\mathbf{e}_r \quad (v)$$

and

$$\text{at } r = r_o, \quad \mathbf{t} = 0 \quad (vi)$$

Thus,

$$A + \frac{B}{r_i^2} = -p_i \quad (vii)$$

$$A + \frac{B}{r_o^2} = 0 \quad (viii)$$

Eqs. (vii) and (viii) give

$$A = \frac{p_i r_i^2}{(r_o^2 - r_i^2)}, \quad B = -\frac{p_i r_i^2 r_o^2}{(r_o^2 - r_i^2)} \quad (ix)$$

thus,

$$T_{rr} = \frac{p r_i^2}{r_o^2 - r_i^2} \left( 1 - \frac{r_o^2}{r^2} \right), \quad T_{\theta\theta} = \frac{p r_i^2}{r_o^2 - r_i^2} \left( 1 + \frac{r_o^2}{r^2} \right) \quad (\text{x})$$

### Example 4.9.2

It is known that the equilibrium stress field in an elastic spherical shell under the action of external and internal pressure in the absence of body forces is of the form

$$\begin{aligned} T_{rr} &= A - \frac{2B}{r^3}, \quad T_{\theta\theta} = T_{\phi\phi} = A + \frac{B}{r^3} \\ T_{\theta r} &= T_{\phi r} = T_{r\theta} = T_{r\phi} = 0 \end{aligned} \quad (\text{i})$$

(a) Verify that the stress field satisfies the equations of equilibrium in the absence of body forces.

(b) Find the stress vector on spherical surface  $r=a$ .

(c) Determine  $A$  and  $B$  if the inner surface of the shell is subjected to a uniform pressure  $p_i$  and the outer surface is free of surface traction.

*Solution.*

(a)

$$r^2 T_{rr} = Ar^2 - \frac{2B}{r}, \quad \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 T_{rr}) = \frac{2A}{r} + \frac{2B}{r^4} \quad (\text{ii})$$

$$\frac{T_{\theta\theta} + T_{\phi\phi}}{r} = \frac{2A}{r} + \frac{2B}{r^4} \quad (\text{iii})$$

Thus, Eq. (4.8.2a) is satisfied when  $B_r = a_r = 0$ , Eqs. (4.8.2b) and (4.8.2c) can be similarly verified. [see Prob.4.38].

(b) The unit normal vector to the spherical surface is  $\mathbf{n} = \mathbf{e}_r$ , thus the stress vector is given by

$$\begin{bmatrix} t_r \\ t_\theta \\ t_\phi \end{bmatrix} = \begin{bmatrix} T_{rr} & T_{r\theta} & T_{r\phi} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta\phi} \\ T_{\phi r} & T_{\phi\theta} & T_{\phi\phi} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{rr} \\ T_{\theta r} \\ T_{\phi r} \end{bmatrix} \quad (\text{iv})$$

i.e.,

$$\mathbf{t} = T_{rr} \mathbf{e}_r + T_{\theta r} \mathbf{e}_\theta + T_{\phi r} \mathbf{e}_\phi = \left( A - \frac{2B}{a^2} \right) \mathbf{e}_r \quad (\text{v})$$

(c)The boundary conditions are

$$\text{At } r = r_i, \quad T_{rr} = -p_i \tag{vi}$$

$$\text{At } r = r_o, \quad T_{rr} = 0 \tag{vii}$$

Thus,

$$A - \frac{2B}{r_i^3} = -p_i \tag{viii}$$

$$A - \frac{2B}{r_o^3} = 0 \tag{ix}$$

From Eqs. (viii) and (ix), we obtain

$$A = \frac{p_i r_i^3}{r_o^3 - r_i^3} \quad B = \frac{p_i r_o^3 r_i^3}{2(r_o^3 - r_i^3)} \tag{x}$$

Thus,

$$T_{rr} = \frac{p_i r_i^3}{r_o^3 - r_i^3} \left(1 - \frac{r_o^3}{r^3}\right) \quad T_{\theta\theta} = \frac{p_i r_i^3}{r_o^3 - r_i^3} \left(1 + \frac{r_o^3}{2r^3}\right) \tag{xi}$$

### 4.10 Piola Kirchhoff Stress Tensors

Let  $dA_o$  be the differential material area with unit normal  $\mathbf{n}_o$  at the reference time  $t_o$  and  $dA$  that at the current time  $t$  of the same material area with unit normal  $\mathbf{n}$ . We may refer to  $dA_o$  as the *undeformed area* and  $dA$  as the *deformed area*. Let  $d\mathbf{f}$  be the force acting on the deformed area  $dA\mathbf{n}$ . In Section 4.1, we defined the Cauchy stress vector  $\mathbf{t}$  and the associated Cauchy stress tensor  $\mathbf{T}$  based on the deformed area  $dA\mathbf{n}$ , that is

$$d\mathbf{f} = \mathbf{t}dA \tag{4.10.1}$$

and

$$\mathbf{t} = \mathbf{T}\mathbf{n} \tag{4.10.2}$$

In this section, we define two other pairs of (pseudo) stress vectors and tensors, based on the *undeformed area*.

#### (A) The First Piola-Kirchhoff Stress Tensor

Let

$$d\mathbf{f} \equiv \mathbf{t}_o dA_o \tag{4.10.3}$$

The stress vector  $\mathbf{t}_o$ , defined by the above equation is a pseudo-stress vector in that, being based on the undeformed area, it does not describe the actual intensity of the force. We note however, that  $\mathbf{t}_o$  has the same direction as the Cauchy stress vector  $\mathbf{t}$ .

The **first Piola-Kirchhoff stress tensor** (also known as the **Lagrangian Stress tensor**) is a linear transformation  $\mathbf{T}_o$  such that

$$\mathbf{t}_o = \mathbf{T}_o \mathbf{n}_o \tag{4.10.4}$$

The relation between the first Piola-Kirchhoff stress tensor and the Cauchy stress tensor can be obtained as follows:

Since

$$d\mathbf{f} = \mathbf{t} dA = \mathbf{t}_o dA_o \tag{i}$$

therefore

$$\mathbf{t}_o = \frac{dA}{dA_o} \mathbf{t} \tag{ii}$$

Using Eqs. (4.10.2) and (4.10.4), Eq. (ii) becomes

$$\mathbf{T}_o \mathbf{n}_o = \left( \frac{dA}{dA_o} \right) \mathbf{T} \mathbf{n} = \frac{\mathbf{T} dA \mathbf{n}}{dA_o} \tag{iii}$$

Using Eq. (3.28.6), i.e.,

$$dA \mathbf{n} = dA_o (\det \mathbf{F}) (\mathbf{F}^{-1})^T \mathbf{n}_o \tag{4.10.5}$$

we have,

$$\mathbf{T}_o \mathbf{n}_o = \mathbf{T} (\det \mathbf{F}) (\mathbf{F}^{-1})^T \mathbf{n}_o \tag{iv}$$

The above equation is to be true for all  $\mathbf{n}_o$ , therefore,

$$\mathbf{T}_o = (\det \mathbf{F}) \mathbf{T} (\mathbf{F}^{-1})^T \tag{4.10.6a}$$

This is the desired relationship.

In Cartesian components, Eq. (4.10.6a) reads

$$(T_o)_{ij} = (\det \mathbf{F}) T_{im} F^{-1}{}^m{}_j \tag{4.10.6b}$$

From Eq. (4.10.6a), we obtain

$$\mathbf{T} = \frac{1}{\det \mathbf{F}} \mathbf{T}_o \mathbf{F}^T \tag{4.10.7a}$$

which in Cartesian components, reads

$$T_{ij} = \frac{1}{\det \mathbf{F}} (T_o)_{im} F_{jm} \tag{4.10.7b}$$

We note that when Cartesian coordinates are used for both the reference and the current configuration,  $F_{im} = \frac{\partial x_i}{\partial X_m}$  and  $F_{im}^{-1} = \frac{\partial X_i}{\partial x_m}$ .

We also note that the first Piola-Kirchhoff stress tensor is in general not symmetric.

**(B) The Second Piola-Kirchhoff Stress Tensor**

Let

$$\tilde{d\mathbf{f}} = \tilde{\mathbf{t}} dA_o \tag{4.10.8a}$$

where

$$d\mathbf{f} = \mathbf{F} \tilde{d\mathbf{f}} \tag{4.10.8b}$$

In Eq. (4.10.8b),  $\tilde{d\mathbf{f}}$  is the (pseudo) differential force which transforms, under the deformation gradient  $\mathbf{F}$  into the (actual) differential force  $d\mathbf{f}$  at the deformed position (one may compare the transformation equation  $d\mathbf{f} = \mathbf{F} \tilde{d\mathbf{f}}$  with  $d\mathbf{x} = \mathbf{F} d\mathbf{X}$ ); thus, the pseudo vector  $\tilde{\mathbf{t}}$  is in general in a different direction than that of the Cauchy stress vector  $\mathbf{t}$ .

The second Piola-Kirchhoff stress tensor is a linear transformation  $\tilde{\mathbf{T}}$  such that

$$\tilde{\mathbf{t}} = \tilde{\mathbf{T}} \mathbf{n}_o \tag{4.10.9}$$

where we recall  $\mathbf{n}_o$  is the normal to the undeformed area. From Eqs. (4.10.8a) (4.10.8b) and (4.10.9), we have

$$d\mathbf{f} = \mathbf{F} \tilde{\mathbf{T}} \mathbf{n}_o dA_o \tag{i}$$

We also have (see Eqs. (4.10.3) and (4.10.4))

$$d\mathbf{f} = \mathbf{t}_o dA_o = \mathbf{T}_o \mathbf{n}_o dA_o \tag{ii}$$

Comparing Eqs. (i) and (ii), we have

$$\tilde{\mathbf{T}} = \mathbf{F}^{-1} \mathbf{T}_o \tag{4.10.10}$$

Equation (4.10.10) gives the relationship between the first Piola-Kirchhoff stress tensor  $\mathbf{T}_o$  and the second Piola-Kirchhoff stress tensor  $\tilde{\mathbf{T}}$ . Now, from Eqs. (4.10.6a) and (4.10.10), one easily obtain the relationship between the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor  $\mathbf{T}$  as

$$\tilde{\mathbf{T}} = (\det \mathbf{F}) \mathbf{F}^{-1} \mathbf{T} (\mathbf{F}^{-1})^T \tag{4.10.11}$$

We note that the second Piola-Kirchhoff stress tensor is always a symmetric tensor if the Cauchy stress tensor is a symmetric one.

## Example 4.10.1

The deformed configuration of a body is described by

$$x_1 = 4X_1, \quad x_2 = -\frac{1}{2}X_2, \quad x_3 = -\frac{1}{2}X_3 \quad (i)$$

If the Cauchy stress tensor for this body is

$$\begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa}$$

- (a) What is the corresponding first Piola-Kirchhoff stress tensor.  
 (b) What is the corresponding second Piola-Kirchhoff stress tensor.

*Solution.* From Eqs. (i), we have

$$[\mathbf{F}] = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \quad \text{and} \quad [\mathbf{F}^{-1}] = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad (ii)$$

$$\det \mathbf{F} = 1 \quad (iii)$$

Thus, from Eq. (4.10.6a), we have, the first Piola-Kirchhoff stress tensor:

$$[\mathbf{T}_o] = (1) \begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa} \quad (v)$$

The second Piola-Kirchhoff stress tensor is, from Eq. (4.10.11)

$$[\tilde{\mathbf{T}}] = [\mathbf{F}]^{-1}[\mathbf{T}_o] = \begin{bmatrix} \frac{25}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa} \quad (vi)$$

## Example 4.10.2

The equilibrium configuration of a body is described by

$$x_1 = \frac{1}{2}X_1, \quad x_2 = -\frac{1}{2}X_3, \quad x_3 = 4X_2 \quad (i)$$

If the Cauchy stress tensor for this body is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 100 \end{bmatrix} \text{ MPa}$$

- (a) What is the corresponding first Piola-Kirchhoff stress tensor.  
 (b) What is the corresponding second Piola-Kirchhoff stress tensor and  
 (c) calculate the pseudo stress vector associated with the first Piola-Kirchhoff stress tensor on the  $\mathbf{e}_3$  - plane in the deformed state.  
 (d) calculate the pseudo-stress vector associated with the second Piola-Kirchhoff stress tensor on the  $\mathbf{e}_3$  - plane in the deformed state.

*Solution.* From Eqs. (i), we have

$$[\mathbf{F}] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & 4 & 0 \end{bmatrix} \quad \text{and} \quad [\mathbf{F}^{-1}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & \frac{1}{4} \\ 0 & -2 & 0 \end{bmatrix} \quad (ii)$$

$$\det \mathbf{F} = 1 \quad (iii)$$

Thus, from Eq. (4.10.6), we have, the first Piola-Kirchhoff stress tensor:

$$[\mathbf{T}_o] = (1) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & \frac{1}{4} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 25 & 0 \end{bmatrix} \text{ MPa} \quad (iv)$$

The second Piola-Kirchhoff stress tensor is, from Eq. (4.10.11)

$$[\tilde{\mathbf{T}}] = [\mathbf{F}]^{-1}[\mathbf{T}_o] = = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{25}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa} \quad (\text{v})$$

(c) For a unit area in the deformed state in the  $\mathbf{e}_3$  direction, its undeformed area  $dA_o \mathbf{n}_o$  is given by Eq. (3.28.6). That is

$$dA_o \mathbf{n}_o = \frac{1}{\det \mathbf{F}} \mathbf{F}^T \mathbf{n} \quad (\text{vi})$$

With  $\det \mathbf{F} = 1$ , and the matrix  $\mathbf{F}$  given above, we obtain

$$dA_o \mathbf{n}_o = 4\mathbf{e}_2 \quad (\text{vii})$$

Thus,  $\mathbf{n}_o = \mathbf{e}_2$  and

$$\mathbf{t}_o = \mathbf{T}_o \mathbf{n}_o$$

gives

$$[\mathbf{t}_o] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 25 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 25 \end{bmatrix}$$

i.e.,  $\mathbf{t}_o = 25\mathbf{e}_3$  MPa. We note that this vector is in the same direction as the Cauchy stress vector, its magnitude is one fourth of that of the Cauchy stress vector, because the undeformed area is 4 times that of the deformed area.

(d) We have, from Eq. (4.10.9)

$$\tilde{\mathbf{t}} = \tilde{\mathbf{T}} \mathbf{n}_o$$

Thus,

$$[\tilde{\mathbf{t}}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{25}{4} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{25}{4} \\ 0 \end{bmatrix}$$

i. e.,  $\tilde{\mathbf{t}} = \frac{25}{4}\mathbf{e}_2$  MPa.

We see that this pseudo stress vector is in a different direction from that of the Cauchy stress vector. (We note that the tensor  $\mathbf{F}$  transforms  $\mathbf{e}_2$  into the direction of  $\mathbf{e}_3$ .)

### 4.11 Equations of Motion Written With Respect to the Reference Configuration.

In this section, we shall show<sup>†</sup> that with respect to the reference configuration, the equations of motion can be written as follows:

$$\frac{\partial(T_o)_{im}}{\partial X_m} + \rho_o B_i = \rho_o a_i \quad (4.11.1)$$

where  $(T_o)_{ij}$  are the Cartesian components of the first Piola-Kirchhoff stress tensor,  $\rho_o$  is the density in the reference configuration,  $X_i$  are the material coordinates and  $B_i$  and  $a_i$  are body force per unit mass and the acceleration components respectively.

From Eq. (4.10.7b), we have

$$T_{ij} = \frac{1}{\det \mathbf{F}} (T_o)_{im} F_{jm} \quad (i)$$

Thus,

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{\partial(T_o)_{im}}{\partial x_j} \frac{F_{jm}}{\det \mathbf{F}} + (T_o)_{im} \frac{\partial}{\partial x_j} \left( \frac{F_{jm}}{\det \mathbf{F}} \right) = \frac{\partial(T_o)_{im}}{\partial x_j} \frac{1}{\det \mathbf{F}} \left( \frac{\partial x_j}{\partial X_m} \right) + (T_o)_{im} \frac{\partial}{\partial x_j} \left( \frac{F_{jm}}{\det \mathbf{F}} \right) \quad (ii)$$

Now,

$$\begin{aligned} \frac{\partial(T_o)_{im}}{\partial x_j} \frac{1}{\det \mathbf{F}} \frac{\partial x_j}{\partial X_m} &= \frac{\partial(T_o)_{im}}{\partial X_n} \frac{\partial X_n}{\partial x_j} \frac{1}{\det \mathbf{F}} \frac{\partial x_j}{\partial X_m} = \frac{\partial(T_o)_{im}}{\partial X_n} \frac{\partial X_n}{\partial X_m} \frac{1}{\det \mathbf{F}} \\ &= \frac{\partial(T_o)_{im}}{\partial X_n} \delta_{nm} \frac{1}{\det \mathbf{F}} = \frac{\partial(T_o)_{im}}{\partial X_m} \frac{1}{\det \mathbf{F}} \end{aligned} \quad (iii)$$

and we can show that the last term of Eq. (ii) is zero as follows:

$$\begin{aligned} \frac{\partial}{\partial x_j} \left( \frac{F_{jm}}{\det \mathbf{F}} \right) &= \frac{1}{(\det \mathbf{F})} \frac{\partial}{\partial x_j} \frac{\partial x_j}{\partial X_m} - F_{jm} \frac{1}{(\det \mathbf{F})^2} \frac{\partial}{\partial x_j} \det \mathbf{F} \\ &= \frac{1}{(\det \mathbf{F})} \frac{\partial^2 \hat{x}_j}{\partial X_n \partial X_m} \frac{\partial X_n}{\partial x_j} - \frac{\partial x_j}{\partial X_m} \frac{1}{(\det \mathbf{F})^2} \left( \frac{\partial}{\partial X_n} \det \mathbf{F} \right) \frac{\partial X_n}{\partial x_j} \end{aligned}$$

<sup>†</sup> In Chapter 7, an alternate shorter proof will be given.

$$= \frac{1}{(\det \mathbf{F})} \frac{\partial^2 \hat{x}_j}{\partial X_n \partial X_m} \frac{\partial X_n}{\partial x_j} - \frac{1}{(\det \mathbf{F})^2} \left( \frac{\partial}{\partial X_n} \det \mathbf{F} \right) \delta_{mn} = \frac{1}{(\det \mathbf{F})} \frac{\partial^2 \hat{x}_j}{\partial X_n \partial X_m} \frac{\partial X_n}{\partial x_j} - \frac{1}{(\det \mathbf{F})^2} \left( \frac{\partial}{\partial X_m} \det \mathbf{F} \right)$$

(iv)

By using the following identity [see Prob. 4.40] for any tensor  $\mathbf{A}(X_1, X_2, X_3)$

$$\frac{\partial}{\partial X_m} \det \mathbf{A} = \det \mathbf{A} (\mathbf{A}^{-1})_{nj} \frac{\partial A_{jn}}{\partial X_m} \quad (4.11.2)$$

we obtain,

$$\frac{\partial}{\partial X_m} \det \mathbf{F} = \det \mathbf{F} \frac{\partial X_n}{\partial x_j} \frac{\partial F_{jn}}{\partial X_m} = \det \mathbf{F} \frac{\partial X_n}{\partial x_j} \frac{\partial^2 \hat{x}_j}{\partial X_n \partial X_m} \quad (v)$$

so that

$$\left( \frac{\partial}{\partial x_j} \right) \left( \frac{F_{jm}}{\det \mathbf{F}} \right) = 0 \quad (vi)$$

Thus,

$$\frac{\partial T_{ij}}{\partial x_j} = \frac{\partial (T_o)_{im}}{\partial X_m} \frac{1}{\det \mathbf{F}} \quad (vii)$$

Substituting Eq. (vii) in the Cauchy's Equation of motion [ Eq.(4.7.2b)], we get

$$\frac{\partial (T_o)_{im}}{\partial X_m} + \rho (\det \mathbf{F}) B_i = \rho (\det \mathbf{F}) a_i \quad (viii)$$

Since  $dV = (\det \mathbf{F}) dV_o$  [See Eq. (3.29.3)], therefore,

$$\rho \det \mathbf{F} = \rho_o \quad (4.11.3)$$

where  $\rho_o$  is the initial density. Thus, we have, in terms of the first Piola-Kirchhoff stress tensor and with respect to the material coordinates, the equations of motion take the following form

$$\frac{\partial (T_o)_{im}}{\partial X_m} + \rho_o B_i = \rho_o a_i \quad (4.11.4)$$

whereas in terms of the Cauchy stress tensor and with respect to the spatial coordinates, the equations of motion take the form

$$\frac{\partial T_{im}}{\partial x_m} + \rho B_i = \rho a_i \quad (4.11.5)$$

In invariant notation, Eq. (4.11.4) reads

$$\text{Div} \mathbf{T}_0 + \rho_0 \mathbf{B} = \rho_0 \mathbf{a} \tag{4.11.6}$$

where Div denotes the divergence with respect to the material coordinates  $\mathbf{X}$  and Eq. (4.11.5) reads

$$\text{div} \mathbf{T} + \rho \mathbf{B} = \rho \mathbf{a} \tag{4.11.7}$$

where div denotes the divergence with respect to the spatial coordinates  $\mathbf{x}$ .

### 4.12 Stress Power

Referring to the infinitesimal rectangular parallelepiped of Fig. 4.8 which is repeated here for convenience, let us compute the rate at which work is done by the stress vectors and body force on the particle as it moves and deforms.

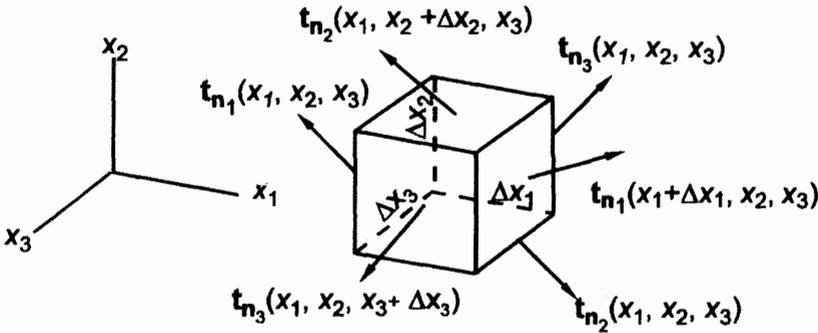


Fig. 4.8 (repeated)

The rate at which work is done by the stress vectors  $\mathbf{t}_{-\mathbf{e}_1}$  and  $\mathbf{t}_{\mathbf{e}_1}$  on the pair of faces having  $-\mathbf{e}_1$  and  $\mathbf{e}_1$  as their respective normal is:

$$[(\mathbf{t}_{\mathbf{e}_1} \cdot \mathbf{v})_{x_1+\Delta x_1, x_2, x_3} - (\mathbf{t}_{-\mathbf{e}_1} \cdot \mathbf{v})_{x_1, x_2, x_3}] \Delta x_2 \Delta x_3 = \frac{\partial}{\partial x_1} (\mathbf{t}_{\mathbf{e}_1} \cdot \mathbf{v}) \Delta x_1 \Delta x_2 \Delta x_3 = \left[ \frac{\partial}{\partial x_1} (v_i T_{i1}) \right] dV \tag{i}$$

where we have used the fact that  $\mathbf{t}_{\mathbf{e}_1} \cdot \mathbf{v} = \mathbf{T} \mathbf{e}_1 \cdot v_i \mathbf{e}_i = v_i \mathbf{e}_i \cdot \mathbf{T} \mathbf{e}_1 = v_i T_{i1}$ , and  $dV = \Delta x_1 \Delta x_2 \Delta x_3$  denotes the differential volume. Similarly, the rate at which work is done by the stress vectors on the other two pairs of faces are:  $\left[ \frac{\partial}{\partial x_2} (v_i T_{i2}) \right] dV$  and  $\left[ \frac{\partial}{\partial x_3} (v_i T_{i3}) \right] dV$ .

Including the rate of work done by the body force ( $\rho \mathbf{B} dV \cdot \mathbf{v} = \rho B_i v_i dV$ ) the total rate of work done on the particle is

$$P = \frac{\partial}{\partial x_j} (v_i T_{ij}) dV + \rho B_i v_i dV \quad (4.12.1)$$

Since

$$\frac{\partial}{\partial x_j} (v_i T_{ij}) = v_i \frac{\partial T_{ij}}{\partial x_j} + T_{ij} \frac{\partial v_i}{\partial x_j} \quad (ii)$$

Eq. (4.12.1) takes the form

$$P = v_i \left[ \frac{\partial T_{ij}}{\partial x_j} + \rho B_i \right] dV + T_{ij} \frac{\partial v_i}{\partial x_j} dV \quad (4.12.2)$$

However, we have from the equations of motion

$$\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = \rho \frac{Dv_i}{Dt} \quad (4.12.3)$$

therefore, we have

$$P = v_i \frac{Dv_i}{Dt} \rho dV + T_{ij} \frac{\partial v_i}{\partial x_j} dV \quad (4.12.4)$$

The first term in the right-hand side of Eq. (4.12.4) represents the rate of change of kinetic energy of the particle as is seen from the following:

$$\frac{D}{Dt}(KE) = \frac{D}{Dt} \left[ \frac{1}{2} (\rho dV) v_i v_i \right] = (\rho dV) v_i \frac{Dv_i}{Dt} + \frac{1}{2} v_i v_i \frac{D}{Dt} (\rho dV) = (\rho dV) v_i \frac{Dv_i}{Dt} \quad (4.12.5)$$

where we note that  $\frac{D}{Dt}(\rho dV) = 0$  on account of the mass conservation principle. Thus, from Eq. (4.12.4)

$$P = \frac{D}{Dt}(KE) + P_s dV \quad (4.12.6)$$

where

$$P_s = T_{ij} \frac{\partial v_i}{\partial x_j} = \text{tr}(\mathbf{T}^T \nabla_{\mathbf{x}} \mathbf{v}) \quad (4.12.7)$$

is known as the **stress power**. It represents the rate at which work is done to change the volume and shape of a particle of unit volume.

For a symmetric stress tensor  $T_{ij} = T_{ji}$  so that  $T_{ij} \frac{\partial v_i}{\partial x_j} = T_{ji} \frac{\partial v_i}{\partial x_j} = T_{ij} \frac{\partial v_j}{\partial x_i}$ . Thus,

$$P_s = \frac{1}{2} T_{ij} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) = T_{ij} D_{ij} = T_{ji} D_{ij} \quad (4.12.8a)$$

where  $D_{ij}$  are the components of the rate of deformation tensor defined in Section 3.13.

Equation (4.12.8a) can be written in the invariant form

$$P_s = \text{tr}(\mathbf{T}\mathbf{D}) \quad (4.12.8b)$$

#### Example 4.12.1

Show that the stress power can be expressed in terms of the first Piola-Kirchhoff stress tensor  $\mathbf{T}_o$  and the deformation gradient  $\mathbf{F}$  as the following

$$P_s = \frac{\rho}{\rho_o} \text{tr}(\mathbf{T}_o^T \frac{D\mathbf{F}}{Dt}) = \frac{\rho}{\rho_o} \left[ (T_o)_{ij} \frac{DF_{ij}}{Dt} \right] \quad (4.12.9)$$

*Solution.* In Sect. 3.12, we obtained [see Eq. (3.12.4)],

$$\frac{D}{Dt} d\mathbf{x} = (\nabla_{\mathbf{x}} \mathbf{v}) d\mathbf{x}$$

Since  $d\mathbf{x} = \mathbf{F} d\mathbf{X}$  [see Eq. (3.7.2)], therefore

$$\frac{D}{Dt} (\mathbf{F} d\mathbf{X}) = (\nabla_{\mathbf{x}} \mathbf{v}) \mathbf{F} d\mathbf{X} \quad (i)$$

Equation (i) is to be true for all  $d\mathbf{X}$ , thus

$$\frac{D\mathbf{F}}{Dt} = (\nabla_{\mathbf{x}} \mathbf{v}) \mathbf{F} \quad (4.12.10a)$$

or,

$$(\nabla_{\mathbf{x}} \mathbf{v}) = \frac{D\mathbf{F}}{Dt} \mathbf{F}^{-1} \quad (4.12.10b)$$

Using Eqs. (4.12.7) and (4.12.10b), the stress power can be written

$$P_s = \text{tr}(\mathbf{T}^T \nabla_{\mathbf{x}} \mathbf{v}) = \text{tr}(\mathbf{T}^T \frac{D\mathbf{F}}{Dt} \mathbf{F}^{-1}) \quad (ii)$$

Since [see Eq. (4.10.7)]

$$\mathbf{T} = \frac{1}{\det \mathbf{F}} \mathbf{T}_o \mathbf{F}^T$$

therefore,

$$P_s = \frac{1}{\det \mathbf{F}} \text{tr}(\mathbf{F} \mathbf{T}_o^T \frac{D\mathbf{F}}{Dt} \mathbf{F}^{-1}) \quad (iii)$$

Using the identity  $\text{tr}(\mathbf{ABCD}) = \text{tr}(\mathbf{BCDA}) = \text{tr}(\mathbf{CDAB})$  and the relation  $\det \mathbf{F} = \frac{\rho_0}{\rho}$ , Eq. (iii) becomes

$$P_s = \frac{\rho}{\rho_0} \text{tr}(\mathbf{T}_o^T \frac{D\mathbf{F}}{Dt}).$$

Example 4.12.2

Show that the stress power can be expressed in terms of the second Piola-Kirchhoff stress tensor  $\tilde{\mathbf{T}}$  and the Lagrange strain tensor  $\mathbf{E}^*$  as follows

$$P_s = \frac{\rho}{\rho_0} \text{tr}(\tilde{\mathbf{T}} \frac{D\mathbf{E}^*}{Dt}) = \frac{\rho}{\rho_0} \tilde{T}_{ij} \frac{DE_{ij}^*}{Dt} \tag{4.12.11}$$

*Solution.* From Eq. (3.13.6),

$$\frac{D}{Dt} ds^2 = 2d\mathbf{x} \cdot \mathbf{D} d\mathbf{x}$$

and Eq. (3.7.2)

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}$$

we obtain

$$\frac{D}{Dt} ds^2 = 2\mathbf{F} d\mathbf{X} \cdot \mathbf{D} \mathbf{F} d\mathbf{X} = 2d\mathbf{X} \cdot \mathbf{F}^T (\mathbf{D}\mathbf{F}) d\mathbf{X} \tag{i}$$

From Eq. (3.24.2), we obtain

$$ds^2 = dS^2 + 2d\mathbf{X} \cdot \mathbf{E}^* d\mathbf{X}$$

so that

$$\frac{D}{Dt} ds^2 = 2d\mathbf{X} \cdot \frac{D\mathbf{E}^*}{Dt} d\mathbf{X} \tag{4.12.12}$$

Compare Eq. (i) with Eq. (4.12.12), we obtain

$$\frac{D\mathbf{E}^*}{Dt} = \mathbf{F}^T \mathbf{D} \mathbf{F} \tag{4.12.13}$$

Using Eq. (4.10.11), that is

$$\mathbf{T} = \frac{1}{\det \mathbf{F}} \mathbf{F} \tilde{\mathbf{T}} \mathbf{F}^T$$

we have for the stress power

$$P_s = \text{tr}(\mathbf{T}\mathbf{D}) = \frac{1}{\det \mathbf{F}} \text{tr}(\mathbf{F} \tilde{\mathbf{T}} \mathbf{F}^T \mathbf{D}) = \frac{1}{\det \mathbf{F}} \text{tr}(\tilde{\mathbf{T}} \mathbf{F}^T \mathbf{D}\mathbf{F}) \tag{ii}$$

Making use of Eq. (4.12.13), Eq. (ii) becomes

$$P_s = \frac{1}{\det \mathbf{F}} \operatorname{tr} \left( \tilde{\mathbf{T}} \frac{D\mathbf{E}}{Dt} \right)^* = \frac{\rho}{\rho_0} \operatorname{tr} \left( \tilde{\mathbf{T}} \frac{D\mathbf{E}}{Dt} \right)^*$$

We note that  $(\mathbf{T}, \mathbf{D})$ ,  $(\mathbf{T}_0, \frac{D\mathbf{F}}{Dt})$  and  $(\tilde{\mathbf{T}}, \frac{D\mathbf{E}}{Dt})^*$  are sometimes known as conjugate pairs.

### 4.13 Rate of Heat Flow Into an Element by Conduction

Let  $\mathbf{q}$  be a vector whose magnitude gives the rate of heat flow across a unit area by conduction and whose direction gives the direction of heat flow, then the net heat flow by conduction  $Q_c$  into a differential element can be computed as follows:

Referring to the infinitesimal rectangular parallelepiped of Fig. 4.10, the rate at which heat flows into the element across the face with  $\mathbf{e}_1$  as its outward normal is  $[(-\mathbf{q} \cdot \mathbf{e}_1)_{x_1+dx_1, x_2, x_3} dx_2 dx_3]$  and that across the face with  $-\mathbf{e}_1$  as its outward normal is  $[(\mathbf{q} \cdot \mathbf{e}_1)_{x_1, x_2, x_3} dx_2 dx_3]$ . Thus, the net rate of heat inflow across the pair of faces is given by

$$-[q_1(x_1+dx_1, x_2, x_3) - q_1(x_1, x_2, x_3)] dx_2 dx_3 = -\left(\frac{\partial q_1}{\partial x_1}\right) dx_1 dx_2 dx_3. \quad (i)$$

where  $q_i \equiv \mathbf{q} \cdot \mathbf{e}_i$ . Similarly, the net rate of heat inflow across the other two pairs of faces is

$$-\left(\frac{\partial q_2}{\partial x_2}\right) dx_1 dx_2 dx_3 \quad \text{and} \quad -\left(\frac{\partial q_3}{\partial x_3}\right) dx_1 dx_2 dx_3$$

so that the total net rate of heat inflow by conduction is

$$Q_c = -\left(\frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2} + \frac{\partial q_3}{\partial x_3}\right) dV = -(\operatorname{div} \mathbf{q}) dV \quad (4.13.1)$$

#### Example 4.13.1

Using the Fourier heat conduction law  $\mathbf{q} = -\kappa \nabla \Theta$ , where  $\nabla \Theta$  is the temperature gradient and  $\kappa$  is the coefficient of thermal conductivity, find the equation governing the steady-state distribution of temperature.

*Solution.* From Eq. (4.13.1), we have, per unit volume, the net rate of heat inflow is given by

$$-\left[\frac{\partial}{\partial x_1} \left(\kappa \frac{\partial \Theta}{\partial x_1}\right) + \frac{\partial}{\partial x_2} \left(\kappa \frac{\partial \Theta}{\partial x_2}\right) + \frac{\partial}{\partial x_3} \left(\kappa \frac{\partial \Theta}{\partial x_3}\right)\right]. \quad (i)$$

Now, if the boundaries of the body are kept at fixed temperature, then when the steady-state is reached, the net rate of heat flow into any element in the body must be zero. Thus, the desired equation is

$$\frac{\partial}{\partial x_1}(\kappa \frac{\partial \Theta}{\partial x_1}) + \frac{\partial}{\partial x_2}(\kappa \frac{\partial \Theta}{\partial x_2}) + \frac{\partial}{\partial x_3}(\kappa \frac{\partial \Theta}{\partial x_3}) = 0 \quad (4.13.2)$$

For constant  $\kappa$ , this reduces to the Laplace equation

$$\frac{\partial^2 \Theta}{\partial x_1^2} + \frac{\partial^2 \Theta}{\partial x_2^2} + \frac{\partial^2 \Theta}{\partial x_3^2} = 0 \quad (4.13.3)$$

#### 4.14 Energy Equation

Consider a particle with a differential volume  $dV$  at the position  $\mathbf{x}$  at time  $t$ . Let  $U$  denote its internal energy,  $KE$  the kinetic energy,  $Q_c$  the net rate of heat flow by conduction into the particle from its surroundings,  $Q_s$  the rate of heat input due to external sources (such as radiation) and  $P$  the rate at which work is done on the particle by body forces and surface forces (i.e.,  $P$  is the mechanical power input). Then, in the absence of other forms of energy input, the fundamental postulate of conservation of energy states that

$$\frac{D}{Dt}(U+KE) = P + Q_c + Q_s \quad (4.14.1)$$

Now, using Eq. (4.12.6) and Eq. (4.13.1), we have

$$P = \frac{D}{Dt}(KE) + T_{ij} \frac{\partial v_i}{\partial x_j} dV \quad (i)$$

$$Q_c = -\frac{\partial q_i}{\partial x_i} dV \quad (ii)$$

thus, Eq. (4.14.1) becomes

$$\frac{DU}{Dt} = T_{ij} \frac{\partial v_i}{\partial x_j} dV - \frac{\partial q_i}{\partial x_i} dV + Q_s \quad (iii)$$

If we let  $u$  be the internal energy per unit mass, then

$$\frac{DU}{Dt} = \frac{D(u \rho dV)}{Dt} = \rho dV \frac{Du}{Dt} \quad (iv)$$

In arriving at the above equation, we have used the conservation of mass principle

$$(D/Dt)(\rho dV) = 0 \quad (v)$$

Thus, the energy equation (4.14.1) becomes

$$\rho \frac{Du}{Dt} = T_{ij} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \rho q_s \quad (4.14.2a)$$

where  $q_s$  is the rate of heat input (known simply as the heat supply) per unit mass by external sources. In invariant notation, Eq. (4.14.2a) is written

$$\rho \frac{Du}{Dt} = \text{tr}(\mathbf{TD}) - \text{div} \mathbf{q} + \rho q_s \quad (4.14.2b)$$

#### 4.15 Entropy Inequality

Let  $\eta(\mathbf{x}, t)$  denote the entropy per unit mass for the continuum. Then the entropy in a volume  $dV$  of the material is  $\rho \eta dV$ , where  $\rho$  is density. The rate of increase of entropy following the volume of material as it is moving is

$$\frac{D}{Dt}(\rho \eta dV) \quad (i)$$

which is equal to  $\rho dV \frac{D\eta}{Dt}$ , because  $\frac{D}{Dt}(\rho dV) = 0$  in accordance with the conservation of mass principle. Thus, per unit volume, the rate of increase of entropy is given by  $\rho \frac{D\eta}{Dt}$

The entropy inequality law states that

$$\rho \frac{D\eta}{Dt} \geq \text{div} \frac{\mathbf{q}}{\Theta} + \rho \frac{q_s}{\Theta} \quad (4.15.1)$$

where  $\Theta$  is the absolute temperature,  $\mathbf{q}$  is the heat flux vector and  $q_s$  is the heat supply.

### Problems

4.1. The state of stress at certain point of a body is given by

$$[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{bmatrix} \mathbf{e}_i$$
 MPa

On each of the coordinate planes (normals  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ ) (a) what is the normal stress and (b) what is the total shearing stress.

4.2. The state of stress at a certain point of a body is given by

$$[\mathbf{T}] = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix} \text{ MPa}$$

(a) Find the stress vector at a point on the plane whose normal is in the direction  $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$ .

(b) Determine the magnitude of the normal and shearing stresses on this plane.

4.3. Do the previous problem for a plane passing through the point and parallel to the plane  $x_1 - 2x_2 + 3x_3 = 4$ .

4.4. The stress distribution in a certain body is given by

$$[\mathbf{T}] = \begin{bmatrix} 0 & 100x_1 & -100x_2 \\ 100x_1 & 0 & 0 \\ -100x_2 & 0 & 0 \end{bmatrix}.$$

Find the stress vector acting on a plane which passes through the point  $(1/2, \sqrt{3}/2, 3)$  and is tangent to the circular cylindrical surface  $x_1^2 + x_2^2 = 1$  at that point.

4.5. Given  $T_{11} = 1$  Mpa,  $T_{22} = -1$  Mpa and all other  $T_{ij} = 0$  at a point in a continuum.

(a) Show that the only plane on which the stress vector is zero is the plane with normal in the  $\mathbf{e}_3$ -direction.

(b) Give three planes on which there is no normal stress acting.

4.6. For the following state of stress

$$[\mathbf{T}] = \begin{bmatrix} 10 & 50 & -50 \\ 50 & 0 & 0 \\ -50 & 0 & 0 \end{bmatrix} \text{ MPa}$$

find  $T_{11}'$  and  $T_{13}'$  where  $\mathbf{e}_1'$  is in the direction of  $\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$  and  $\mathbf{e}_2'$  is in the direction of  $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$ .

4.7. Consider the following stress distribution

$$[\mathbf{T}] = \begin{bmatrix} \alpha x_2 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where  $\alpha$  and  $\beta$  are constants.

(a) Determine and sketch the distribution of the stress vector acting on the square in the  $x_1 = 0$  plane with vertices located at  $(0,1,1)$ ,  $(0,-1,1)$ ,  $(0,1,-1)$ ,  $(0,-1,-1)$ .

(b) Find the total resultant force and moment about the origin of the stress vectors acting on the square of part (a).

4.8. Do the previous problem if the stress distribution is given by

$$T_{11} = \alpha x_2^2$$

and all other  $T_{ij} = 0$ .

4.9. Do problem 4.7 for the stress distribution

$$T_{11} = \alpha, \quad T_{21} = T_{12} = \alpha x_3$$

and all other  $T_{ij} = 0$ .

4.10. Consider the following stress distribution for a circular cylindrical bar

$$[\mathbf{T}] = \begin{bmatrix} 0 & -\alpha x_3 & +\alpha x_2 \\ -\alpha x_3 & 0 & 0 \\ \alpha x_2 & 0 & 0 \end{bmatrix}$$

(a) What is the distribution of the stress vector on the surfaces defined by  $x_2^2 + x_3^2 = 4$ ,  $x_1 = 0$  and  $x_1 = l$ ?

(b) Find the total resultant force and moment on the end face  $x_1 = l$ .

4.11. An elliptical bar with lateral surface defined by  $x_2^2 + 2x_3^2 = 1$  has the following stress distribution

$$[\mathbf{T}] = \begin{bmatrix} 0 & -2x_3 & x_2 \\ -2x_3 & 0 & 0 \\ x_2 & 0 & 1 \end{bmatrix} \text{ MPa}$$

(a) Show that the stress vector any point  $(x_1, x_2, x_3)$  on the lateral surface is zero.

(b) Find the resultant force and resultant moment about the origin O of the stress vector on the left end face  $x_1 = 0$ .

$$\text{Note: } \int x_2^2 dA = \frac{\pi}{4\sqrt{2}} \quad \text{and} \quad \int x_3^2 dA = \frac{\pi}{8\sqrt{2}}.$$

4.12. For any stress state  $\mathbf{T}$ , we define the deviatoric stress  $\mathbf{S}$  to be

$$\mathbf{S} = \mathbf{T} - \left( \frac{T_{kk}}{3} \right) \mathbf{I}$$

where  $T_{kk}$  is the first invariant of the stress tensor  $\mathbf{T}$ .

- (a) Show that the first invariant of the deviatoric stress vanishes.  
 (b) Given the stress tensor

$$[\mathbf{T}] = 100 \begin{bmatrix} 6 & 5 & -2 \\ 5 & 3 & 4 \\ -2 & 4 & 9 \end{bmatrix} \text{ kPa}$$

evaluate  $\mathbf{S}$

- (c) Show that the principal direction of the stress and the deviatoric stress coincide.  
 (d) Find a relation between the principal values of the stress and the deviatoric stress.

**4.13.** An octahedral stress plane is defined to make equal angles with each of the principal axes of stress.

- (a) How many independent octahedral planes are there at each point?  
 (b) Show that the normal stress on an octahedral plane is given by one-third the first stress invariant.  
 (c) Show that the shearing stress on the octahedral plane is given by

$$T_s = \frac{1}{3} [(T_1 - T_2)^2 + (T_2 - T_3)^2 + (T_1 - T_3)^2]^{1/2},$$

where  $T_1, T_2, T_3$  are the principal values of the stress tensor.

**4.14.** (a) Let  $\mathbf{m}$  and  $\mathbf{n}$  be two unit vectors that define two planes  $M$  and  $N$  that pass through a point  $P$ . For an arbitrary state of stress defined at the point  $P$ , show that the component of the stress vector  $\mathbf{t}_{\mathbf{m}}$  in the  $\mathbf{n}$ -direction is equal to the component of the stress vector  $\mathbf{t}_{\mathbf{n}}$  in the  $\mathbf{m}$ -direction.

- (b) If  $\mathbf{m} = \mathbf{e}_1$  and  $\mathbf{n} = \mathbf{e}_2$ , what does the result of part (a) reduce to?

**4.15.** Let  $\mathbf{m}$  be a unit vector that defines a plane  $M$  passing through a point  $P$ . Show that the stress vector on any plane that contains the stress traction  $\mathbf{t}_{\mathbf{m}}$  lies in the  $M$ -plane.

**4.16.** Let  $\mathbf{t}_{\mathbf{m}}$  and  $\mathbf{t}_{\mathbf{n}}$  be stress vectors on planes defined by the unit vectors  $\mathbf{m}$  and  $\mathbf{n}$  and pass through the point  $P$ . Show that if  $\mathbf{k}$  is a unit vector that determines a plane that contains  $\mathbf{t}_{\mathbf{m}}$  and  $\mathbf{t}_{\mathbf{n}}$ , then  $\mathbf{t}_{\mathbf{m}}$  is perpendicular to  $\mathbf{m}$  and  $\mathbf{n}$ .

**4.17.** True or false

- (i) Symmetry of stress tensor is not valid if the body has an angular acceleration.  
 (ii) On the plane of maximum normal stress, the shearing stress is always zero.

**4.18.** True or false

- (i) On the plane of maximum shearing stress, the normal stress is zero.  
 (ii) A plane with its normal in the direction of  $\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3$  has a stress vector  $\mathbf{t} = 50\mathbf{e}_1 + 100\mathbf{e}_2 - 100\mathbf{e}_3$  MPa. It is a principal plane.

4.19. Why can the following two matrices not represent the same stress tensor?

$$\begin{bmatrix} 100 & 200 & 40 \\ 200 & 0 & -30 \\ 40 & -30 & -50 \end{bmatrix} \text{ MPa} \quad \begin{bmatrix} 40 & 100 & 60 \\ 100 & 100 & 0 \\ 60 & 0 & 20 \end{bmatrix} \text{ MPa.}$$

4.20. Given a

$$[\mathbf{T}] = \begin{bmatrix} 0 & 100 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ Mpa}$$

- (a) Find the magnitude of shearing stress on the plane whose normal is in the direction of  $\mathbf{e}_1 + \mathbf{e}_2$ .  
 (b) Find the maximum and minimum normal stresses and the planes on which they act.  
 (c) Find the maximum shearing stress and the plane on which it acts.

4.21. The stress components at a point are given by

$$T_{11} = 100 \text{ MPa}, T_{22} = 300 \text{ MPa}, T_{33} = 400 \text{ MPa}, T_{12} = T_{13} = T_{23} = 0$$

- (a) Find the maximum shearing stress and the planes on which it acts.  
 (b) Find the normal stress on these planes.  
 (c) Are there any plane/planes on which the normal stress is 500 MPa?

4.22. The principal values of a stress tensor  $\mathbf{T}$  are:  $T_1 = 10$  MPa,  $T_2 = -10$  MPa and  $T_3 = 30$  MPa. If the matrix of the stress is given by

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & T_{33} \end{bmatrix} \times 10 \text{ Mpa}$$

find the value of  $T_{11}$  and  $T_{33}$ .

4.23. If the state of stress at a point is

$$[\mathbf{T}] = \begin{bmatrix} 300 & 0 & 0 \\ 0 & -200 & 0 \\ 0 & 0 & 400 \end{bmatrix} \text{ kPa}$$

find (a) the magnitude of the shearing stress on the plane whose normal is in the direction of  $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$ , and (b) the maximum shearing stress.

4.24. Given

$$[\mathbf{T}] = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ MPa.}$$

- (a) Find the stress vector on the plane whose normal is in the direction  $\mathbf{e}_1 + \mathbf{e}_2$ .
- (b) Find the normal stress on the same plane.
- (c) Find the magnitude of the shearing stress on the same plane.
- (d) Find the maximum shearing stress and the planes on which this maximum shearing stress acts.

**4.25.** The stress state in which the only non-vanishing stress components are a single pair of shearing stresses is called simple shear. Take  $T_{12} = T_{21} = \tau$  and all other  $T_{ij} = 0$ .

- (a) Find the principal values and principal directions of this stress state.
- (b) Find the maximum shearing stress and the plane on which it acts.

**4.26.** The stress state in which only the three normal stress components do not vanish is called tri-axial stress state. Take  $T_{11} = \sigma_1$ ,  $T_{22} = \sigma_2$ ,  $T_{33} = \sigma_3$  with  $\sigma_1 > \sigma_2 > \sigma_3$  and all other  $T_{ij} = 0$ . Find the maximum shearing stress and the plane on which it acts.

**4.27.** Show that the symmetry of the stress tensor is not valid if there are body moments per unit volume, as in the case of a polarized anisotropic dielectric solid.

**4.28.** Given the following stress distribution

$$[\mathbf{T}] = \begin{bmatrix} x_1 + x_2 & T_{12}(x_1, x_2) & 0 \\ T_{12}(x_1, x_2) & x_1 - 2x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix}$$

find  $T_{12}$  so that the stress distribution is in equilibrium with zero body force and so that the stress vector on  $x_1 = 1$  is given by  $\mathbf{t} = (1 + x_2)\mathbf{e}_1 + (5 - x_2)\mathbf{e}_2$ .

**4.29.** Suppose the body force vector is  $\mathbf{B} = -g\mathbf{e}_3$ , where  $g$  is a constant. Consider the following stress tensor

$$[\mathbf{T}] = \alpha \begin{bmatrix} x_2 & -x_3 & 0 \\ -x_3 & 0 & -x_2 \\ 0 & -x_2 & T_{33} \end{bmatrix}$$

and find an expression for  $T_{33}$  such that  $\mathbf{T}$  satisfies the equations of equilibrium.

**4.30.** In the absence of body forces, the equilibrium stress distribution for a certain body is

$$[\mathbf{T}] = \begin{bmatrix} Ax_2 & x_1 & 0 \\ x_1 & Bx_1 + Cx_2 & 0 \\ 0 & 0 & \frac{1}{2}(T_{11} + T_{22}) \end{bmatrix}$$

(a) Find the value of  $C$ .

(b) The boundary plane  $x_1 - x_2 = 0$  for the body is free of stress. Determine the values of  $A$  and  $B$ .

**4.31.** In the absence of body forces, do the stress components

$$T_{11} = \alpha [x_2^2 + \nu (x_1^2 - x_2^2)], \quad T_{22} = \alpha [x_1^2 + \nu (x_2^2 - x_1^2)],$$

$$T_{33} = \alpha \nu (x_1^2 + x_2^2), \quad T_{12} = -2\alpha \nu x_1 x_2, \quad T_{13} = T_{23} = 0.$$

satisfy the equations of equilibrium?

**4.32.** Repeat the previous problem for the stress distribution

$$[\mathbf{T}] = \alpha \begin{bmatrix} x_1 + x_2 & 2x_1 - x_2 & 0 \\ 2x_1 - x_2 & x_1 - 3x_2 & 0 \\ 0 & 0 & x_1 \end{bmatrix}$$

**4.33.** Suppose that the stress distribution has the form (called plane stress)

$$[\mathbf{T}] = \alpha \begin{bmatrix} T_{11}(x_1, x_2) & T_{12}(x_1, x_2) & 0 \\ T_{12}(x_1, x_2) & T_{22}(x_1, x_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(a) What are the equilibrium equations in this special case?

(b) If we introduce a function  $\varphi(x_1, x_2)$  such that

$$T_{11} = \frac{\partial^2 \varphi}{\partial x_2^2}, \quad T_{22} = \frac{\partial^2 \varphi}{\partial x_1^2}, \quad T_{12} = -\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}$$

will this stress distribution be in equilibrium with zero body force?

**4.34.** In cylindrical coordinates  $(r, \theta, z)$ , consider a differential volume of material bounded by the three pairs of faces  $r = r_o$ ,  $r = r_o + dr$ ,  $\theta = \theta_o$ ,  $\theta = \theta_o + d\theta$  and  $z = z_o$ ,  $z = z_o + dz$ . Derive the equations of motion in cylindrical coordinates and compare the equations with those given in Section 4.8.

**4.35.** Verify that the stress field of Example 4.8.1 satisfies the  $z$ -equation of equilibrium in the absence of body forces.

**4.36.** Given the following stress field in cylindrical coordinates

$$T_{rr} = -\frac{3Pr^2z}{2\pi R^5}, \quad T_{\theta\theta} = 0, \quad T_{zz} = -\frac{3Pz^3}{2\pi R^5}$$

$$T_{rz} = -\frac{3Prz^2}{2\pi R^5}, \quad T_{r\theta} = T_{z\theta} = 0, \quad R^2 = r^2 + z^2$$

Verify that the state of stress satisfies the equations of equilibrium in the absence of body force.

4.37. For the stress field given in Example 4.9.1, determine the constants  $A$  and  $B$  if the inner cylindrical wall is subjected to a uniform pressure  $p_o$  and the outer cylindrical wall is subjected to a uniform pressure  $p_o$ .

4.38. Verify that Eq. (4.8.2b) and (4.8.2c) are satisfied by the stress field given in Example 4.9.2.

4.39. In Example 4.9.2, if the spherical shell is subjected to an inner pressure of  $p_i$  and an outer pressure of  $p_o$ , determine the constant  $A$  and  $B$

4.40. Prove that for any tensor  $A(X_1, X_2, X_3)$

$$\frac{\partial}{\partial X_m} \det A = \det A (A^{-1})_n^j \frac{\partial A_{jn}}{\partial X_m}$$

4.41. The equilibrium configuration of a body is described by

$$x_1 = 16X_1, \quad x_2 = -\frac{1}{4}X_2, \quad x_3 = -\frac{1}{4}X_3$$

If the Cauchy stress tensor is given by  $T_{11} = 1000 \text{ MPa}$ ., all other  $T_{ij} = 0$ .

(a) Calculate the first Piola-Kirchoff stress tensor.

(b) Calculate the second Piola-Kirchoff stress tensor.

4.42. The equilibrium configuration of a body is described by

$$x_1 = -\frac{1}{2}X_1, \quad x_2 = \frac{1}{2}X_3, \quad x_3 = -4X_2 \tag{i}$$

If the Cauchy stress tensor for this body is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -100 \end{bmatrix} \text{ MPa}$$

(a) What is the corresponding first Piola-Kirchoff stress tensor?

(b) What is the corresponding second Piola-Kirchoff stress tensor?

(c) Calculate the pseudo stress vector associated with the first Piola-Kirchoff stress tensor on the  $e_3$  - plane in the deformed state.

(d) Calculate the pseudo stress vector associated with the second Piola-Kirchoff stress tensor on the  $e_3$  - plane in the deformed state.

## The Elastic Solid

So far we have studied the kinematics of deformation, the description of the state of stress and four basic principles of continuum physics: the principle of conservation of mass [Eq. (3.15.2)], the principle of linear momentum [Eq. (4.7.2)], the principle of moment of momentum [Eq. (4.4.1)] and the principle of conservation of energy [Eq. (4.14.1)]. All these relations are valid for every continuum, indeed no mention was made of any material in the derivations.

These equations are however not sufficient to describe the response of a specific material due to a given loading. We know from experience that under the same loading conditions, the response of steel is different from that of water. Furthermore, for a given material, it varies with different loading conditions. For example, for moderate loadings, the deformation in steel caused by the application of loads disappears with the removal of the loads. This aspect of the material behavior is known as **elasticity**. Beyond a certain level of loading, there will be permanent deformations, or even fracture exhibiting behavior quite different from that of elasticity. In this chapter, we shall study idealized materials which model the elastic behavior of real solids. The linear isotropic elastic model will be presented in part A, followed by the linear anisotropic elastic model in part B and an incompressible isotropic nonlinear elastic model in part C.

### 5.1 Mechanical Properties

We want to establish some appreciation of the mechanical behavior of solid materials. To do this, we perform some thought experiments modeled after real laboratory experiments.

Suppose from a block of material, we cut out a slender cylindrical test specimen of cross-sectional area  $A$ . The bar is now statically tensed by an axially applied load  $P$ , and the elongation  $\Delta l$ , over some axial gage length  $l$ , is measured. A typical plot of tensile force against elongation is shown in Fig. 5.1. Within the linear portion  $OA$  (sometimes called the proportional range), if the load is reduced to zero (i.e., unloading), then the line  $OA$  is retraced back to  $O$  and the specimen has exhibited an elasticity. Applying a load that is greater than  $A$  and then unloading, we typically traverse  $OABC$  and find that there is a "permanent elongation"  $OC$ . Reapplication of the load from  $C$  indicates elastic behavior with the same slope as  $OA$ , but with an increased proportional limit. The material is said to have work-hardened.

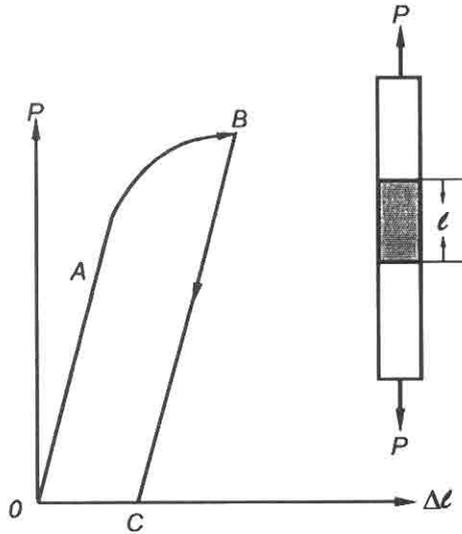


Fig. 5.1

The load-elongation diagram in Fig. 5.1 depends on the cross-section of the specimen and the axial gage length  $l$ . In order to have a representation of material behavior which is independent of specimen size and variables introduced by the experimental setup, we may plot the stress  $\sigma = P/A_o$ , where  $A_o$  is the undeformed area of the cross-section versus the axial strain  $\epsilon_a = \Delta l/l$  as shown in Fig. 5.2. In this way, the test results appear in a form which is not dependent on the specimen dimensions. The slope of the line  $OA$  will therefore be a material coefficient which is called the **Young's modulus** (or, **modulus of elasticity**)

$$E_Y = \frac{\sigma}{\epsilon_a} \tag{5.1.1}$$

The numerical value of  $E_Y$  for steel is around 207 GPa ( $30 \times 10^6$  psi). This means for a steel bar of cross-sectional area  $32.3 \text{ cm}^2$  ( $5 \text{ in}^2$ ) that carries a load of 667,000 N (150,000 lbs), the axial strain is

$$\epsilon_a = \frac{667200 / (32.3 \times 10^{-4})}{207 \times 10^9} \approx 10^{-3} \tag{i}$$

As expected, the strains in the linear elastic range of metals are quite small and we can therefore, use infinitesimal strain theory to describe the deformation of metals.

In the tension test, we can also measure changes in the lateral dimension. If the bar is of circular cross section with an initial diameter  $d$ , it will remain, under certain conditions circular, decreasing in diameter as the tensile load is increased. Letting  $\epsilon_d$  be the lateral strain

(equal to  $\Delta d/d$ ), we find that the ratio  $-\varepsilon_d/\varepsilon_a$  is a constant if the strains are small. We call this constant Poisson's ratio and denote it by  $\nu$ . A typical value of  $\nu$  for steel is 0.3.

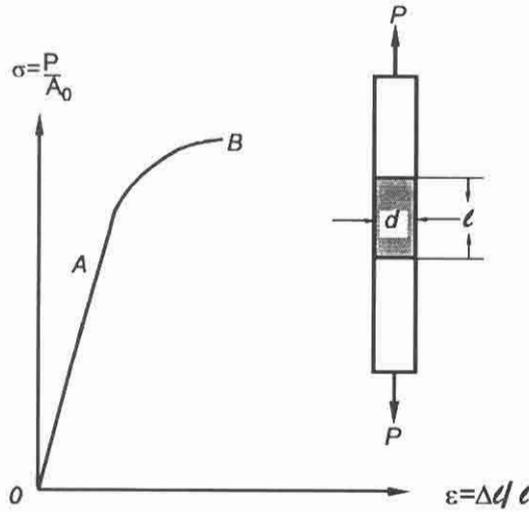


Fig. 5.2

So far we have only been considering a single specimen out of the block of material. It is conceivable that the modulus of elasticity  $E_Y$ , as well as Poisson's ratio  $\nu$  may depend on the orientation of the specimen relative to the block. In this case, the material is said to be **anisotropic** with respect to its elastic properties. Anisotropic properties are usually exhibited by materials with a definite internal structure such as wood or a rolled steel plate or composite materials. If the specimens, cut at different orientations at a sufficiently small neighborhood, show the same stress-strain diagram, we can conclude that the material is **isotropic** with respect to its elastic properties in that neighborhood.

In addition to a possible dependence on orientation of the elastic properties, we may also find that they may vary from one neighborhood to the other. In this case, we call the material **inhomogeneous**. If there is no change in the test results for specimens at different neighborhoods, we say the material is **homogeneous**.

Previously, we stated that the circular cross-section of a bar can remain circular in the tension test. This is true when the material is homogeneous and isotropic with respect to its elastic properties.

Other characteristic tests with an elastic material are also possible. In one case, we may be interested in the change of volume of a block of material under hydrostatic stress  $\sigma$  for which the stress state is

$$T_{ij} = \sigma \delta_{ij} \quad (5.1.2)$$

In a suitable experiment, we measure the relation between  $\sigma$ , the applied stress and  $e$ , the change in volume per initial volume (also known as **dilatation**, see Eq. (3.10.2)). For an elastic material, a linear relation exists for small  $e$  and we define the **bulk modulus**  $k$ , as

$$k = \frac{\sigma}{e} \quad (5.1.3)$$

A typical value of  $k$  for steel is 138 GPa ( $20 \times 10^6$  psi).

A torsion experiment yields another elastic constant. For example, we may subject a cylindrical steel bar of circular cross-section of radius  $r$  to a torsional moment  $M_t$  along the cylinder axis. The bar has a length  $l$  and will twist by an angle  $\theta$  upon the application of the moment  $M_t$ . A linear relation between the angle of twist  $\theta$  and the applied moment will be obtained for small  $\theta$ . We define a **shear modulus**  $\mu$

$$\mu = \frac{M_t l}{I_p \theta} \quad (5.1.4)$$

where  $I_p = \pi r^4/2$  (the polar area moment of inertia). A typical value of  $\mu$  for steel is 76 GPa ( $11 \times 10^6$  psi).

For an anisotropic elastic solid, the values of these material coefficients (or material constants) depend on the orientation of the specimen prepared from the block of material. Inasmuch as there are infinitely many orientations possible, an important and interesting question is how many coefficients are required to define completely the mechanical behavior of a particular elastic solid. We shall answer this question in the following section.

## 5.2 Linear Elastic Solid

Within certain limits, the experiments cited in Section 5.1 have the following features in common:

- (a) The relation between the applied loading and a quantity measuring the deformation is linear
- (b) The rate of load application does not have an effect.
- (c) Upon removal of the loading, the deformations disappear completely.
- (d) The deformations are very small.

The characteristics (a) through (d) are now used to formulate the constitutive equation of an ideal material, the linear elastic or Hookean elastic solid. The constitutive equation relates the stress to relevant quantities of deformation. In this case, deformations are small and the rate of load application has no effect. We therefore can write

$$\mathbf{T} = \mathbf{T}(\mathbf{E}) \quad (5.2.1)$$



Furthermore, we shall assume that the concept of “elasticity” is associated with the existence of a **stored energy function**  $U(E_{ij})$ , also known as the **strain energy function**, which is a positive definite<sup>†</sup> function of the strain components such that

$$T_{ij} = \frac{\partial U}{\partial E_{ij}} \quad (5.2.7)$$

With such an assumption, (the motivation for Eq. (5.2.7) is given in Example 5.2.1), it can be shown (see Example 5.2.2 below) that

$$C_{ijkl} = C_{klij} \quad (5.2.8)$$

Equations (5.2.8) reduces the number of elastic coefficients from 36 to 21.

### Example 5.2.1

(a) In the infinitesimal theory of elasticity, both the displacement components and the components of the displacement gradient are assumed to be very small. Show that under these assumptions, the rate of deformation tensor  $\mathbf{D}$  can be approximated by  $D\mathbf{E}/Dt$ , where  $\mathbf{E}$  is the infinitesimal strain tensor.

(b) Show that if  $T_{ij}$  is given by  $T_{ij} = C_{ijkl} E_{kl}$  [Eq. (5.2.2b)], then the rate of work done  $P_s$  by the stress components in a body is given by

$$P_s = \frac{DU}{Dt} \quad (5.2.9)$$

where  $U$  is the strain energy function defined by Eq. (5.2.7).

*Solution.* (a) From  $2E_{ij} = \frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}$ , we have,

$$2 \frac{DE_{ij}}{Dt} = \frac{\partial}{\partial X_j} \frac{D u_i}{Dt} + \frac{\partial}{\partial X_i} \frac{D u_j}{Dt} = \frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \quad (i)$$

Since  $x_i = x_i(X_1, X_2, X_3, t)$ , we can obtain

$$\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} = \frac{\partial v_i}{\partial x_m} \frac{\partial x_m}{\partial X_j} + \frac{\partial v_j}{\partial x_m} \frac{\partial x_m}{\partial X_i} \quad (ii)$$

Now, from  $x_m = X_m + u_m$ , where  $u_m$  are the displacement components, we have

---

<sup>†</sup> By positive definite is meant that the function is zero if and only if all the strain components are zero, otherwise, it is always positive.

$$\frac{\partial x_m}{\partial X_i} = \delta_{mi} + \frac{\partial u_m}{\partial X_i} \text{ and } \frac{\partial x_m}{\partial X_j} = \delta_{mj} + \frac{\partial u_m}{\partial X_j} \quad (\text{iii})$$

where  $\frac{\partial u_m}{\partial X_i}$  are infinitesimal. Thus,

$$\frac{\partial v_i}{\partial X_j} + \frac{\partial v_j}{\partial X_i} \approx \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} = 2 D_{ij} \quad (\text{iv})$$

That is,

$$\frac{D E_{ij}}{Dt} \approx D_{ij} \quad (\text{v})$$

(b) In Section 4.12, we derived the formula for computing the rate of work done by the stress components (the stress power) as

$$P_s = T_{ij} D_{ij} \quad (\text{vi})$$

Using Eq. (v), we have

$$P_s = T_{ij} \frac{D E_{ij}}{Dt} \quad (\text{vii})$$

Now if  $T_{ij} = \frac{\partial U}{\partial E_{ij}}$  [Eq. (5.2.7)], then,

$$P_s = \frac{\partial U}{\partial E_{ij}} \frac{D E_{ij}}{Dt} = \frac{\partial U}{\partial E_{ij}} \left( \frac{\partial E_{ij}}{\partial t} \right)_{X_i-\text{fixed}} = \left( \frac{\partial U}{\partial t} \right)_{X_i-\text{fixed}} = \frac{DU}{Dt} \quad (\text{viii})$$

That is, with the assumption given by Eq. (5.2.7), the rate at which the strain energy increases is completely determined by the rate at which the stress components are doing work and if  $P_s$  is zero then the strain energy remains a constant (i.e., stored). This result provides the motivation for assuming the existence of a positive definite energy function through Eq. (5.2.7) in association with the concept of “elasticity”<sup>†</sup>.

### Example 5.2.2

Show that if  $T_{ij} = \frac{\partial U}{\partial E_{ij}}$  for a linearly elastic solid, then

---

† We are dealing here with a purely mechanical theory where temperature and entropy play no part in the model. However, within the frame work of thermoelastic model, it can be proved that a stored energy function exists if the deformation process is either isothermal or isentropic.

(a)

$$C_{ijkl} = C_{klij} \quad (5.2.10)$$

(b) the strain energy function  $U$  is given by

$$U = \frac{1}{2} T_{ij} E_{ij} = \frac{1}{2} C_{ijkl} E_{ij} E_{kl} \quad (5.2.11)$$

*Solution.* (a) Since for linearly elastic solid  $T_{ij} = C_{ijkl} E_{kl}$ , therefore

$$\frac{\partial T_{ij}}{\partial E_{rs}} = C_{ijrs} \quad (i)$$

Thus, from Eq. (5.2.7), i.e.,  $T_{ij} = \frac{\partial U}{\partial E_{ij}}$ , we have

$$C_{ijrs} = \frac{\partial^2 U}{\partial E_{rs} \partial E_{ij}} \quad (ii)$$

Now, since  $\frac{\partial^2 U}{\partial E_{rs} \partial E_{ij}} = \frac{\partial^2 U}{\partial E_{ij} \partial E_{rs}}$ 

therefore,

$$C_{ijrs} = C_{rsij} \text{ or } C_{ijkl} = C_{klij} \quad (5.2.12)$$

(b) From

$$T_{ij} = \frac{\partial U}{\partial E_{ij}} \quad (iii)$$

we have

$$T_{ij} dE_{ij} = \frac{\partial U}{\partial E_{ij}} dE_{ij} = dU \quad (iv)$$

i.e.,

$$dU = C_{ijkl} E_{kl} dE_{ij} \quad (v)$$

Changing the dummy indices, we obtain

$$dU = C_{klij} E_{ij} dE_{kl} \quad (vi)$$

But,  $C_{klij} = C_{ijkl}$ , therefore

$$dU = C_{ijkl} E_{ij} dE_{kl} \quad (vii)$$

Adding Eqs. (v) and (vii), we obtain

$$2dU = C_{ijkl}(E_{kl} dE_{ij} + E_{ij} dE_{kl}) = C_{ijkl} d(E_{ij} E_{kl}) \tag{viii}$$

from which,

$$U = \frac{1}{2} C_{ijkl} E_{ij} E_{kl}$$

In the following, we first show that if the material is isotropic, then the number of independent coefficients reduces to only 2. Later, in Part B, the constitutive equations for anisotropic elastic solid involving 13 coefficients (monoclinic elastic solid), 9 coefficients (orthotropic elastic solid) and 5 coefficients (transversely isotropic solid), will be discussed.

### PART A Linear Isotropic Elastic Solid

#### 5.3 Linear Isotropic Elastic Solid

A material is said to be **isotropic** if its mechanical properties can be described without reference to direction. When this is not true, the material is said to be anisotropic. Many structural metals such as steel and aluminum can be regarded as isotropic without appreciable error.

We had, for a linear elastic solid, with respect to the  $e_i$  basis,

$$T_{ij} = C_{ijkl} E_{kl} \tag{i}$$

and with respect to the  $e_i'$  basis,

$$T'_{ij} = C'_{ijkl} E'_{kl} \tag{ii}$$

If the material is isotropic, then the components of the elasticity tensor must remain the same regardless of how the rectangular basis are rotated and reflected. That is

$$C'_{ijkl} = C_{ijkl} \tag{5.3.1}$$

under all orthogonal transformation of basis. A tensor having the same components with respect to every orthonormal basis is known as an **isotropic tensor**. For example, the identity tensor  $\mathbf{I}$  is obviously an isotropic tensor since its components  $\delta_{ij}$  are the same for any Cartesian basis. Indeed, it can be proved (see Prob. 5.1) that except for a scalar multiple, the identity tensor is the only isotropic second tensor. From  $\delta_{ij}$ , we can form the following three independent isotropic fourth-order tensors

$$A_{ijkl} \equiv \delta_{ij} \delta_{kl} \tag{5.3.2}$$

$$B_{ijkl} \equiv \delta_{ik} \delta_{jl} \tag{5.3.3}$$

$$H_{ijkl} \equiv \delta_{il} \delta_{jk} \tag{5.3.4}$$

It can be shown that any isotropic fourth order tensor can be represented as a linear combination of the above three isotropic fourth order tensors (we omit the rather lengthy proof here. In part B of this chapter, we shall give the detail reductions of the general  $C_{ijkl}$  to the isotropic case). Thus, for an isotropic linearly elastic material, the elasticity tensor  $C_{ijkl}$  can be written as a linear combination of  $A_{ijkl}$ ,  $B_{ijkl}$ , and  $H_{ijkl}$ .

$$C_{ijkl} = \lambda A_{ijkl} + \alpha B_{ijkl} + \beta H_{ijkl} \quad (5.3.5)$$

where  $\lambda$ ,  $\alpha$ , and  $\beta$  are constants. Substituting Eq. (5.3.5) into Eq. (i) and since

$$A_{ijkl} E_{kl} = \delta_{ij} \delta_{kl} E_{kl} = \delta_{ij} E_{kk} = \delta_{ij} e \quad (iii)$$

$$B_{ijkl} E_{kl} = \delta_{ik} \delta_{jl} E_{kl} = E_{ij} \quad (iv)$$

$$H_{ijkl} E_{kl} = \delta_{il} \delta_{jk} E_{kl} = E_{ji} = E_{ij} \quad (v)$$

we have

$$T_{ij} = C_{ijkl} E_{kl} = \lambda e \delta_{ij} + (\alpha + \beta) E_{ij} \quad (vi)$$

Or, denoting  $\alpha + \beta$  by  $2\mu$ , we have

$$T_{ij} = \lambda e \delta_{ij} + 2\mu E_{ij} \quad (5.3.6a)$$

or, in direct notation

$$\mathbf{T} = \lambda e \mathbf{I} + 2\mu \mathbf{E} \quad (5.3.6b)$$

where  $e = E_{kk}$  = first scalar invariant of  $\mathbf{E}$ . In long form, Eqs. (5.3.6) are given by

$$T_{11} = \lambda (E_{11} + E_{22} + E_{33}) + 2\mu E_{11} \quad (5.3.6c)$$

$$T_{22} = \lambda (E_{11} + E_{22} + E_{33}) + 2\mu E_{22} \quad (5.3.6d)$$

$$T_{33} = \lambda (E_{11} + E_{22} + E_{33}) + 2\mu E_{33} \quad (5.3.6e)$$

$$T_{12} = 2\mu E_{12} \quad (5.3.6f)$$

$$T_{13} = 2\mu E_{13} \quad (5.3.6g)$$

$$T_{23} = 2\mu E_{23} \quad (5.3.6h)$$

Equations (5.3.6) are the constitutive equations for a linear isotropic elastic solid. The two material constants  $\lambda$  and  $\mu$  are known as **Lame's coefficients**, or, **Lame's constants**. Since  $E_{ij}$  are dimensionless,  $\lambda$  and  $\mu$  are of the same dimension as the stress tensor, force per unit area. For a given real material, the values of the Lame's constants are to be determined from suitable experiments. We shall have more to say about this later.

## Example 5.3.1

Find the components of stress at a point if the strain matrix is

$$[\mathbf{E}] = \begin{bmatrix} 30 & 50 & 20 \\ 50 & 40 & 0 \\ 20 & 0 & 30 \end{bmatrix} \times 10^{-6}$$

and the material is steel with  $\lambda = 119.2 \text{ GPa}$  ( $17.3 \times 10^6 \text{ psi}$ ) and  $\mu = 79.2 \text{ GPa}$  ( $11.5 \times 10^6 \text{ psi}$ ).

*Solution.* We use Hooke's law  $T_{ij} = \lambda e \delta_{ij} + 2\mu E_{ij}$ , by first evaluating the dilatation  $e = 100 \times 10^{-6}$ . The stress components can now be obtained

$$T_{11} = \lambda e + 2\mu E_{11} = 1.67 \times 10^{-2} \text{ GPa}$$

$$T_{22} = \lambda e + 2\mu E_{22} = 1.83 \times 10^{-2} \text{ GPa}$$

$$T_{33} = \lambda e + 2\mu E_{33} = 1.67 \times 10^{-2} \text{ GPa}$$

$$T_{12} = T_{21} = 2\mu E_{12} = 7.92 \times 10^{-2} \text{ GPa}$$

$$T_{13} = T_{31} = 2\mu E_{13} = 3.17 \times 10^{-3} \text{ GPa}$$

$$T_{23} = T_{32} = 0 \text{ GPa}$$

## Example 5.3.2

(a) For an isotropic Hookean material, show that the principal directions of stress and strain coincide.

(b) Find a relation between the principal values of stress and strain

*Solution.* (a) Let  $\mathbf{n}_1$  be an eigenvector of the strain tensor  $\mathbf{E}$  (i.e.,  $\mathbf{E}\mathbf{n}_1 = E_1 \mathbf{n}_1$ ). Then, by Hooke's law we have

$$\mathbf{T}\mathbf{n}_1 = 2\mu\mathbf{E}\mathbf{n}_1 + \lambda e \mathbf{I}\mathbf{n}_1 = (2\mu E_1 + \lambda e) \mathbf{n}_1$$

Therefore,  $\mathbf{n}_1$  is also an eigenvector of the tensor  $\mathbf{T}$ .

(b) Let  $E_1, E_2, E_3$  be the eigenvalues of  $\mathbf{E}$  then  $e = E_1 + E_2 + E_3$ , and from Eq. (5.3.6b),

$$T_1 = 2\mu E_1 + \lambda (E_1 + E_2 + E_3).$$

In a similar fashion,

$$T_2 = 2\mu E_2 + \lambda (E_1 + E_2 + E_3).$$

$$T_3 = 2\mu E_3 + \lambda (E_1 + E_2 + E_3).$$

## Example 5.3.3

For an isotropic material

- (a) Find a relation between the first invariants of stress and strain.  
 (b) Use the result of part (a) to invert Hooke's law so that strain is a function of stress

*Solution.* (a) By adding Eqs. (5.3.6c,d,e), we have

$$T_{kk} = (2\mu + 3\lambda)E_{kk} = (2\mu + 3\lambda)e \quad (5.3.7)$$

- (b) We now invert Eq. (5.3.6b) as

$$\mathbf{E} = \frac{1}{2\mu}\mathbf{T} - \frac{\lambda}{2\mu}e\mathbf{I} = \frac{1}{2\mu}\mathbf{T} - \frac{\lambda T_{kk}}{2\mu(2\mu + 3\lambda)}\mathbf{I} \quad (5.3.8)$$

## 5.4 Young's Modulus, Poisson's Ratio, Shear Modulus, and Bulk Modulus

Equations (5.3.6) express the stress components in terms of the strain components. These equations can be inverted, as was done in Example 5.3.3, to give

$$E_{ij} = \frac{1}{2\mu} \left[ T_{ij} - \frac{\lambda}{3\lambda + 2\mu} T_{kk} \delta_{ij} \right] \quad (5.4.1)$$

We also have, from Eq. (5.3.7)

$$e = \left( \frac{1}{2\mu + 3\lambda} \right) T_{kk} \quad (5.4.2)$$

If the state of stress is such that only one normal stress component is not zero, we call it a **uniaxial stress** state. The uniaxial stress state is a good approximation to the actual state of stress in the cylindrical bar used in the tensile test described in Section 5.1. If we take the  $\mathbf{e}_1$  direction to be axial with  $T_{11} \neq 0$  and all other  $T_{ij} = 0$ , then Eqs. (5.4.1) give

$$E_{11} = \frac{1}{2\mu} \left[ T_{11} - \frac{\lambda}{3\lambda + 2\mu} T_{11} \right] = \frac{\lambda + \mu}{\mu(3\lambda + 2\mu)} T_{11} \quad (5.4.3)$$

$$E_{33} = E_{22} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)} T_{11} = -\frac{\lambda}{2(\lambda + \mu)} E_{11} \quad (5.4.4)$$

$$E_{12} = E_{13} = E_{23} = 0 \quad (5.4.5)$$

The ratio  $T_{11}/E_{11}$ , corresponding to the ratio  $\sigma/\epsilon_a$  of the tensile test described in Section 5.1, is the **Young's modulus** or the **modulus of elasticity**  $E_Y$ . Thus, from Eq. (5.4.3),

$$E_Y = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad (5.4.6)$$

The ratio  $-E_{22}/E_{11}$  and  $-E_{33}/E_{11}$ , corresponding to the ratio  $-\varepsilon_d/\varepsilon_a$  of the same tensile test, is the **Poisson's ratio**. Thus, from Eq. (5.4.4)

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad (5.4.7)$$

Using Eqs. (5.4.6) and (5.4.7) we write Eq. (5.4.1) in the frequently used engineering form

$$E_{11} = \frac{1}{E_Y} [T_{11} - \nu (T_{22} + T_{33})] \quad (5.4.8a)$$

$$E_{22} = \frac{1}{E_Y} [T_{22} - \nu (T_{33} + T_{11})] \quad (5.4.8b)$$

$$E_{33} = \frac{1}{E_Y} [T_{33} - \nu (T_{11} + T_{22})] \quad (5.4.8c)$$

$$E_{12} = \frac{1}{2\mu} T_{12} \quad (5.4.8d)$$

$$E_{13} = \frac{1}{2\mu} T_{13} \quad (5.4.8e)$$

$$E_{23} = \frac{1}{2\mu} T_{23} \quad (5.4.8f)$$

Even though there are three material constants in Eq. (5.4.8), it is important to remember that only two of them are independent for the isotropic material. In fact, by eliminating  $\lambda$  from Eqs. (5.4.6) and (5.4.7), we have the important relation

$$\mu = \frac{E_Y}{2(1 + \nu)}. \quad (5.4.9)$$

Using this relation, we can also write Eq. (5.4.1) as

$$E_{ij} = \frac{1}{E_Y} [(1 + \nu)T_{ij} - \nu (T_{kk})\delta_{ij}] \quad (5.4.10)$$

If the state of stress is such that only one pair of shear stresses is not zero, it is called a **simple shear stress** state. This state of stress may be described by  $T_{12} = T_{21} = \tau$  and Eq. (5.4.8d) gives

$$E_{12} = E_{21} = \frac{\tau}{2\mu} \quad (5.4.11)$$

Defining the **shear modulus**  $G$ , as the ratio of the shearing stress  $\tau$  in simple shear to the small decrease in angle between elements that are initially in the  $\mathbf{e}_1$  and  $\mathbf{e}_2$  directions, we have

$$\frac{\tau}{2 E_{12}} \equiv G \tag{5.4.12}$$

Comparing Eq. (5.4.12) with (5.4.11), we see that the Lamé’s constant  $\mu$  is also the **shear modulus**  $G$ .

A third stress state, called the **hydrostatic stress**, is defined by the stress tensor  $\mathbf{T} = \sigma \mathbf{I}$ . In this case, Eq. (5.3.7) gives

$$e = \frac{3 \sigma}{2 \mu + 3 \lambda} \tag{5.4.13}$$

As mentioned in Section 5.1, the bulk modulus  $k$ , is defined as the ratio of the hydrostatic normal stress  $\sigma$ , to the unit volume change, we have

$$k = \frac{\sigma}{e} = \frac{2 \mu + 3 \lambda}{3} = \lambda + \frac{2}{3} \mu \tag{5.4.14}$$

From, Eqs. (5.4.6),(5.4.7), (5.4.9) and (5.4.14) we see that the Lamé’s constants, the Young’s modulus, the shear modulus, the Poisson’s ratio and the bulk modulus are all interrelated. Only two of them are independent for a linear, elastic isotropic material. Table 5.1 expresses the various elastic constants in terms of two basic pairs. Table 5.2 gives some numerical values for some common materials.

**Table 5.1** Conversion of constants for an isotropic elastic material

	$\lambda, \mu$	$E_Y, \nu$	$\mu, \nu$	$E_Y, \mu$	$k, \nu$
$\lambda$	$\lambda$	$\frac{\nu E_Y}{(1+\nu)(1-2\nu)}$	$\frac{2\mu\nu}{1-2\nu}$	$\frac{\mu(E_Y-2\mu)}{3\mu-E_Y}$	$\frac{3k\nu}{1+\nu}$
$\mu$	$\mu$	$\frac{E_Y}{2(1+\nu)}$	$\mu$	$\mu$	$\frac{3k(1-2\nu)}{2(1+\nu)}$
$k$	$\lambda + \frac{2}{3}\mu$	$\frac{E_Y}{3(1-2\nu)}$	$\frac{2\mu(1+\nu)}{3(1-2\nu)}$	$\frac{\mu E_Y}{3(3\mu-E_Y)}$	$k$
$E_Y$	$\frac{\mu(3\lambda+2\mu)}{\lambda+\mu}$	$E_Y$	$2\mu(1+\nu)$	$E_Y$	$3k(1-2\nu)$
$\nu$	$\frac{\lambda}{2(\lambda+\mu)}$	$\nu$	$\nu$	$\frac{E_Y}{2\mu}-1$	$\nu$

**Table 5.2** Elastic constants for isotropic materials at room temperature‡.

Material	Composition	Modulus of Elasticity $E_Y$		Poisson's Ratio $\nu$	Shear Modulus $\mu$		Lamé Constant $\lambda$		Bulk Modulus $k$	
		$10^6$ psi	GPa		$10^6$ psi	GPa	$10^6$ psi	GPa	$10^6$ psi	GPa
Aluminum	Pure and alloy	9.9-11.4	68.2-78.5	0.32-0.34	3.7-3.85	25.5-26.53	6.7-9.1	46.2-62.7	9.2-11.7	63.4-80.6
Brass	60-70% Cu, 40-30% Zn	14.5-15.9	99.9-109.6	0.33-0.36	5.3-6.0	36.6-41.3	10.6-15.0	73.0-103.4	14.1-19.0	97.1-130.9
Copper		17-18	117-124	0.33-0.36	5.8-6.7	40.0-46.2	12.4-19.0	85.4-130.9	16.3-21.5	112.3-148.1
Iron, cast	2.7-3.6% C	13-21	90-145	0.21-0.30	5.2-8.2	35.8-56.5	3.9-12.1	26.9-83.4	7.4-17.6	51.0-121.3
Steel	Carbon and low alloy	5.4-16.6	106.1-114.4	0.34	6.0	41.3	12.2-13.2	84.1-90.9	16.2-17.2	111.6-118.5
Stainless steel	18% Cr, 8% Ni	28-30	193-207	0.30	10.6	73.0	16.2-17.3	111.6-119.2	23.2-24.4	160.5-168.1
Titanium	Pure and alloy	15.4-16.6	106.1-114.4	0.34	6.0	41.3	12.2-13.2	84.1-90.9	16.2-17.2	111.6-118.5
Glass	Various	7.2-11.5	49.6-79.2	0.21-0.27	3.8-4.7	26.2-32.4	2.2-5.3	15.2-36.5	4.7-8.4	32.4-57.9
Methyl methacrylate		0.35-0.5	2.41-3.45	--	--	--	--	--	--	--
Polyethylene		0.02-0.055	0.14-0.38	--	--	--	--	--	--	--
Rubber		0.00011- 0.00060	0.00076- 0.00413	0.50	0.00004- 0.00020	0.00028- 0.00138	$\infty^\dagger$	$\infty^\dagger$	$\infty^\dagger$	$\infty^\dagger$

† As  $\nu$  approaches 0.5 the ratio of  $k/E_Y$  and  $\lambda/\mu \rightarrow \infty$ . The actual value of  $k$  and  $\lambda$  for some rubbers may be close to the values of steel.

‡ Partly from “an Introduction to the Mechanics of Solids,” S.H. Crandall and N.C. Dahl, (Eds.), McGraw-Hill, 1959. (Used with permission of McGraw-Hill Book Company.)

## Example 5.4.1

(a) If for a specific material the ratio of the bulk modulus to Young's modulus is very large, find the approximate value of Poisson's ratio.

(b) Indicate why the material of part(a) can be called incompressible.

*Solution.* (a) In terms of Lamé's constants, we have

$$\frac{k}{E_Y} = \frac{1}{3} \left( \frac{\lambda}{\mu} + 1 \right) \quad (5.4.15)$$

$$\frac{\lambda}{\mu} = \frac{2\nu}{1 - 2\nu} \quad (5.4.16)$$

Combining these two equation gives

$$\frac{k}{E_Y} = \frac{1}{3(1 - 2\nu)} \quad (5.4.17)$$

Therefore, if  $\frac{k}{E_Y} \rightarrow \infty$ , then Poisson's ratio  $\nu \rightarrow \frac{1}{2}$ .

(b) For an arbitrary stress state, the dilatation or unit volume change is given by

$$e = \frac{T_{ii}}{3k} = \left( \frac{1 - 2\nu}{E_Y} \right) T_{ii} \quad (5.4.18)$$

If  $\nu \rightarrow \frac{1}{2}$ , then  $e \rightarrow 0$ . That is, the material is incompressible. It has never been observed in real material that hydrostatic compression results in an increase of volume, therefore, the upper limit of Poisson's ratio is  $\nu = \frac{1}{2}$ .

## 5.5 Equations of the Infinitesimal Theory of Elasticity

In section 4.7, we derived the Cauchy's equation of motion, to be satisfied by any continuum, in the following form

$$\rho a_i = \rho B_i + \frac{\partial T_{ij}}{\partial x_j} \quad (5.5.1)$$

where  $\rho$  is the density,  $a_i$  the acceleration component,  $\rho B_i$  the component of body force per unit volume and  $T_{ij}$  the Cauchy stress components. All terms in the equation are quantities associated with a particle which is currently at the position  $(x_1, x_2, x_3)$ .

We shall consider only the case of small motions, that is, motions such that every particle is always in a small neighborhood of the natural state. † More specifically, if  $X_i$  denotes the position in the natural state of a typical particle, we assume that

$$x_i \approx X_i$$

and that the magnitude of the components of the displacement gradient  $\partial u_i / \partial X_j$ , is also very small.

Since

$$x_1 = X_1 + u_1, \quad x_2 = X_2 + u_2, \quad x_3 = X_3 + u_3 \quad (5.5.2)$$

therefore, the velocity component

$$v_1 = \frac{Dx_1}{Dt} = \left( \frac{\partial u_1}{\partial t} \right)_{x_i \text{-fixed}} + v_1 \frac{\partial u_1}{\partial x_1} + v_2 \frac{\partial u_1}{\partial x_2} + v_3 \frac{\partial u_1}{\partial x_3} \quad (5.5.3)$$

where  $v_i$  are the small velocity components associated with the small displacement components. Neglecting the small quantities of higher order, we obtain the velocity component

$$v_1 \approx \left( \frac{\partial u_1}{\partial t} \right)_{x_i \text{-fixed}} \quad (i)$$

and the acceleration component

$$a_1 \approx \left( \frac{\partial^2 u_1}{\partial t^2} \right)_{x_i \text{-fixed}} \quad (ii)$$

Similar approximations are obtained for the other acceleration components. Thus,

$$a_i \approx \left( \frac{\partial^2 u_i}{\partial t^2} \right)_{x_i \text{-fixed}} \quad (5.5.4)$$

Furthermore, since the differential volume  $dV$  is related to the initial volume  $dV_o$  by the equation [See Sect. 3.10]

$$dV = (1 + E_{kk})dV_o \quad (iii)$$

therefore, the densities are related by

$$\rho = (1 + E_{kk})^{-1} \rho_o \approx (1 - E_{kk}) \rho_o \quad (5.5.5)$$

† We assume the existence of a state, called natural state, in which the body is unstressed

Again, neglecting small quantities of higher order, we have

$$\rho a_i \approx \rho_o \left( \frac{\partial^2 u_i}{\partial t^2} \right)_{x_i - \text{fixed}} \quad (5.5.6)$$

Thus, one can replace the equations of motion

$$\rho \frac{D^2 u_i}{Dt^2} = \rho B_i + \frac{\partial T_{ij}}{\partial x_j} \quad (\text{iv})$$

with

$$\rho_o \frac{\partial^2 u_i}{\partial t^2} = \rho_o B_i + \frac{\partial T_{ij}}{\partial x_j} \quad (5.5.7)$$

In Eq. (5.5.7) all displacement components are regarded as functions of the spatial coordinates and the equations simply state that for infinitesimal motions, there is no need to make the distinction between the spatial coordinates  $x_i$  and the material coordinates  $X_i$ . In the following sections in part A and B of this chapter, all displacement components will be expressed as functions of the spatial coordinates.

A displacement field  $u_i$  is said to describe a possible motion in an elastic medium with small deformation if it satisfies Eq. (5.5.7). When a displacement field  $u_i = u_i(x_1, x_2, x_3, t)$  is given, to make sure that it is a possible motion, we can first compute the strain field  $E_{ij}$  from Eq. (3.7.10), i.e.,

$$E_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (5.5.8)$$

and then the corresponding elastic stress field  $T_{ij}$  from Eq. (5.3.6a), i.e.,

$$T_{ij} = \lambda e \delta_{ij} + 2\mu E_{ij} \quad (5.5.9)$$

The substitution of  $u_i$  and  $T_{ij}$  in Eq. (5.5.7) will then verify whether or not the given motion is possible. If the motion is found to be possible, the surface tractions, on the boundary of the body, needed to maintain the motion are given by Eq. (4.9.1), i.e.,

$$t_i = T_{ij} n_j \quad (5.5.10)$$

On the other hand, if the boundary conditions are prescribed (e.g., certain boundaries of the body must remain fixed at all times and other boundaries must remain traction-free at all times, etc.) then, in order that  $u_i$  be the solution to the problem, it must meet the prescribed conditions on the boundary.

## Example 5.5.1

Combine Eqs. (5.5.7), (5.5.8) and (5.5.9) to obtain the Navier's equations of motion in terms of the displacement components only.

*Solution.* From

$$T_{ij} = \lambda e \delta_{ij} + 2\mu E_{ij} = \lambda e \delta_{ij} + \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (i)$$

we have

$$\frac{\partial T_{ij}}{\partial x_j} = \lambda \frac{\partial e}{\partial x_j} \delta_{ij} + \mu \left( \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_j \partial x_i} \right) \quad (ii)$$

Now,

$$\frac{\partial e}{\partial x_j} \delta_{ij} = \frac{\partial e}{\partial x_i} \quad (iii)$$

$$\frac{\partial^2 u_j}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_i} \left( \frac{\partial u_j}{\partial x_j} \right) = \frac{\partial e}{\partial x_i} \quad (iv)$$

Therefore, the equation of motion, Eq. (5.5.7), becomes

$$\rho_o \frac{\partial^2 u_i}{\partial t^2} = \rho_o B_i + (\lambda + \mu) \frac{\partial e}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad (5.5.11)$$

In long form, Eqs. (5.5.11) read

$$\rho_o \frac{\partial^2 u_1}{\partial t^2} = \rho_o B_1 + (\lambda + \mu) \frac{\partial e}{\partial x_1} + \mu \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) u_1 \quad (5.5.11a)$$

$$\rho_o \frac{\partial^2 u_2}{\partial t^2} = \rho_o B_2 + (\lambda + \mu) \frac{\partial e}{\partial x_2} + \mu \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) u_2 \quad (5.5.11b)$$

$$\rho_o \frac{\partial^2 u_3}{\partial t^2} = \rho_o B_3 + (\lambda + \mu) \frac{\partial e}{\partial x_3} + \mu \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) u_3 \quad (5.5.11c)$$

where

$$e = \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \quad (5.5.12)$$

In invariant form, the Navier equations of motion take the form

$$\rho_o \frac{\partial^2 \mathbf{u}}{\partial t^2} = \rho_o \mathbf{B} + (\lambda + \mu) \nabla e + \mu \operatorname{div} \nabla \mathbf{u} \quad (5.5.13)$$

$$e = \operatorname{div} \mathbf{u} \quad (5.5.14)$$

## 5.6 Navier Equations in Cylindrical and Spherical Coordinates

In cylindrical coordinates, with  $u_r, u_\theta, u_z$  denoting the displacement in  $(r, \theta, z)$  direction, Hooke's law takes the form of [See Sect. 2D2 for components of  $\nabla f, \nabla \mathbf{u}$  and  $\operatorname{div} \mathbf{u}$  in cylindrical coordinates]

$$T_{rr} = \lambda e + 2\mu \frac{\partial u_r}{\partial r} \quad (5.6.1a)$$

$$T_{\theta\theta} = \lambda e + 2\mu \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) \quad (5.6.1b)$$

$$T_{zz} = \lambda e + 2\mu \frac{\partial u_z}{\partial z} \quad (5.6.1c)$$

$$T_{r\theta} = \mu \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) = T_{\theta r} \quad (5.6.1d)$$

$$T_{\theta z} = \mu \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) = T_{z\theta} \quad (5.6.1e)$$

$$T_{zr} = \mu \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) = T_{rz} \quad (5.6.1f)$$

where

$$e = \frac{\partial u}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} \right) + \frac{\partial u_z}{\partial z} \quad (5.6.1g)$$

and the Navier's equations of motion are:

$$\rho_o \frac{\partial^2 u_r}{\partial t^2} = \rho_o B_r + (\lambda + \mu) \frac{\partial e}{\partial r} + \mu \left[ \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \right] \quad (5.6.2a)$$

$$\rho_0 \frac{\partial^2 u_\theta}{\partial t^2} = \rho_0 B_\theta + \frac{(\lambda + \mu)}{r} \frac{\partial e}{\partial \theta} + \mu \left[ \frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial u_\theta}{\partial r} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right] \quad (5.6.2b)$$

$$\rho_0 \frac{\partial^2 u_z}{\partial t^2} = \rho_0 B_z + (\lambda + \mu) \frac{\partial e}{\partial z} + \mu \left[ \frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2} + \frac{1}{r} \frac{\partial u_z}{\partial r} \right] \quad (5.6.2c)$$

In spherical coordinates, with  $u_r, u_\theta, u_\phi$  denoting the displacement components in  $(r, \theta, \phi)$  direction, Hooke's law take the form of [See Sect. 2D3 for components of  $\nabla f, \nabla \mathbf{u}$  and  $\text{div} \mathbf{u}$  in spherical coordinates]

$$T_{rr} = \lambda e + 2\mu \frac{\partial u_r}{\partial r} \quad (5.6.3a)$$

$$T_{\theta\theta} = \lambda e + 2\mu \left( \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \right) \quad (5.6.3b)$$

$$T_{\phi\phi} = \lambda e + 2\mu \left( \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_r}{r} + \frac{u_\theta \cot \theta}{r} \right) \quad (5.6.3c)$$

$$T_{r\theta} = \mu \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \quad (5.6.3d)$$

$$T_{\theta\phi} = \mu \left( \frac{1}{r \sin \theta} \frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi \cot \theta}{r} + \frac{1}{r} \frac{\partial u_\phi}{\partial \theta} \right) \quad (5.6.3e)$$

$$T_{\phi r} = \mu \left( \frac{1}{r \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r} \right) \quad (5.6.3f)$$

where

$$e = \frac{\partial u}{\partial r} + \frac{2u_r}{r} + \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial u_\phi}{\partial \phi} + \frac{u_\theta \cot \theta}{r} \quad (5.6.3g)$$

and the Navier's equations of motion are

$$\begin{aligned} \rho_0 \frac{\partial^2 u_r}{\partial t^2} = & \rho_0 B_r + (\lambda + \mu) \frac{\partial e}{\partial r} + \mu \left[ \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) \right) \right. \\ & \left. + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial u_r}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u_r}{\partial \phi^2} - \frac{2}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) - \frac{2}{r^2 \sin \theta} \frac{\partial u_\phi}{\partial \phi} \right] \end{aligned} \quad (5.6.4a)$$

$$\rho_0 \frac{\partial^2 u_\theta}{\partial t^2} = \rho_0 B_\theta + \frac{\lambda + \mu}{r} \frac{\partial e}{\partial \theta} + \mu \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u_\theta}{\partial r} \right) \right.$$

$$+ \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (u_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial u_\phi}{\partial \phi} \quad (5.6.4b)$$

$$\rho_o \frac{\partial^2 v_\phi}{\partial t^2} = \rho_o B_\phi + \frac{\lambda + \mu}{r \sin \theta} \frac{\partial e}{\partial \phi} + \mu \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (u_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin \theta} \frac{\partial u_r}{\partial \phi} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial u_\theta}{\partial \phi} \right] \quad (5.6.4c)$$

## 5.7 Principle of Superposition

Let  $u_i^{(1)}$  and  $u_i^{(2)}$  be two possible displacement fields corresponding to two body force fields  $B_i^{(1)}$  and  $B_i^{(2)}$ . Let  $T_{ij}^{(1)}$  and  $T_{ij}^{(2)}$  be the corresponding stress fields. Then

$$\rho_o \frac{\partial^2 u_i^{(1)}}{\partial t^2} = \rho_o B_i^{(1)} + \frac{\partial T_{ij}^{(1)}}{\partial x_j} \quad (i)$$

and

$$\rho_o \frac{\partial^2 u_i^{(2)}}{\partial t^2} = \rho_o B_i^{(2)} + \frac{\partial T_{ij}^{(2)}}{\partial x_j} \quad (ii)$$

Adding the two equations, we get

$$\rho_o \frac{\partial^2}{\partial t^2} (u_i^{(1)} + u_i^{(2)}) = \rho_o (B_i^{(1)} + B_i^{(2)}) + \frac{\partial}{\partial x_j} (T_{ij}^{(1)} + T_{ij}^{(2)})$$

It is clear from the linearity of Eqs. (5.5.8) and (5.5.9) that  $T_{ij}^{(1)} + T_{ij}^{(2)}$  is the stress field corresponding to the displacement field  $u_i^{(1)} + u_i^{(2)}$ . Thus,  $u_i^{(1)} + u_i^{(2)}$  is also a possible motion under the body force field  $(B_i^{(1)} + B_i^{(2)})$ . The corresponding stress fields are given by  $T_{ij}^{(1)} + T_{ij}^{(2)}$  and the surface tractions needed to maintain the total motion are given by  $t_i^{(1)} + t_i^{(2)}$ . This is the **principle of superposition**. One application of this principle is that in a given problem, we shall often assume that the body force is absent having in mind that its effect, if not negligible, can always be obtained separately and then superposed onto the solution of vanishing body force.

## 5.8 Plane Irrotational Wave

In this section, and in the following three sections, we shall present some simple but important elastodynamic problems using the model of linear isotropic elastic material.

Consider the motion

$$u_1 = \varepsilon \sin \frac{2\pi}{l} (x_1 - c_L t), \quad u_2 = 0, \quad u_3 = 0 \quad (5.8.1)$$

representing an infinite train of sinusoidal plane waves. In this motion, every particle executes simple harmonic oscillations of small amplitude  $\varepsilon$  around its natural state, the motion being always parallel to the  $\mathbf{e}_1$  direction. All particles on a plane perpendicular to  $\mathbf{e}_1$  are at the same phase of the harmonic motion at any one time [i.e., the same value of  $(2\pi/l)(x_1 - c_L t)$ ]. A particle which at time  $t$  is at  $x_1 + dx_1$  acquires at  $t + dt$  the same phase of motion of the particle which is at  $x_1$  at time  $t$  if  $(x_1 + dx_1) - c_L(t + dt) = x_1 - c_L t$ , i.e.,  $dx_1/dt = c_L$ . Thus  $c_L$  is known as the **phase velocity** (the velocity with which the sinusoidal disturbance of wavelength  $l$  is moving in the  $\mathbf{e}_1$  direction). Since the motions of the particles are parallel to the direction of the propagation of wave, it is a **longitudinal wave**.

We shall now consider if this wave is a possible motion in an elastic medium.

The strain components corresponding to the  $u_i$  given in Eq. (5.8.1) are

$$E_{11} = \varepsilon \left( \frac{2\pi}{l} \right) \cos \frac{2\pi}{l} (x_1 - c_L t) \quad (i)$$

$$E_{22} = E_{23} = E_{12} = E_{13} = E_{33} = 0 \quad (ii)$$

The stress components are (note  $e = E_{11} + 0 + 0 = E_{11}$ )

$$T_{11} = (\lambda + 2\mu)E_{11} = (\lambda + 2\mu) \frac{\partial u_1}{\partial x_1} \quad (iii)$$

$$T_{22} = T_{33} = \lambda E_{11} = \lambda \frac{\partial u_1}{\partial x_1} \quad (iv)$$

$$T_{12} = T_{13} = T_{23} = 0 \quad (v)$$

Substituting  $T_{ij}$  and  $u_i$  into the equations of motion in the absence of body forces, i.e.,

$$\rho_0 \frac{\partial^2 u_i}{\partial t^2} = \frac{\partial T_{ij}}{\partial x_j} \quad (5.8.2)$$

we easily see that the second and third equations of motion are automatically satisfied ( $0=0$ ) and the first equation demands that

$$\rho_0 \frac{\partial^2 u_1}{\partial t^2} = \frac{\partial T_{11}}{\partial x_1} = (\lambda + 2\mu) \frac{\partial^2 u_1}{\partial x_1^2} \quad (vi)$$

or

$$-\rho_o \varepsilon \left( \frac{2\pi}{l} \right)^2 c_L^2 \sin \frac{2\pi}{l} (x_1 - c_L t) = -(\lambda + 2\mu) \varepsilon \left( \frac{2\pi}{l} \right)^2 \sin \frac{2\pi}{l} (x_1 - c_L t) \quad (\text{vii})$$

so that the phase velocity  $c_L$  is obtained to be

$$c_L = \left( \frac{\lambda + 2\mu}{\rho_o} \right)^{1/2} \quad (5.8.3)$$

Thus, we see that with  $c_L$  given by Eq. (5.8.3), the wave motion considered is a possible one. Since for this motion, the components of the rotation tensor

$$\frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) \quad (\text{viii})$$

are zero at all times, it is known as a plane **irrotational wave**. As a particle oscillates, its volume also changes harmonically [the dilatation  $e = E_{11} = \varepsilon(2\pi/l)\cos(2\pi/l)(x_1 - c_L t)$ ], the wave is thus also known as a **dilatational wave**.

From Eq. (5.8.3), we see that for the plane wave discussed, the phase velocity  $c_L$  depends only on the material properties and not on the wave length  $l$ . Thus any disturbance represented by the superposition of any number of one-dimensional plane irrotational wave trains of different wavelengths propagates, without changing the form of the disturbance (no longer sinusoidal), with the velocity equal to the phase velocity  $c_L$ . In fact, it can be easily seen [from Eq. (5.5.11)] that any irrotational disturbance given by

$$u_1 = u_1(x_1, t), \quad u_2 = u_3 = 0 \quad (5.8.4)$$

is a possible motion in the absence of body forces provided that  $u_1(x_1, t)$  is a solution of the simple wave equation

$$\frac{\partial^2 u_1}{\partial t^2} = c_L^2 \frac{\partial^2 u_1}{\partial x_1^2} \quad (5.8.5)$$

It can be easily verified that  $u_1 = f(s)$ , where  $s = x_1 \pm c_L t$  satisfies the above equation for any function  $f$ , so that disturbances of any form given by  $f(s)$  propagate without changing its form with wave speed  $c_L$ . In other words, the phase velocity is also the rate of advance of a finite train of waves, or, of any arbitrary disturbance, into an undisturbed region.

### Example 5.8.1

Consider a displacement field

$$u_1 = \alpha \sin \frac{2\pi}{l} (x_1 - c_L t) + \beta \cos \frac{2\pi}{l} (x_1 - c_L t) \quad u_2 = u_3 = 0 \quad (\text{i})$$

for a material half-space that lies to the right of the plane  $x_1 = 0$ .

(a) Determine  $\alpha$ ,  $\beta$ , and  $l$  if the applied displacement on the plane  $x_1 = 0$  is given by  $\mathbf{u} = (b \sin \omega t) \mathbf{e}_1$

(b) Determine  $\alpha$ ,  $\beta$ , and  $l$  if the applied surface traction on  $x_1 = 0$  is given by  $\mathbf{t} = (d \sin \omega t) \mathbf{e}_1$ .

*Solution.* The given displacement field is the superposition of two longitudinal elastic waves having the same velocity of propagation  $c_L$  in the positive  $x_1$  direction and is therefore a possible elastic solution.

(a) To satisfy the displacement boundary condition, one simply sets

$$u_1(0, t) = b \sin \omega t \quad (\text{ii})$$

or

$$-\alpha \sin\left(\frac{2\pi c_L t}{l}\right) + \beta \cos\left(\frac{2\pi c_L t}{l}\right) = b \sin \omega t \quad (\text{iii})$$

Since this relation must be satisfied for all time  $t$ , we have

$$\beta = 0, \quad \alpha = -b, \quad l = \frac{2\pi c_L}{\omega} \quad (\text{iv})$$

and the elastic wave has the form

$$u_1 = -b \sin \frac{\omega}{c_L} (x_1 - c_L t). \quad (\text{v})$$

Note that the wavelength is inversely proportional to the forcing frequency  $\omega$ . That is, the higher the forcing frequency the smaller the wavelength of the elastic wave.

(b) To satisfy the traction boundary condition on  $x_1 = 0$ , one requires that

$$\mathbf{t} = \mathbf{T}(-\mathbf{e}_1) = -(T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3) = (d \sin \omega t) \mathbf{e}_1 \quad (\text{vi})$$

that is, at  $x_1 = 0$ ,  $T_{11} = -d \sin \omega t$ ,  $T_{21} = T_{31} = 0$ . For the assumed displacement field

$$T_{11} = (2\mu + \lambda) \frac{\partial u_1}{\partial x_1}, \quad T_{21} = T_{31} = 0, \quad (\text{vii})$$

therefore,

$$-d \sin \omega t = (2\mu + \lambda) \left[ \alpha \left( \frac{2\pi}{l} \right) \cos \frac{2\pi}{l} (x_1 - c_L t) - \beta \left( \frac{2\pi}{l} \right) \sin \frac{2\pi}{l} (x_1 - c_L t) \right]_{x_1=0} \quad (\text{viii})$$

i.e.,

$$-d \sin \omega t = (2\mu + \lambda) \frac{2\pi}{l} \left[ \alpha \cos \frac{2\pi}{l} c_L t + \beta \sin \frac{2\pi}{l} c_L t \right] \quad (\text{ix})$$

To satisfy this relation for all time  $t$ , we have

$$\alpha = 0, \quad \beta = \frac{-d}{2\mu + \lambda} \left( \frac{l}{2\pi} \right), \quad \omega = \frac{2\pi c_L}{l} \quad (\text{x})$$

or,

$$\alpha = 0, \quad \beta = -\frac{d c_L}{(2\mu + \lambda)\omega}, \quad l = \frac{2\pi c_L}{\omega} \quad (\text{xi})$$

and the resulting wave has the form,

$$u_1 = \frac{-d c_L}{(2\mu + \lambda)\omega} \cos \frac{\omega}{c_L} (x_1 - c_L t) \quad (\text{xii})$$

We note, that not only the wavelength but the amplitude of the resulting wave is inversely proportional to the forcing frequency.

### 5.9 Plane Equivoluminal Wave

Consider the motion

$$u_1 = 0, \quad u_2 = \varepsilon \sin \frac{2\pi}{l} (x_1 - c_L t), \quad u_3 = 0 \quad (\text{5.9.1})$$

This infinite train of plane harmonic wave differs from that discussed in Section 5.8 in that it is a transverse wave: the particle motion is parallel to  $\mathbf{e}_2$  direction, whereas the disturbance is propagating in the  $\mathbf{e}_1$  direction.

For this motion, the strain components are

$$E_{11} = E_{22} = E_{33} = E_{13} = E_{23} = 0 \quad (\text{i})$$

and

$$E_{12} = \frac{\varepsilon}{2} \left( \frac{2\pi}{l} \right) \cos \frac{2\pi}{l} (x_1 - c_T t). \quad (\text{ii})$$

and the stress components are

$$T_{12} = \mu \varepsilon \left( \frac{2\pi}{l} \right) \cos \frac{2\pi}{l} (x_1 - c_T t) \quad (\text{iii})$$

Substitution of  $T_{ij}$  and  $u_i$  in the equations of motion, neglecting body forces, gives the phase velocity  $c_T$  to be

$$c_T = \sqrt{\mu/\rho_0} \quad (\text{5.9.2})$$

Since, in this motion, the dilatation  $e$  is zero at all times, it is known as an **equivoluminal wave**. It is also called a **shear wave**.

Here again the phase velocity  $c_T$  is independent of the wavelength  $l$ , so that it again has the additional significance of being the wave velocity of a finite train of equivoluminal waves, or of any arbitrary equivoluminal disturbance into an undisturbed region.

The ratio of the two phase velocities  $c_L$  and  $c_T$  is

$$\frac{c_L}{c_T} = \left( \frac{\lambda + 2\mu}{\mu} \right)^{1/2} \quad (5.9.3)$$

Since  $\lambda = 2\mu \nu / (1 - 2\nu)$ , the ratio is found to depend only on  $\nu$ , in fact

$$\frac{c_L}{c_T} = \left[ \frac{2(1 - \nu)}{1 - 2\nu} \right]^{1/2} = \left[ \left( 1 + \frac{1}{1 - 2\nu} \right) \right]^{1/2} \quad (5.9.4)$$

For steel with  $\nu = 0.3$ ,  $c_L/c_T = \sqrt{7/2} = 1.87$ . We note that since  $\nu < \frac{1}{2}$ ,  $c_L$  is always greater than  $c_T$ .

#### Example 5.9.1

Consider a displacement field

$$u_2 = \alpha \sin \frac{2\pi}{l} (x_1 - c_T t) + \beta \cos \frac{2\pi}{l} (x_1 - c_T t), \quad u_1 = u_3 = 0 \quad (i)$$

for a material half-space that lies to the right of the plane  $x_1 = 0$

- (a) Determine  $\alpha$ ,  $\beta$  and  $l$  if the applied displacement on  $x_1 = 0$  is given by  $\mathbf{u} = (b \sin \omega t)\mathbf{e}_2$   
 (b) Determine  $\alpha$ ,  $\beta$  and  $l$  if the applied surface traction on  $x_1 = 0$  is  $\mathbf{t} = (d \sin \omega t)\mathbf{e}_2$

*Solution.* The problem is analogous to that of the previous example.

- (a) Using  $u_2(0, t) = b \sin \omega t$ , we have

$$\beta = 0, \quad \alpha = -b, \quad l = \frac{2\pi c_T}{\omega} \quad (ii)$$

and

$$u_2 = -b \sin \frac{\omega}{c_T} (x_1 - c_T t) \quad (iii)$$

- (b) Using  $\mathbf{t} = -T_{21}\mathbf{e}_2 = (d \sin \omega t)\mathbf{e}_2$  gives

$$\alpha = 0, \quad \beta = -\frac{dc_T}{\mu \omega}, \quad l = \frac{2\pi c_T}{\omega}$$

and

$$u_2 = -\frac{dc_T}{\mu \omega} \cos \frac{\omega}{c_T} (x_1 - c_T t)$$


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## Example 5.9.2

Consider the displacement field

$$u_3 = \alpha \cos p x_2 \cos \frac{2\pi}{l} (x_1 - c t), \quad u_2 = u_1 = 0 \quad (\text{i})$$

(a) Show that this is an equivoluminal motion.

(b) From the equation of motion, determine the phase velocity  $c$  in terms of  $p$ ,  $l$ ,  $\rho_0$  and  $\mu$  (assuming no body forces).

(c) This displacement field is used to describe a type of wave guide that is bounded by the plane  $x_2 = \pm h$ . Find the phase velocity  $c$  if these planes are traction free.

*Solution.* (a) Since

$$\text{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 + 0 + 0 = 0 \quad (\text{ii})$$

thus, there is no change of volume at any time.

(b) For convenience, let  $k = \frac{2\pi}{l}$  and  $\omega = kc = \frac{2\pi c}{l}$ , then

$$u_3 = \alpha \cos p x_2 \cos(kx_1 - \omega t), \quad (\text{iii})$$

where  $k$  is known as the wave number and  $\omega$  is the circular frequency. The only nonzero stresses are given by (note:  $u_1 = u_2 = 0$ )

$$T_{13} = T_{31} = \mu \frac{\partial u_3}{\partial x_1} = \alpha \mu k [-\cos p x_2 \sin(kx_1 - \omega t)], \quad (\text{iv})$$

$$T_{23} = T_{32} = \mu \frac{\partial u_3}{\partial x_2} = \alpha \mu p [-\sin p x_2 \cos(kx_1 - \omega t)], \quad (\text{v})$$

The substitution of the stress components into the third equation of motion yields (the first two equations are trivially satisfied)

$$\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} = (\mu k^2 + \mu p^2)(-u_3) = \rho_0 \frac{\partial^2 u_3}{\partial t^2} = \rho_0 \omega^2 (-u_3) \quad (\text{vi})$$

Therefore, with  $c_T^2 = \mu/\rho_0$ ,

$$k^2 = \frac{\rho_o}{\mu} \omega^2 - p^2 = \left( \frac{\omega}{c_T} \right)^2 - p^2 \quad (\text{vii})$$

Since  $k = 2\pi/l$ , and  $\omega = 2\pi c/l$ , therefore

$$c = c_T \left[ \left( \frac{lp}{2\pi} \right)^2 + 1 \right]^{\frac{1}{2}} \quad (\text{viii})$$

(c) to satisfy the traction free boundary condition at  $x_2 = \pm h$ , we require that

$$\mathbf{t} = \pm \mathbf{T}\mathbf{e}_2 = \pm (T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3) = \pm T_{32}\mathbf{e}_3 = 0 \quad (\text{ix})$$

therefore,  $T_{32}|_{x_2=\pm h} = -\pm u p \alpha \sin ph \cos(kx_1 - \omega t) = 0$ . In order for this relation to be satisfied for all  $x_1$  and  $t$ , we must have

$$\sin ph = 0 \quad (\text{x})$$

Thus,

$$p = \frac{n\pi}{h}, \quad n = 0, 1, 2, \dots \quad (\text{xi})$$

Each value of  $n$  determines a possible displacement field, and the phase velocity  $c$  corresponding to each mode is given by

$$c = c_r \left[ \left( \frac{nl}{2h} \right)^2 + 1 \right]^{\frac{1}{2}} \quad (\text{xii})$$

This result indicates that the equivoluminal wave is propagating with a speed  $c$  greater than the speed of a plane equivoluminal wave  $c_T$ . Note that when  $p = 0$ ,  $c = c_T$  as expected.

### Example 5.9.3

An infinite train of harmonic plane waves propagates in the direction of the unit vector  $\mathbf{e}_n$ . Express the displacement field in vector form for (a) a longitudinal wave, (b) a transverse wave.

**Solution.** Let  $\mathbf{x}$  be the position vector of any point on a plane whose normal is  $\mathbf{e}_n$  and whose distance from the origin is  $d$  (Fig. 5.3). Then  $\mathbf{x} \cdot \mathbf{e}_n = d$ . Thus, in order that the particles on the plane be at the same phase of the harmonic oscillation at any one time, the argument of

sine (or cosine) must be of the form  $(2\pi/l)(\mathbf{x} \cdot \mathbf{e}_n - ct - \eta)$ , where  $\eta$  is an arbitrary constant.

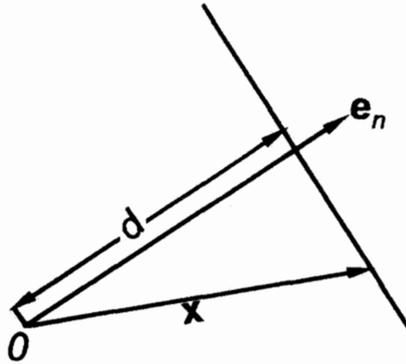


Fig. 5.3

a) For longitudinal waves,  $\mathbf{u}$  is parallel to  $\mathbf{e}_n$ , thus

$$\mathbf{u} = \varepsilon \sin \left[ \frac{2\pi}{l} (\mathbf{x} \cdot \mathbf{e}_n - c_L t - \eta) \right] \mathbf{e}_n \tag{i}$$

In particular, if  $\mathbf{e}_n = \mathbf{e}_1$ ,

$$u_1 = \varepsilon \sin \left[ \frac{2\pi}{l} (x_1 - c_L t - \eta) \right], \quad u_2 = u_3 = 0 \tag{ii}$$

(b) For transverse waves,  $\mathbf{u}$  is perpendicular to  $\mathbf{e}_n$ . Let  $\mathbf{e}_t$  be a unit vector perpendicular to  $\mathbf{e}_n$ . Then

$$\mathbf{u} = \varepsilon \left[ \sin \frac{2\pi}{l} (\mathbf{x} \cdot \mathbf{e}_n - c_T t - \eta) \right] \mathbf{e}_t \tag{iii}$$

The plane of  $\mathbf{e}_t$  and  $\mathbf{e}_n$  is known as the plane of polarization. In particular, if  $\mathbf{e}_n = \mathbf{e}_1$ ,  $\mathbf{e}_t = \mathbf{e}_2$ , then

$$u_1 = 0, \quad u_2 = \varepsilon \sin \frac{2\pi}{l} (x_1 - c_T t - \eta), \quad u_3 = 0 \tag{iv}$$

Example 5.9.4

In Fig. 5.4, all three unit vectors  $\mathbf{e}_{n_1}, \mathbf{e}_{n_2}$  and  $\mathbf{e}_{n_3}$  lie in the  $x_1 x_2$  plane. Express the displacement components with respect to the  $x_i$  coordinates of plane harmonic waves for

(a) a transverse wave of amplitude  $\varepsilon_1$  wavelength  $l_1$  polarized in the  $x_1 x_2$  plane and propagating in the direction of  $\mathbf{e}_{n_1}$ .

(b) a transverse wave of amplitude  $\varepsilon_2$  wavelength  $l_2$  polarized in the  $x_1 x_2$  plane and propagating in the direction of  $\mathbf{e}_{n_2}$ .

(c) a longitudinal wave of amplitude  $\varepsilon_3$  wavelength  $l_3$  propagating in the direction of  $\mathbf{e}_{n_3}$

*Solution.* Using the results of Example 5.9.3, we have, (a)

$$\mathbf{e}_{n_1} = \sin\alpha_1 \mathbf{e}_1 - \cos\alpha_1 \mathbf{e}_2, \quad \mathbf{x} \cdot \mathbf{e}_{n_1} = x_1 \sin\alpha_1 - x_2 \cos\alpha_1, \quad \mathbf{e}_{t_1} = \pm(\cos\alpha_1 \mathbf{e}_1 + \sin\alpha_1 \mathbf{e}_2) \quad (i)$$

Thus,

$$\begin{aligned} u_1 &= \cos \alpha_1 \varepsilon_1 \sin[2\pi/l_1 (x_1 \sin\alpha_1 - x_2 \cos\alpha_1 - c_T t - \eta_1)] \\ u_2 &= \sin \alpha_1 \varepsilon_1 \sin[2\pi/l_1 (x_1 \sin\alpha_1 - x_2 \cos\alpha_1 - c_T t - \eta_1)] \\ u_3 &= 0 \end{aligned} \quad (ii)$$

(b)

$$\mathbf{e}_{n_2} = \sin \alpha_2 \mathbf{e}_1 + \cos \alpha_2 \mathbf{e}_2, \quad \mathbf{x} \cdot \mathbf{e}_{n_2} = x_1 \sin\alpha_2 + x_2 \cos\alpha_2, \quad \mathbf{e}_{t_2} = \pm(\cos\alpha_2 \mathbf{e}_1 - \sin\alpha_2 \mathbf{e}_2) \quad (iii)$$

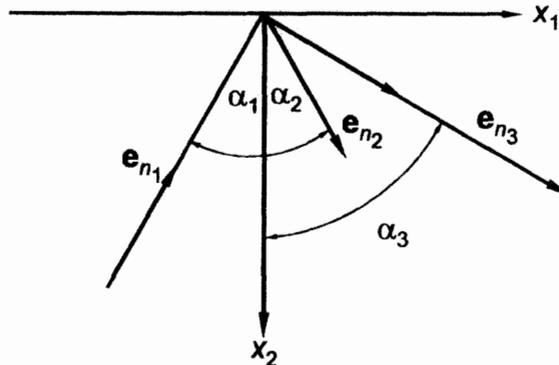


Fig. 5.4

$$\begin{aligned} u_1 &= \cos \alpha_2 \varepsilon_2 \sin[2\pi/l_2 (x_1 \sin\alpha_2 + x_2 \cos\alpha_2 - c_T t - \eta_2)] \\ u_2 &= -\sin \alpha_2 \varepsilon_2 \sin[2\pi/l_2 (x_1 \sin\alpha_2 + x_2 \cos\alpha_2 - c_T t - \eta_2)] \\ u_3 &= 0 \end{aligned} \quad (iv)$$

(c)

$$\mathbf{e}_{n_3} = \sin \alpha_3 \mathbf{e}_1 + \cos \alpha_3 \mathbf{e}_2, \quad \mathbf{x} \cdot \mathbf{e}_{n_3} = x_1 \sin\alpha_3 + x_2 \cos\alpha_3 \quad (v)$$

$$u_1 = \sin \alpha_3 \varepsilon_3 \sin[2\pi/l_3 (x_1 \sin\alpha_3 + x_2 \cos\alpha_3 - c_L t - \eta_3)]$$

$$u_2 = \cos \alpha_3 \varepsilon_3 \sin[2\pi/l_3 (x_1 \sin \alpha_3 + x_2 \cos \alpha_3 - c_L t - \eta_3)] \tag{vi}$$

$$u_3 = 0$$

**5.10 Reflection of Plane Elastic Waves.**

In Fig. 5.5, the plane  $x_2 = 0$  is the free boundary of an elastic medium, occupying the lower half-space  $x_2 \geq 0$ . We wish to study how an incident plane wave is reflected by the boundary. Consider an incident transverse wave of wavelength  $l_1$ , polarized in the plane of incidence with an incident angle  $\alpha_1$ , (see Fig. 5.5). Since  $x_2 = 0$  is a free boundary, the surface traction on the plane is zero at all times. Thus, the boundary will generate reflection waves in such a way that when they are superposed on the incident wave, the stress vector on the boundary vanishes at all times.

Let us superpose on the incident transverse wave two reflection waves (see Fig. 5.5), one transverse, the other longitudinal, both oscillating in the plane of incidence. The reason for superposing not only a reflected transverse wave but also a longitudinal one is that if only one is superposed, the stress-free condition on the boundary in general cannot be met, as will become obvious in the following derivation.

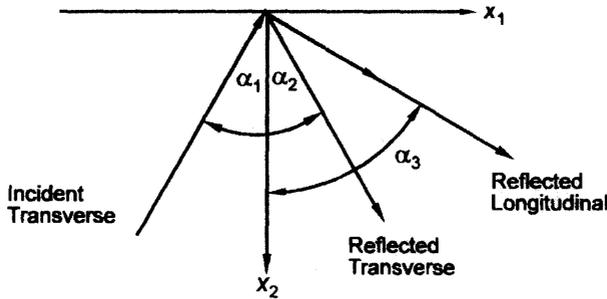


Fig. 5.5

Let  $u_i$  denote the displacement components of the superposition of the three waves, then from the results of Example 5.9.4, we have

$$u_1 = \cos \alpha_1 \varepsilon_1 \sin \varphi_1 + \cos \alpha_2 \varepsilon_2 \sin \varphi_2 + \sin \alpha_3 \varepsilon_3 \sin \varphi_3$$

$$u_2 = \sin \alpha_1 \varepsilon_1 \sin \varphi_1 - \sin \alpha_2 \varepsilon_2 \sin \varphi_2 + \cos \alpha_3 \varepsilon_3 \sin \varphi_3 \tag{i}$$

$$u_3 = 0$$

where

$$\varphi_1 = \frac{2\pi}{l_1} (x_1 \sin \alpha_1 - x_2 \cos \alpha_1 - c_T t - \eta_1)$$

$$\varphi_2 = \frac{2\pi}{l_2} (x_1 \sin \alpha_2 + x_2 \cos \alpha_2 - c_T t - \eta_2) \tag{ii}$$

$$\varphi_3 = \frac{2\pi}{l_3} (x_1 \sin \alpha_3 + x_2 \cos \alpha_3 - c_L t - \eta_3)$$

On the free boundary ( $x_2 = 0$ ), where  $\mathbf{n} = -\mathbf{e}_2$ , the condition  $\mathbf{t} = \mathbf{0}$  leads to

$$\mathbf{T}\mathbf{e}_2 = \mathbf{0} \quad (\text{iii})$$

i.e.,

$$T_{12} = T_{22} = T_{32} = 0 \quad (\text{iv})$$

Using Hooke's law, and noting that  $u_3 = 0$  and  $u_2$  does not depend on  $x_3$ , we easily see that the condition  $T_{32} = 0$  is automatically satisfied. The other two conditions, in terms of displacement components, are

$$\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0 \quad \text{on } x_2 = 0 \quad (\text{v})$$

$$(\lambda + 2\mu) \frac{\partial u_2}{\partial x_2} + \lambda \frac{\partial u_1}{\partial x_1} = 0 \quad \text{on } x_2 = 0 \quad (\text{vi})$$

Performing the required differentiation, we obtain from Eqs. (v) and (vi)

$$\frac{\varepsilon_1}{l_1} (\sin^2 \alpha_1 - \cos^2 \alpha_1) \cos \varphi_1 + \frac{\varepsilon_2}{l_2} (\cos^2 \alpha_2 - \sin^2 \alpha_2) \cos \varphi_2 + \frac{\varepsilon_3}{l_3} (\sin 2\alpha_3) \cos \varphi_3 = 0 \quad (\text{vii})$$

$$\frac{\varepsilon_1}{l_1} \mu \sin 2\alpha_1 \cos \varphi_1 + \frac{\varepsilon_2}{l_2} \mu \sin 2\alpha_2 \cos \varphi_2 - \frac{\varepsilon_3}{l_3} (\lambda + 2\mu \cos^2 \alpha_3) \cos \varphi_3 = 0 \quad (\text{viii})$$

Since these equations are to be satisfied on  $x_2 = 0$  for whatever values of  $x_1$  and  $t$ , we must have

$$\cos \varphi_1 = \cos \varphi_2 = \cos \varphi_3 \quad \text{on } x_2 = 0 \quad (\text{ix})$$

so that they drop out from Eq. (vii) and (viii). Thus, at

$$x_2 = 0, \quad \varphi_1 = \varphi_2 \pm 2p\pi = \varphi_3 \pm 2q\pi$$

where  $p$  and  $q$  are integers, i.e.,

$$\begin{aligned} \frac{2\pi}{l_1} (x_1 \sin \alpha_1 - c_T t - \eta_1) &= \frac{2\pi}{l_2} (x_1 \sin \alpha_2 - c_T t - \eta_2') \\ &= \frac{2\pi}{l_3} (x_1 \sin \alpha_3 - c_L t - \eta_3') \end{aligned} \quad (\text{x})$$

where  $\eta_2' = \eta_2 - (\pm p l_2)$  and  $\eta_3' = \eta_3 - (\pm p l_3)$

Equation (x) can be satisfied for whatever values of  $x_1$  and  $t$  only if

$$\frac{\sin \alpha_1}{l_1} = \frac{\sin \alpha_2}{l_2} = \frac{\sin \alpha_3}{l_3} \quad (\text{xi})$$

$$\frac{c_T}{l_1} = \frac{c_T}{l_2} = \frac{c_L}{l_3} \quad (\text{xii})$$

and

$$\frac{\eta_1}{l_1} = \frac{\eta_2'}{l_2} = \frac{\eta_3'}{l_3} \quad (\text{xiii})$$

Thus,

$$l_2 = l_1, \quad n l_3 = l_1 \quad (\text{xiv})$$

where

$$\frac{1}{n} = \frac{c_L}{c_T} = \left( \frac{\lambda + 2\mu}{\mu} \right)^{1/2} \quad (\text{xv})$$

$$\alpha_1 = \alpha_2 \quad (\text{xvi})$$

$$n \sin \alpha_3 = \sin \alpha_1 \quad (\text{xvii})$$

$$\eta_2' = \eta_1, \quad n \eta_3' = \eta_1 \quad (\text{xviii})$$

That is, the reflected transverse wave has the same wavelength as that of the incident transverse wave and the angle of reflection is the same as the incident angle, the longitudinal wave has a different wavelength and a different reflection angle depending on the so-called "refraction index  $n$ ."

With  $\cos \varphi_i$  dropped out, and in view of Eqs. (xiv) to (xviii), the boundary conditions (vii) and (viii) now become

$$\varepsilon_1 (\sin^2 \alpha_1 - \cos^2 \alpha_1) + \varepsilon_2 (\cos^2 \alpha_1 - \sin^2 \alpha_1) + \varepsilon_3 n \sin 2 \alpha_3 = 0 \quad (\text{xix})$$

$$\varepsilon_1 (\mu \sin 2 \alpha_1) + \varepsilon_2 (\mu \sin 2 \alpha_1) - \varepsilon_3 n (2 \mu \cos^2 \alpha_3 + \lambda) = 0 \quad (\text{xx})$$

These two equations uniquely determine the amplitudes of the reflected waves in terms of the incident amplitude ( which is arbitrary ). In fact

$$\varepsilon_3 = \frac{n \sin 4 \alpha_1}{\cos^2 2 \alpha_1 + n^2 \sin 2 \alpha_1 \sin 2 \alpha_3} \varepsilon_1 \quad (\text{xxi})$$

$$\varepsilon_2 = \frac{\cos^2 2 \alpha_1 - n^2 \sin 2 \alpha_1 \sin 2 \alpha_3}{\cos^2 2 \alpha_1 + n^2 \sin 2 \alpha_1 \sin 2 \alpha_3} \varepsilon_1 \quad (\text{xxii})$$

Thus, the problem of the reflection of a transverse wave polarized in the plane of incidence is solved. We mention that if the incident transverse wave is polarized normal to the plane of incidence, no longitudinal component occurs. Also, when an incident longitudinal wave is reflected, in addition to the regularly reflected longitudinal wave, there is also a transverse wave polarized in the plane of incidence.

Equation (xvii) is analogous to Snell's law in optics, except here we have reflection instead of refraction. If  $\sin \alpha_1 > n$ , then  $\sin \alpha_3 > 1$  and there is no longitudinal reflected wave but rather, waves of a more complicated nature will be generated. The angle  $\alpha_1 = \sin^{-1} n$  is called the critical angle.

### 5.11 Vibration of an Infinite Plate

Consider an infinite plate bounded by the planes  $x_1 = 0$  and  $x_1 = l$ . These plane faces may have either a prescribed motion or a prescribed surface traction.

The presence of these two boundaries indicates the possibility of a vibration ( a standing wave). We begin by assuming the vibration to be of the form

$$u_1 = u_1(x_1, t), \quad u_2 = u_3 = 0 \quad (5.11.1)$$

and, just as for longitudinal waves, the displacement must satisfy the equation

$$c_L^2 \frac{\partial^2 u_1}{\partial x_1^2} = \frac{\partial^2 u_1}{\partial t^2} \quad (5.11.2)$$

A steady-state vibration solution to this equation is of the form

$$u_1 = (A \cos \lambda x_1 + B \sin \lambda x_1)(C \cos c_L \lambda t + D \sin c_L \lambda t) \quad (5.11.3)$$

where the constant  $A, B, C, D$ , and  $\lambda$  are determined by the boundary conditions. This vibration mode is sometimes termed a **thickness stretch** vibration because the plate is being stretched through its thickness. It is analogous to acoustic vibration of organ pipes and to the longitudinal vibration of slender rods.

Another vibration mode can be obtained by assuming the displacement field

$$u_2 = u_2(x_1, t), \quad u_1 = u_3 = 0 \quad (5.11.4)$$

In this case, the displacement field must satisfy the equation

$$c_T^2 \frac{\partial^2 u_2}{\partial x_1^2} = \frac{\partial^2 u_2}{\partial t^2} \quad (5.11.5)$$

and the solution is of the same form as in the previous case. This vibration is termed **thickness-shear** and it is analogous to the vibrating string.

## Example 5.11.1

(a) Find the thickness-stretch vibration of a plate, where the left face ( $x_1 = 0$ ) is subjected to a forced displacement  $\mathbf{u} = (\alpha \cos \omega t)\mathbf{e}_1$  and the right face ( $x_1 = l$ ) is fixed.

(b) Determine the values of  $\omega$  that give resonance.

*Solution.* (a) Using Eq. (5.11.3) and the first boundary condition, we have

$$\alpha \cos \omega t = u_1(0, t) = AC \cos c_L \lambda t + AD \sin c_L \lambda t \quad (\text{i})$$

Therefore

$$AC = \alpha, \quad \lambda = \frac{\omega}{c_L}, \quad D = 0 \quad (\text{ii})$$

The second boundary condition gives

$$0 = u_1(l, t) = \left( \alpha \cos \frac{\omega l}{c_L} + BC \sin \frac{\omega l}{c_L} \right) \cos \omega t \quad (\text{iii})$$

Therefore

$$BC = -\alpha \cot \frac{\omega l}{c_L} \quad (\text{iv})$$

and the vibration is given by

$$u_1(x_1, T) = \alpha \left[ \cos \frac{\omega}{c_L} x_1 - \frac{\sin \frac{\omega}{c_L} x_1}{\tan \frac{\omega l}{c_L}} \right] \cos \omega t \quad (\text{v})$$

(b) Resonance is indicated by unbounded displacements. This occurs for forcing frequencies corresponding to  $\tan \omega l / c_L = 0$ , that is, when<sup>†</sup>

$$\omega = \frac{n \pi c_L}{l}, \quad n = 1, 2, 3, \dots$$

## Example 5.11.2

(a) Find the thickness-shear vibration of an infinite plate which has an applied surface traction  $\mathbf{t} = -(\beta \cos \omega t)\mathbf{e}_2$  on the plane  $x_1 = 0$  and is fixed at the plane  $x_1 = l$ .

† These values of frequencies correspond to the natural free vibration frequencies with both faces fixed.

(b) Determine the resonance frequencies.

*Solution.* The traction on  $x_1 = 0$  determines the stress  $T_{12}|_{x_1=0} = \beta \cos \omega t$ . This shearing stress forces a vibration of the form

$$u_2 = (A \cos \lambda x_1 + B \sin \lambda x_1)(C \cos c_T \lambda t + D \sin c_T \lambda t).$$

Using Hooke's law, we have

$$T_{12} |_{x_1=0} = \mu \frac{\partial u_2}{\partial x_1} |_{x_1=0} = \beta \cos \omega t \quad (i)$$

or,

$$\frac{\beta}{\mu} \cos \omega t = \lambda BC \cos c_T \lambda t + \lambda BD \sin c_T \lambda t \quad (ii)$$

Thus,

$$\lambda = \frac{\omega}{c_T}, \quad D = 0, \quad BC = \frac{\beta c_T}{\omega \mu} \quad (iii)$$

The boundary condition at  $x_1 = l$  gives

$$u_2(l, t) = 0 = \left( AC \cos \frac{\omega l}{c_T} + \frac{\beta c_T}{\omega \mu} \sin \frac{\omega l}{c_T} \right) \cos \omega t \quad (iv)$$

Thus,

$$AC = -\frac{\beta c_T}{\omega \mu} \tan \frac{\omega l}{c_T} \quad (v)$$

and

$$u_2(x_1, t) = \frac{\beta c_T}{\omega \mu} \left( \sin \frac{\omega}{c_T} x_1 - \tan \frac{\omega l}{c_T} \cos \frac{\omega}{c_T} x_1 \right) \cos \omega t \quad (vi)$$

(b) Resonance occurs for

$$\tan \frac{\omega l}{c_T} = \infty$$

or

$$\omega = \frac{n \pi c_T}{2l}, \quad n = 1, 3, 5, \dots \quad (vii)$$

We remark that these values of  $\omega$  correspond to free vibration natural frequencies with one face traction-free and one face fixed.

5.12 Simple Extension

In this section and the following several sections, we shall present some examples of elastostatic problems. We begin by considering the problem of simple extension. Again, in all these problems, we assume small deformations so that there is no need to make a distinction between the spatial coordinates and the material coordinates in the equations of motion and in the boundary conditions.

A cylindrical elastic bar of arbitrary cross-section (Fig. 5.6) is under the action of equal and opposite normal traction  $\sigma$  distributed uniformly at its end faces. Its lateral surface is free from any surface traction and body forces are assumed to be absent.

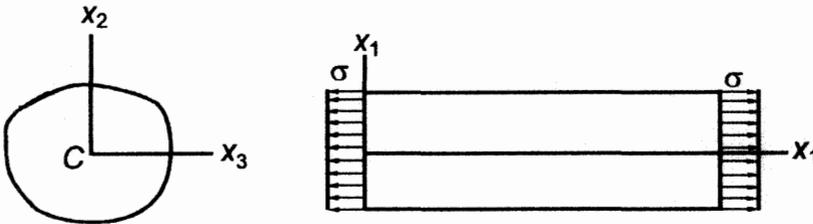


Fig. 5.6

Intuitively, one expects that the state of stress at any point will depend neither on the length of the bar nor on its lateral dimension. In other words, the state of stress in the bar is expected to be the same everywhere. Guided by the boundary conditions that on the planes  $x_1 = 0$  and  $x_1 = l$   $T_{11} = \sigma, T_{21} = T_{31} = 0$  and on the planes  $x_2 = a$  a constant and tangent to the lateral surface,  $T_{12} = T_{22} = T_{32} = 0$ , it seems reasonable to assume that for the whole bar

$$T_{11} = \sigma, \quad T_{22} = T_{23} = T_{12} = T_{13} = T_{23} = 0 \tag{5.12.1}$$

We now proceed to show that this state of stress is indeed the solution to our problem. We need to show that (i) all the equations of equilibrium are satisfied (ii) all the boundary conditions are satisfied and (iii) there exists a displacement field which corresponds to the assumed stress field.

(i) Since the stress components are all constants (either  $\sigma$  or zero), it is obvious that in the absence of body forces, the equations of equilibrium  $\partial T_{ij} / \partial x_j = 0$  are identically satisfied.

(ii) The boundary condition on each of the end faces is obviously satisfied. On the lateral cylindrical surface,

$$\mathbf{n} = 0\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3 \tag{i}$$

and

$$\mathbf{t} = \mathbf{Tn} = n_2(\mathbf{Te}_2) + n_3(\mathbf{Te}_3) = n_2(\mathbf{0}) + n_3(\mathbf{0}) = \mathbf{0} \quad (\text{ii})$$

Thus, the traction-free condition on the whole lateral surface is satisfied.

(iii) From Hooke's law, the strain components are obtained to be

$$E_{11} = \frac{1}{E_Y} [T_{11} - \nu (T_{22} + T_{33})] = \frac{\sigma}{E_Y} \quad (5.12.2a)$$

$$E_{22} = \frac{1}{E_Y} [T_{22} - \nu (T_{33} + T_{11})] = -\nu \frac{\sigma}{E_Y} \quad (5.12.2b)$$

$$E_{33} = \frac{1}{E_Y} [T_{33} - \nu (T_{11} + T_{22})] = -\nu \frac{\sigma}{E_Y} \quad (5.12.2c)$$

$$E_{12} = E_{13} = E_{23} = 0 \quad (5.12.2d)$$

These strain components are constants, therefore, the equations of compatibility are automatically satisfied. In fact it is easily verified that the following single-valued continuous displacement field corresponds to the strain field of Eq. (5.12.2)

$$u_1 = \left( \frac{\sigma}{E_Y} \right) x_1, \quad u_2 = -\left( \nu \frac{\sigma}{E_Y} \right) x_2, \quad u_3 = -\left( \nu \frac{\sigma}{E_Y} \right) x_3 \quad (5.12.3)$$

Thus, we have completed the solution of the problem of simple extension ( $\sigma > 0$ ) or compression ( $\sigma < 0$ ). We note that Eq. (5.12.3) is the unique solution to Eqs. (5.12.2) if rigid body displacement fields (translation and rotation) are excluded.

If the constant cross-sectional area of the bar is  $A$ , the surface traction  $\sigma$  on either end face gives rise to a resultant force of magnitude

$$P = \sigma A \quad (5.12.4)$$

passing through the centroid of the area  $A$ . Thus, in terms of  $P$  and  $A$ , the stress components in the bar are

$$[\mathbf{T}] = \begin{bmatrix} \frac{P}{A} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.12.5)$$

Since the matrix is diagonal, we know from Chapter 2, that the principal stresses are  $P/A, 0, 0$ . Thus, the maximum normal stress is

$$(T_n)_{\max} = \frac{P}{A} \quad (5.12.6)$$

It acts on normal cross-sectional planes, and the maximum shearing stress is

$$(T_s)_{\max} = \frac{P}{2A} \quad (5.12.7)$$

and it acts on planes making  $45^\circ$  with the normal cross-sectional plane.

Let the undeformed length of the bar be  $l$  and let  $\Delta l$  be its elongation. Then  $E_{11} = \frac{\Delta l}{l}$  and from Eqs. (5.12.2a) and (5.12.4), we have

$$\Delta l = \frac{Pl}{AE_Y} \quad (5.12.8)$$

Also, if  $d$  is the undeformed length of a line in the transverse direction, its elongation  $\Delta d$  is given by

$$\Delta d = -\frac{\nu Pd}{AE_Y} \quad (5.12.9)$$

The minus sign indicates the expected contraction of the lateral dimension for a bar under tension.

In reality, when a bar is pulled, the exact nature of the distribution of surface traction is often not known, only the resultant force is known. The question naturally arises under what conditions can an elasticity solution such as the one we just obtained for simple extension be applicable to real problems. The answer to the question is given by the so-called **St. Venant's Principle** which can be stated as follows:

*If some distribution of forces acting on a portion of the surface of body is replaced by a different distribution of forces acting on the same portion of the body, then the effects of the two different distributions on the parts of the body sufficiently far removed from the region of application of the forces are essentially the same, provided that the two distributions of forces have the same resultant force and the same resultant couple.*

The validity of the principle can be demonstrated in specific instances and a number of sufficient conditions have been established. We state only that in most cases the principle has been proven to be in close agreement with experiments.

By invoking Saint-Venant's principle, we now regard the solution we just obtained for "simple extension" to be valid at least in most part of a slender bar, provided the resultant force on either end passes through the centroid of the cross-sectional area.

#### Example 5.12.1

A steel circular bar, 2 ft (0.61 m) long, 1 in. (2.54 cm) radius, is pulled by equal and opposite axial forces  $P$  at its ends. Find the maximum normal and shear stresses if  $P = 10,000$  lbs (44.5 kN).  $E_Y = 30 \times 10^6$  psi (207 GPa.) and  $\nu = 0.3$ .

*Solution.* The maximum normal stress is

$$(T_n)_{\max} = \frac{P}{A} = \frac{10,000}{\pi} = 3180 \text{ psi. (21.9 Mpa.)}$$

The maximum shearing stress is

$$(T_s)_{\max} = \frac{3180}{2} = 1590 \text{ psi. (11.0 Mpa.)}$$

and the total elongation is

$$\Delta l = \frac{Pl}{AE_Y} = \frac{(10,000)(2 \times 12)}{\pi(30 \times 10^6)} = 2.54 \times 10^{-3} \text{ in. (64.5 } \mu\text{m.)}$$

The diameter will contract by an amount

$$-\Delta d = \frac{\nu P}{E_Y A} d = \frac{(0.3)(10,000)(2)}{(30 \times 10^6)(\pi)} = 0.636 \times 10^{-4} \text{ in. (1.61 } \mu\text{m.)}$$

### Example 5.12.2

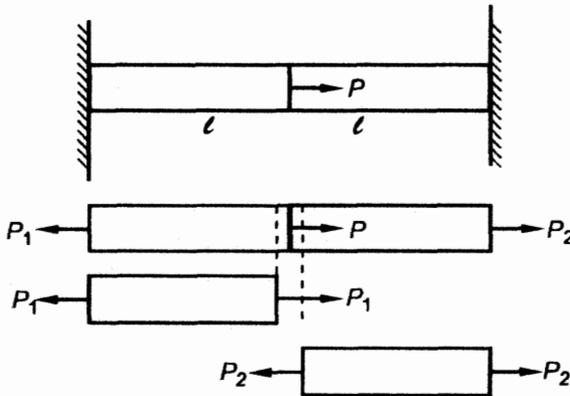


Fig. 5.7

A composite bar, formed by welding two slender bars of equal length and equal cross-sectional area, is loaded by an axial force  $P$  as shown in Fig. 5.7. If Young's moduli of the two portions are  $E_Y^{(1)}$  and  $E_Y^{(2)}$ , find how the applied force is distributed between the two halves.

*Solution.* Taking the whole bar as a free body, the equation of static equilibrium requires that

$$P = P_1 - P_2 \quad (i)$$

Statics alone does not determine the distribution of the load (a statically indeterminate problem), so we must consider the deformation induced by the load  $P$ . In this problem, there is no net elongation of the composite bar, therefore

$$\frac{P_1 l}{A E_Y^{(1)}} + \frac{P_2 l}{A E_Y^{(2)}} = 0 \quad (ii)$$

Combining Eqs. (i) and (ii), we obtain

$$P_1 = \frac{P}{1 + (E_Y^{(2)}/E_Y^{(1)})}, \quad P_2 = \frac{-P}{1 + (E_Y^{(1)}/E_Y^{(2)})} \quad (iii)$$

If in particular, Young's moduli are  $E_Y^{(1)} = 207$  GPa (steel) and  $E_Y^{(2)} = 69$  GPa.(aluminum), then

$$P_1 = \frac{3P}{4}, \quad P_2 = \frac{-P}{4} \quad (iv)$$

### 5.13 Torsion of a Circular Cylinder

Let us consider the elastic deformation of a cylindrical bar of circular cross-section (of radius  $a$  and length  $l$ ), twisted by equal and opposite end moments  $M_t$  (see Fig. 5.8). We choose the  $x_1$  axis to coincide with axis of the cylinder and the left and right faces to correspond to the plane  $x_1 = 0$  and  $x_1 = l$  respectively

By the symmetry of the problem, it is reasonable to assume that the motion of each cross-sectional plane induced by the end moments is a rigid body rotation about the  $x_1$  axis. This motion is similar to that of a stack of coins in which each coin is rotated by a slightly different angle than the previous coin. It is the purpose of this section to demonstrate that for a circular cross-section, this assumption of the deformation leads to an exact solution within the linear theory of elasticity.

Denoting the small rotation angle by  $\theta$ , we evaluate the associated displacement field as

$$\mathbf{u} = (\theta \mathbf{e}_1) \times \mathbf{r} = (\theta \mathbf{e}_1) \times (x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3) = \theta (x_2 \mathbf{e}_3 - x_3 \mathbf{e}_2) \quad (5.13.1a)$$

or,

$$u_1 = 0, \quad u_2 = -\theta x_3, \quad u_3 = \theta x_2 \quad (5.13.1b)$$

where  $\theta = \theta(x_1)$

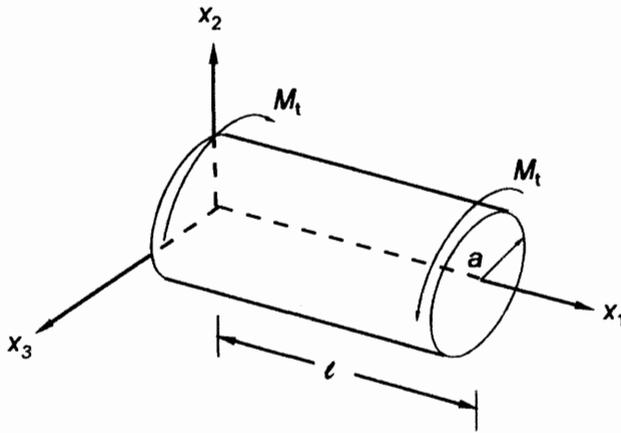


Fig. 5.8

Corresponding to this displacement field are the nonzero strains

$$E_{12} = E_{21} = -\frac{1}{2}x_3 \frac{\partial \theta}{\partial x_1} \tag{5.13.2a}$$

$$E_{13} = E_{31} = \frac{1}{2}x_2 \frac{\partial \theta}{\partial x_1} \tag{5.13.2b}$$

The nonzero stress components are, from Hooke's law

$$T_{12} = T_{21} = -\mu x_3 \frac{\partial \theta}{\partial x_1} \tag{5.13.3a}$$

$$T_{13} = T_{31} = \mu x_2 \frac{\partial \theta}{\partial x_1} \tag{5.13.3b}$$

To determine if this is a possible state of stress in the absence of body forces, we check the equilibrium equations  $\partial T_{ij}/\partial x_j = 0$ . The first equation is identically satisfied, whereas from the second and third equations we have

$$-\mu x_3 \left( \frac{d^2 \theta}{d x_1^2} \right) = 0 \tag{5.13.4a}$$

$$+\mu x_2 \left( \frac{d^2 \theta}{d x_1^2} \right) = 0 \tag{5.13.4b}$$

Thus,

$$\frac{d\theta}{dx_1} \equiv \theta' = \text{constant.} \tag{5.13.5}$$

Interpreted physically, we satisfy equilibrium if the increment in angular rotation (i.e, twist per unit length) is a constant. Now that the displacement field has been shown to generate a possible stress field, we must determine the surface tractions that correspond to the stress field.

On the lateral surface (see Fig. 5.9 ) we have a unit normal vector  $\mathbf{n} = (1/a)(x_2\mathbf{e}_2 + x_3\mathbf{e}_3)$ . Therefore, the surface traction on the lateral surface

$$[\mathbf{t}] = [\mathbf{T}] [\mathbf{n}] = \frac{1}{a} \begin{bmatrix} 0 & T_{12} & T_{13} \\ T_{21} & 0 & 0 \\ T_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{a} \begin{bmatrix} x_2 T_{12} + x_3 T_{13} \\ 0 \\ 0 \end{bmatrix} \tag{5.13.6}$$

Substituting from Eqs. (5.13.3) and (5.13.5), we have

$$\mathbf{t} = \frac{\mu}{a} (-x_2 x_3 \theta' + x_2 x_3 \theta') \mathbf{e}_1 = \mathbf{0} \tag{5.13.7}$$

Thus, in agreement with the fact that the bar is twisted by end moments only, the lateral surface is traction free.

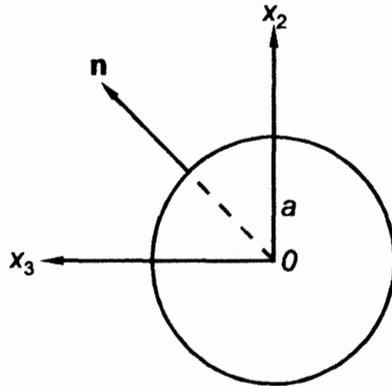


Fig. 5.9

On the face  $x_1 = l$ , we have a unit normal  $\mathbf{n} = \mathbf{e}_1$  and a surface traction

$$\mathbf{t} = \mathbf{T}\mathbf{e}_1 = T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3 \tag{5.13.8}$$

This distribution of surface traction on the end face gives rise to the following resultant (Fig. 5.10)

$$R_1 = \int T_{11} dA = 0 \tag{5.13.9a}$$

$$R_2 = \int T_{21} dA = -\mu \theta' \int x_3 dA = 0 \tag{5.13.9b}$$

$$R_3 = \int T_{31} dA = \mu \theta' \int x_2 dA = 0 \tag{5.13.9c}$$

$$M_1 = \int (x_2 T_{31} - x_3 T_{21}) dA = \mu \theta' \int (x_2^2 + x_3^2) dA = \mu \theta' I_p \tag{5.13.9d}$$

$$M_2 = M_3 = 0 \tag{5.13.9e}$$

where  $I_p = \pi a^4/2$  is the polar moment of inertia of the cross-sectional area. We also note that  $\int x_2 dA = \int x_3 dA = 0$  because the area is symmetrical with respect to the axes.

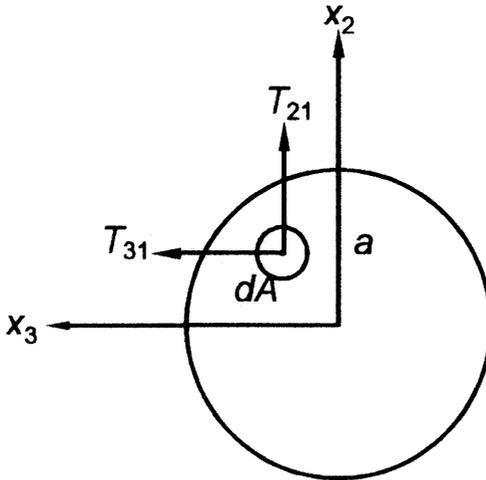


Fig. 5.10

The resultant force system on the face  $x_1 = 0$  will similarly give rise to a counter-balancing couple  $-\mu \theta' I_p$ . Therefore, the resultant force system on either end face is a twisting couple  $M_1 = M_t$  and it induces a twist per unit length given by

$$\theta' = \frac{M_t}{\mu I_p} \tag{5.13.10}$$

This indicates that we can, as indicated in Section 5.1, determine the shear modulus from a simple torsion experiment.

In terms of the twisting couple  $M_t$ , the stress tensor becomes

$$[\mathbf{T}] = \begin{bmatrix} 0 & \frac{M_t x_3}{I_p} & \frac{M_t x_2}{I_p} \\ \frac{-M_t x_3}{I_p} & 0 & 0 \\ \frac{M_t x_2}{I_p} & 0 & 0 \end{bmatrix} \quad (5.13.11)$$

In reality, when a bar is twisted the exact distribution of the applied forces is rarely, if ever known. Invoking St. Venant’s principle, we conclude that as long as the resultants of the applied forces on the two ends of a slender bar are equal and opposite couples of strength  $M_t$ , the state of stress inside the bar is given by Eq. (5.13.11).

Example 5.13.1

For a circular bar of radius  $a$  in torsion (a) find the magnitude and location of the greatest normal and shearing stresses throughout the bar; (b) find the principal direction at the position  $x_2 = 0, x_3 = a$ .

*Solution.* (a) We first evaluate the principal stresses as a function of position by solving the characteristic equation

$$\lambda^3 - \lambda \left( \frac{M_t}{I_p} \right)^2 (x_2^2 + x_3^2) = 0 \quad (i)$$

Thus, the principal values at any point are

$$\lambda = 0 \quad \text{and} \quad \lambda = \pm \frac{M_t}{I_p} (x_2^2 + x_3^2)^{1/2} = \pm \frac{M_t r}{I_p} \quad (ii)$$

where  $r$  is the distance from the axis of the bar.

In this case, the magnitude of the maximum shearing and normal stress at any point are equal and are proportional to the distance  $r$ . Therefore, the greatest shearing and normal stress both occur on the boundary,  $r = a$  with

$$(T_n)_{\max} = (T_s)_{\max} = \frac{M_t a}{I_p} \quad (5.13.12)$$

(b) For the principal value  $\lambda = M_t a/I_p$  at the boundary points  $(x_1, 0, a)$  the eigenvector equation becomes

$$-\frac{M_t a}{I_p} n_1 - \frac{M_t a}{I_p} n_2 = 0 \quad (iii)$$

$$-\frac{M_t a}{I_p} n_1 - \frac{M_t a}{I_p} n_2 = 0 \quad (\text{iv})$$

$$-\frac{M_t a}{I_p} n_3 = 0 \quad (\text{v})$$

Therefore, the eigenvector is given by  $\mathbf{n} = (\sqrt{2}/2)(\mathbf{e}_1 - \mathbf{e}_2)$ . This normal determines a plane perpendicular to the lateral face which makes a  $45^\circ$  angle with the  $x_1$ -axis. Frequently, a crack along a helix inclined at  $45^\circ$  to the axis of a circular cylinder under torsion is observed. This is especially true for brittle materials such as cast iron.

### Example 5.13.2

In Fig. 5.11, a twisting torque  $M_t$  is applied to the rigid disc  $A$ . Find the twisting moments transmitted to the circular shafts on either side of the disc.

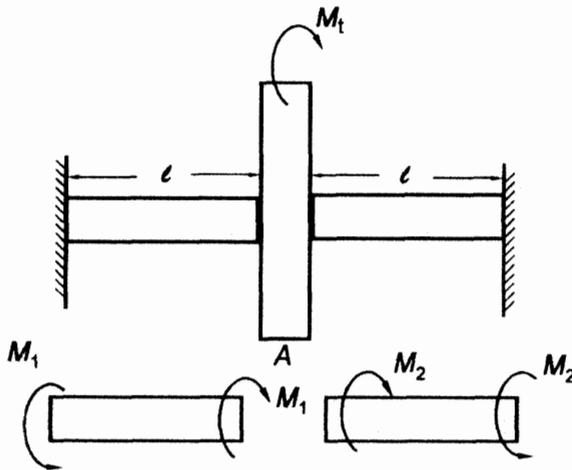


Fig. 5.11

*Solution.* Let  $M_1$  be the twisting moment transmitted to the left shaft and  $M_2$  that to the right shaft. Then, the equilibrium of the disc demands that

$$M_1 + M_2 = M_t \quad (\text{i})$$

In addition, because the disc is assumed to be rigid, the angle of twist of the left and right shaft must be equal:

$$\frac{M_t l_1}{\mu I_p} = \frac{M_t l_2}{\mu I_p} \quad (\text{ii})$$

Thus,

$$M_1 l_1 = M_2 l_2 \quad (\text{iii})$$

From Eqs. (i) and (iii), we then obtain

$$M_1 = \left( \frac{l_2}{l_1 + l_2} \right) M_t, \quad M_2 = \left( \frac{l_1}{l_1 + l_2} \right) M_t \quad (\text{iv})$$

### Example 5.13.3

Consider the angle of twist for a circular cylinder under torsion to be a function of  $x_1$  and time  $t$ , i.e.,  $\theta = \theta(x_1, t)$ .

(a) Determine the differential equation that  $\theta$  must satisfy for it to be a possible solution in the absence of body forces. What are the boundary conditions that  $\theta$  must satisfy (b) if the plane  $x_1 = 0$  is a fixed end; (c) if the plane  $x_1 = 0$  is a free end.

*Solution.* (a) From the displacements

$$u_1 = 0, \quad u_2 = -\theta(x_1, t) x_3, \quad u_3 = \theta(x_1, t) x_2 \quad (\text{i})$$

we find the stress to be

$$T_{12} = T_{21} = 2\mu E_{12} = -\mu x_3 \frac{\partial \theta}{\partial x_1} \quad (\text{ii a})$$

$$T_{13} = T_{31} = 2\mu E_{13} = \mu x_2 \frac{\partial \theta}{\partial x_1} \quad (\text{ii b})$$

and

$$T_{11} = T_{22} = T_{33} = T_{23} = 0 \quad (\text{ii c})$$

The second and third equations of motion give

$$-\mu x_3 \frac{\partial^2 \theta}{\partial x_1^2} = -\rho_0 x_3 \frac{\partial^2 \theta}{\partial t^2} \quad (\text{iii a})$$

$$\mu x_2 \frac{\partial^2 \theta}{\partial x_1^2} = \rho_0 x_2 \frac{\partial^2 \theta}{\partial t^2} \quad (\text{iii b})$$

Therefore,  $\theta(x_1, t)$  must satisfy the equation

$$c_T^2 \frac{\partial^2 \theta}{\partial x_1^2} = \frac{\partial^2 \theta}{\partial t^2} \quad (\text{iv})$$

where  $c_T = \sqrt{\mu/\rho_0}$ .

(b) At the fixed end  $x_1 = 0$ , there is no displacement, therefore,

$$\theta(0, t) = 0 \quad (\text{v})$$

(c) At the traction-free end  $x_1 = 0$ ,  $\mathbf{t} = -\mathbf{T}\mathbf{e}_1 = 0$ . Thus,  $T_{21}|_{x_1=0} = 0$ ,  $T_{31}|_{x_1=0} = 0$ , therefore,

$$\frac{\partial \theta}{\partial x_1}(0, t) = 0 \quad (\text{vi})$$

#### Example 5.13.4

A cylindrical bar of square cross-section (see Fig. 5.12) is twisted by end moments. Show that the displacement field of the torsion of the circular bar does not give a correct solution to this problem.

*Solution.* The displacement field for the torsion of circular cylinders has already been shown to generate an equilibrium stress field. We therefore check if the surface traction of the lateral surface vanishes. The unit vector on the plane  $x_3 = a$  is  $\mathbf{e}_3$ , so that the surface traction for the stress tensor of Eq. (5.13.1) is given by

$$\mathbf{t} = \mathbf{T}\mathbf{e}_3 = T_{13}\mathbf{e}_1 = \frac{M_t x_2}{I_p} \mathbf{e}_1$$

Similarly, there will be surface tractions in the  $\mathbf{e}_1$  direction on the remainder of the lateral surface. Thus, the previously assumed displacement field must be altered. To obtain the actual solution for twisting by end moments only, we must somehow remove these axial surface tractions. As will be seen in the next section, this will cause the cross-sectional planes to warp.

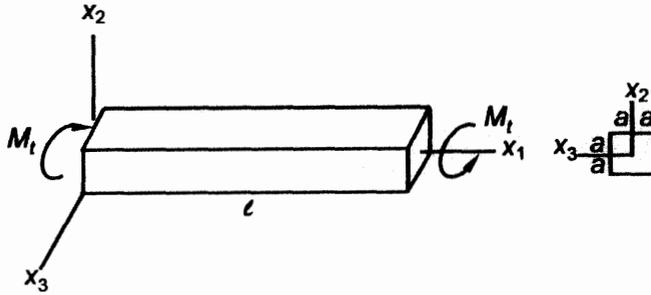


Fig. 5.12

**5.14 Torsion of a Noncircular Cylinder**

For cross-sections other than circular, the simple displacement field of Section 5.13 will not satisfy the tractionless lateral surface boundary condition (see Example 5.13.4). We will show that in order to satisfy this boundary condition, the cross-sections will not remain plane.

We begin by assuming a displacement field that still rotates each cross-section by a small angle  $\theta$ , but in addition there may be a displacement in the axial direction. This warping of the cross-sectional plane will be defined by  $u_1 = \varphi(x_2, x_3)$ . Our displacement field now has the form

$$u_1 = \varphi(x_2, x_3), \quad u_2 = -x_3 \theta(x_1), \quad u_3 = x_2 \theta(x_1) \tag{5.14.1}$$

The associated nonzero strains and stresses are given by

$$T_{12} = T_{21} = 2 \mu E_{12} = -\mu x_3 \theta' + \mu \frac{\partial \varphi}{\partial x_2} \tag{5.14.2a}$$

$$T_{13} = T_{31} = 2 \mu E_{13} = \mu x_2 \theta' + \mu \frac{\partial \varphi}{\partial x_3} \tag{5.14.2b}$$

The second and third equilibrium equations are still satisfied if  $\theta' = \text{constant}$ . However, the first equilibrium equation requires that

$$\frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 \varphi}{\partial x_3^2} = 0 \tag{5.14.3}$$

Therefore, the displacement field of Eq. (5.14.1) will generate a possible state of stress if  $\varphi$  satisfies Eq. (5.14.3). Now, we compute the traction on the lateral surface. Since the bar is

cylindrical, the unit normal to the lateral surface has the form  $\mathbf{n} = n_2\mathbf{e}_2 + n_3\mathbf{e}_3$  and the associated surface traction is given by

$$\begin{aligned} \mathbf{t} = \mathbf{T}\mathbf{n} &= \left[ \mu \theta'(-n_2x_3 + n_3x_2) + \mu \left( \frac{\partial\varphi}{\partial x_2}n_2 + \frac{\partial\varphi}{\partial x_3}n_3 \right) \right] \mathbf{e}_1 \\ &= \left[ \mu \theta'(-n_2x_3 + n_3x_2) + \mu(\nabla\varphi) \cdot \mathbf{n} \right] \mathbf{e}_1 \end{aligned} \tag{i}$$

We require that the lateral surface be traction-free, i.e.,  $\mathbf{t} = 0$ , so that on the boundary the function  $\varphi$  must satisfy the condition

$$\frac{d\varphi}{dn} = (\nabla\varphi) \cdot \mathbf{n} = \theta' (n_2x_3 - n_3x_2) \tag{5.14.4}$$

Equations(5.14.3) and (5.14.4) define a well-known boundary-value problem<sup>†</sup> which is known to admit an exact solution for the function  $\varphi$ . Here, we will only consider the torsion of an elliptic cross-section by demonstrating that

$$\varphi = A x_2x_3 \tag{5.14.5}$$

gives the correct solution.

Taking  $A$  as a constant, this choice of  $\varphi$  obviously satisfy the equilibrium equation [Eq. (5.14.3)]. To check the boundary condition we begin by defining the elliptic boundary by the equation

$$f(x_2, x_3) = \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1 \tag{ii}$$

The unit normal vector is given by

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2}{|\nabla f|} \left[ \frac{x_2}{a^2} \mathbf{e}_2 + \frac{x_3}{b^2} \mathbf{e}_3 \right] \tag{iii}$$

and the boundary condition of Eq. (5.14.4) becomes

$$\left( \frac{\partial\varphi}{\partial x_2} \right) b^2 x_2 + \left( \frac{\partial\varphi}{\partial x_3} \right) a^2 x_3 = \theta' x_2x_3 (b^2 - a^2) \tag{iv}$$

Substituting our choice of  $\varphi$  into this equation, we find that

$$A = \theta' \left( \frac{b^2 - a^2}{a^2 + b^2} \right) \tag{5.14.6}$$

---

<sup>†</sup> It is known as a Neumann problem

Because  $\mathcal{A}$  does turn out to be a constant, we have satisfied both Eq. (5.14.3) and (5.14.4). Substituting the value of  $\varphi$  into Eq. (5.14.2), we obtain the associated stresses

$$T_{21} = T_{12} = -\left(\frac{2\mu a^2}{a^2 + b^2}\right) \theta' x_3 \tag{5.14.7a}$$

$$T_{31} = T_{13} = \left(\frac{2\mu b^2}{a^2 + b^2}\right) \theta' x_2 \tag{5.14.7b}$$

This distribution of stress gives a surface traction on the end face,  $x_1 = l$

$$\mathbf{t} = T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3 \tag{v}$$

and the following resultant force system

$$R_1 = R_2 = R_3 = M_2 = M_3 = 0 \tag{vi}$$

$$\begin{aligned} M_1 &= \int (x_2 T_{31} - x_3 T_{21}) dA = \frac{2\mu \theta'}{a^2 + b^2} \left[ a^2 \int x_3^2 dA + b^2 \int x_2^2 dA \right] \\ &= \frac{2\mu \theta'}{a^2 + b^2} (a^2 I_{22} + b^2 I_{33}) \end{aligned} \tag{vii}$$

Denoting  $M_1 = M_t$  and recalling that for an ellipse  $I_{33} = \pi a^3 b/4$  and  $I_{22} = \pi b^3 a/4$ , we obtain

$$\theta' = \frac{a^2 + b^2}{\pi a^3 b^3 \mu} M_t \tag{5.14.8}$$

Similarly the resultant on the other end face  $x_1 = 0$  will give rise to a counterbalancing couple.

In terms of the twisting moment, the stress tensor becomes

$$[\mathbf{T}] = \begin{bmatrix} 0 & \frac{-2M_t x_3}{\pi a b^3} & \frac{2M_t x_2}{\pi a^3 b} \\ \frac{-2M_t x_3}{\pi a b^3} & 0 & 0 \\ \frac{2M_t x_2}{\pi a^3 b} & 0 & 0 \end{bmatrix} \tag{5.14.9}$$

## Example 5.14.1

For an elliptic cylindrical bar in torsion, (a) find the magnitude of the maximum normal and shearing stress at any point of the bar, and (b) find the ratio of the maximum shearing stresses at the extremities of the elliptic minor and major axes.

*Solution.* As in Example 5.13.1, we first solve the characteristic equation

$$\lambda^3 - \lambda \left( \frac{2M_t}{\pi ab} \right)^2 \left[ \frac{x_2^2}{a^4} + \frac{x_3^2}{b^4} \right] = 0 \quad (\text{i})$$

The principal values are

$$\lambda = 0, \quad \text{and} \quad \lambda = \pm \frac{2M_t}{\pi ab} \left( \frac{x_2^2}{a^4} + \frac{x_3^2}{b^4} \right)^{1/2} \quad (\text{ii})$$

which determines the maximum normal and shearing stresses:

$$(T_s)_{\max} = (T_n)_{\max} = \frac{2M_t}{\pi ab} \left( \frac{x_2^2}{a^4} + \frac{x_3^2}{b^4} \right)^{1/2} \quad (5.14.10)$$

(b) Supposing that  $b > a$ , we have at the end of the minor axis ( $x_2 = a, x_3 = 0$ ),

$$(T_s)_{\max} = \left( \frac{2M_t}{\pi ab} \right) \left( \frac{1}{a} \right) \quad (\text{iii})$$

and at the end of major axis ( $x_2 = 0, x_3 = b$ )

$$(T_n)_{\max} = \left( \frac{2M_t}{\pi ab} \right) \left( \frac{1}{b} \right) \quad (\text{iv})$$

The ratio of the maximum stresses is therefore  $b/a$  and the greater stress occurs at the end of the minor axis.

## 5.15 Pure Bending of a Beam

A beam is a bar acted on by forces or couples in an axial plane, which chiefly cause bending of the bar. When a beam or portion of a beam is acted on by end couples only, it is said to be in **pure bending** or **simple bending**. We shall consider the case of cylindrical bar of arbitrary cross-section that is in pure bending.

Figure 5.13 shows a bar of uniform cross-section. We choose the  $x_1$  axis to pass through the cross-sectional centroids and let  $x_1 = 0$  and  $x_1 = l$  correspond to the left- and right-hand faces of the bar.

For the pure bending problem, we seek the state of stress that corresponds to a tractionless lateral surface and some distribution of normal surface tractions on the end faces that is statically equivalent to bending couples  $\mathbf{M}_R = M_2\mathbf{e}_2 + M_3\mathbf{e}_3$  and  $\mathbf{M}_L = -\mathbf{M}_R$  (note that the  $M_1$  component is absent because  $M_1$  is a twisting couple). Guided by the state of stress associated with simple extension, we tentatively assume that  $T_{11}$  is the only nonzero stress component and that it is an arbitrary function of  $x_1$ .

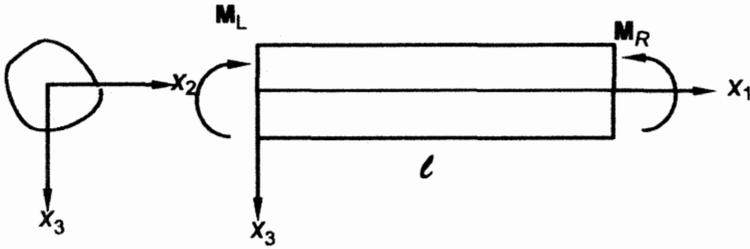


Fig. 5.13

To satisfy equilibrium, we require

$$\frac{\partial T_{11}}{\partial x_1} = 0 \tag{i}$$

i.e.,  $T_{11} = T_{11}(x_2, x_3)$ . The corresponding strains are

$$E_{11} = \frac{1}{E_Y} T_{11}, \quad E_{22} = E_{33} = -\frac{\nu}{E_Y} T_{11}, \tag{ii}$$

$$E_{12} = E_{13} = E_{23} = 0 \tag{iib}$$

Since we have begun with an assumption on the state of stress, we must check whether these strains are compatible. Substituting the strains into the compatibility equations [Eq. (3.16.7-12)] we obtain

$$\frac{\partial^2 T_{11}}{\partial x_2^2} = 0, \quad \frac{\partial^2 T_{11}}{\partial x_3^2} = 0, \quad \frac{\partial^2 T_{11}}{\partial x_3 \partial x_2} = 0 \tag{iii}$$

which can be satisfied only if  $T_{11}$  is at most a linear function of the form

$$T_{11} = \alpha + \beta x_2 + \gamma x_3 \tag{iv}$$

Now that we have a possible stress distribution, let us consider the nature of the boundary tractions. As is the case with simple extension, the lateral surface is obviously traction-free. On the end face  $x_1 = l$ , we have a surface traction

$$\mathbf{t} = \mathbf{T}\mathbf{e}_1 = T_{11} \mathbf{e}_1 \quad (\text{v})$$

which gives a resultant force system

$$R_1 = \int T_{11} dA = \alpha \int dA + \beta \int x_2 dA + \gamma \int x_3 dA = \alpha A \quad (\text{vi})$$

$$R_2 = R_3 = 0 \quad (\text{vii})$$

$$M_1 = 0 \quad (\text{viii})$$

$$\begin{aligned} M_2 &= \int x_3 T_{11} dA = \alpha \int x_3 dA + \beta \int x_2 x_3 dA + \gamma \int x_3^2 dA \\ &= \beta I_{23} + \gamma I_{22} \end{aligned} \quad (\text{ix})$$

$$\begin{aligned} M_3 &= -\int x_2 T_{11} dA = -\alpha \int x_2 dA - \beta \int x_2^2 dA - \gamma \int x_2 x_3 dA \\ &= -\beta I_{33} - \gamma I_{23} \end{aligned} \quad (\text{x})$$

where  $A$  is the cross-sectional area,  $I_{22}$ ,  $I_{33}$ , and  $I_{23}$  are the moments and product of inertia of the cross-sectional area. On the face  $x_1 = 0$ , the resultant force system is equal and opposite to that given above.

we will set  $\alpha = 0$  to make  $R_1 = 0$  so that there is no axial forces acting at the end faces. We now assume, without any loss in generality, that we have chosen the  $x_2$  and  $x_3$  axis to coincide with the principal axes of the cross-sectional area (e.g., along lines of symmetry) so that  $I_{23} = 0$ . In this case, from Eqs. (ix) and (x), we have  $\beta = -M_3/I_{33}$  and  $\gamma = M_2/I_{22}$  so that the stress distribution for the cylindrical bar is given by

$$T_{11} = \frac{M_2}{I_{22}} x_3 - \frac{M_3}{I_{33}} x_2 \quad (5.15.1)$$

and all other  $T_{ij} = 0$ .

To investigate the nature of the deformation that is induced by bending moments, for simplicity we let  $M_3 = 0$ . The corresponding strains are

$$E_{11} = \frac{M_2}{I_{22} E_Y} x_3, \quad E_{22} = E_{33} = -\frac{\nu M_2}{I_{22} E_Y} x_3 \quad (5.15.2a)$$

$$E_{12} = E_{13} = E_{23} = 0 \quad (5.15.2b)$$

These equations can be integrated (we are assured that this is possible since the strains are compatible) to give the following displacement field:

$$u_1 = \frac{M_2}{E_Y I_{22}} x_1 x_3 - \alpha_3 x_2 + \alpha_2 x_3 + \alpha_4 \quad (5.15.3a)$$

$$u_2 = -\nu \frac{M_2}{E_Y I_{22}} x_2 x_3 + \alpha_3 x_1 - \alpha_1 x_3 + \alpha_5 \quad (5.15.3b)$$

$$u_3 = -\frac{M_2}{2E_Y I_{22}} [x_1^2 - \nu (x_2^2 - x_3^2)] - \alpha_2 x_1 + \alpha_1 x_2 + \alpha_6 \quad (5.15.3c)$$

where  $\alpha_i$  are constants of integration. In fact,  $\alpha_4$ ,  $\alpha_5$ ,  $\alpha_6$  define an overall rigid body translation of the bar and  $\alpha_1$ ,  $\alpha_2$ ,  $\alpha_3$  being constant parts of the antisymmetric part of the displacement gradient, define an overall small rigid body rotation. For convenience, we let all the  $\alpha_i = 0$  [ note that this corresponds to requiring  $\mathbf{u} = \mathbf{0}$  and  $(\nabla u)^A = \mathbf{0}$  at the origin ]. The displacements are therefore,

$$u_1 = \frac{M_2}{E_Y I_{22}} x_1 x_3, \quad u_2 = -\frac{\nu M_2}{E_Y I_{22}} x_2 x_3 \quad (5.15.4a)$$

$$u_3 = -\frac{M_2}{2E_Y I_{22}} [x_1^2 - \nu (x_2^2 - x_3^2)] \quad (5.15.4b)$$

Considering the cross-sectional plane  $x_1 = \text{constant}$ , we note that the displacement perpendicular to the plane is given by

$$u_1 = \left( \frac{M_2 x_1}{E_Y I_{22}} \right) x_3 \quad (5.15.5)$$

Since  $u_1$  is a linear function of  $x_3$ , the cross-sectional plane remains plane and is rotated about the  $x_2$  axis (see Fig. 5.14) by an angle

$$\theta \approx \tan \theta = \frac{u_1}{x_3} = \frac{M_2 x_1}{E_Y I_{22}} \quad (5.15.6)$$

In addition, consider the displacement of the material that is initially along the  $x_1$  axis ( $x_2 = x_3 = 0$ )

$$u_1 = u_2 = 0, \quad u_3 = -\frac{M_2 x_1^2}{2E_Y I_{22}} \quad (5.15.7)$$

The displacement of this material element ( often called the neutral axis or neutral fiber ) is frequently used to define the deflection of the beam. Note that since

$$-\frac{du_3}{dx_1} = \frac{M_2 x_1}{E_Y I_{22}} = \tan \theta \quad (5.15.8)$$

the cross-sectional planes remain perpendicular to the neutral axis. This is a result of the absence of shearing stress in pure bending.

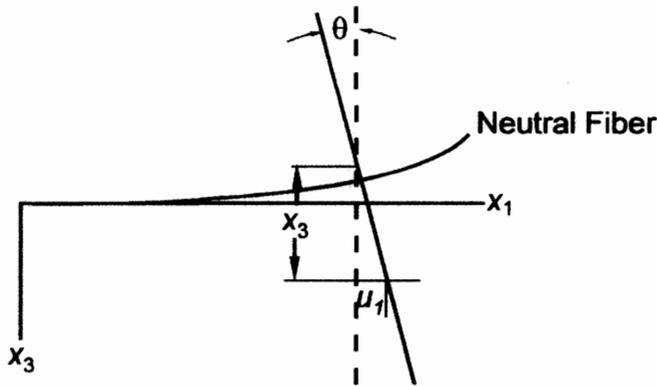


Fig. 5.14

Example 5.15.1

Figure 5.15 shows the right end face of a rectangular beam of width 15 cm and height 20 cm. The beam is subjected to pure bending couples at its ends. The right-hand couple is given as  $M = 7000e_2$  Nm. Find the greatest normal and shearing stresses throughout the beam.

*Solution.* We have

$$T_{11} = \frac{M_2 x_3}{I_{22}} \tag{i}$$

and the remaining stress components vanish. Therefore, at any point

$$(T_n)_{\max} = \frac{M_2 x_3}{I_{22}} \tag{ii}$$

and

$$(T_s)_{\max} = \frac{M_2 x_3}{2I_{22}} \tag{iii}$$

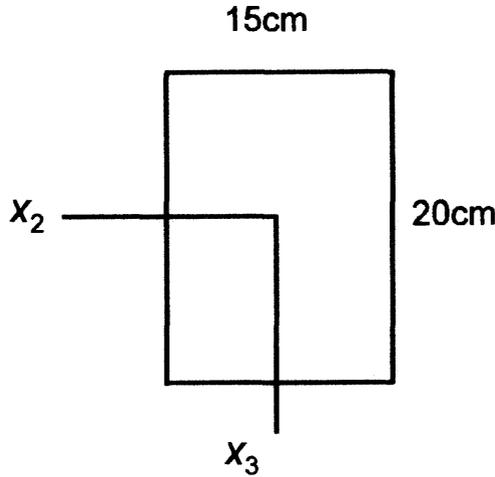


Fig. 5.15

The greatest value will be at the boundary, i.e.,  $x_3 = 10^{-1}$  m. To obtain a numerical answer, we have

$$I_{22} = \frac{1}{12} (15 \times 10^{-2}) (20 \times 10^{-2})^3 = 10^{-4} \text{ m}^4$$

and the greatest stresses are

$$(T_n)_{\max} = \frac{(7000)(10^{-1})}{10^{-4}} = 7 \times 10^6 \text{ Pa.}$$

$$(T_s)_{\max} = 3.5 \times 10^6 \text{ Pa}$$

Example 5.15.2

For the beam of Example 5.15.1, if the right end couple is  $M = 7000 (\mathbf{e}_2 + \mathbf{e}_3)$  Nm and the left end couple is equal and opposite, find the maximum normal stress.

*Solution.* We have

$$I_{33} = 0.563 \times 10^{-4} \text{ m}^4, \quad I_{22} = 10^{-4} \text{ m}^4$$

$$T_{11} = \frac{M_2 x_3}{I_{22}} - \frac{M_3 x_2}{I_{33}} = (70x_3 - 124x_2) \times 10^6 \text{ Pa}$$

The maximum normal stress occurs at  $x_2 = -7.5 \times 10^{-2}$  m and  $x_3 = 10^{-1}$  m with

$$T_{11} = 16.3 \text{ MPa}$$

### 5.16 Plane Strain

If the deformation of a cylindrical body is such that there is no axial components of the displacement and that the other components do not depend on the axial coordinate, then the body is said to be in a state of plane strain. Such a state of strain exists for example in a cylindrical body whose end faces are prevented from moving axially and whose lateral surface are acted on by loads that are independent of the axial position and without axial components.

Letting the  $e_3$  direction correspond to the cylindrical axis, we have

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \quad u_3 = 0 \quad (5.16.1)$$

The strain components corresponding to this displacement field are:

$$E_{11} = \frac{\partial u_1}{\partial x_1}, \quad E_{22} = \frac{\partial u_2}{\partial x_2}, \quad E_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \quad (5.16.2a)$$

$$E_{13} = E_{23} = E_{33} = 0 \quad (5.16.2b)$$

and the nonzero stress components are  $T_{11}$ ,  $T_{12}$ ,  $T_{22}$ ,  $T_{33}$ , where

$$T_{33} = \nu(T_{11} + T_{22}) \quad (5.16.3)$$

This last equation is obtained from the Hooke's law, Eq. (5.4.8c) and the fact that  $E_{33} = 0$  for the plane strain problem.

Considering a static stress field with no body forces, the equilibrium equations reduce to

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} = 0. \quad (5.16.4a)$$

$$\frac{\partial T_{12}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} = 0 \quad (5.16.4b)$$

$$\frac{\partial T_{33}}{\partial x_3} = 0 \quad (5.16.4c)$$

Because  $T_{33} = T_{33}(x_1, x_2)$ , the third equation is trivially satisfied. It can be easily verified that for any arbitrary scalar function  $\varphi$ , if we compute the stress components from the following equations

$$T_{11} = \frac{\partial^2 \varphi}{\partial x_2^2}, \quad T_{12} = -\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}, \quad T_{22} = \frac{\partial^2 \varphi}{\partial x_1^2} \quad (5.16.5)$$

then the first two equations are automatically satisfied. However, not all stress components obtained this way are acceptable as a possible solution because the strain components derived from them may not be compatible; that is, there may not exist displacement components which

correspond to the strain components. To ensure the compatibility of the strain components, we obtain the strain components in terms of  $\varphi$  from Hooke's law Eqs. (5.4.8) [and using Eq. (5.16.3)]

$$E_{11} = \frac{1}{E_Y} [(1 - \nu^2)T_{11} - \nu(1 + \nu)T_{22}] = \frac{1}{E_Y} [(1 - \nu^2) \frac{\partial^2 \varphi}{\partial x_2^2} - \nu(1 + \nu) \frac{\partial^2 \varphi}{\partial x_1^2}] \quad (5.16.6a)$$

$$E_{22} = \frac{1}{E_Y} [(1 - \nu^2)T_{22} - \nu(1 + \nu)T_{11}] = \frac{1}{E_Y} [(1 - \nu^2) \frac{\partial^2 \varphi}{\partial x_1^2} - \nu(1 + \nu) \frac{\partial^2 \varphi}{\partial x_2^2}] \quad (5.16.6b)$$

$$E_{12} = \frac{1}{E_Y} (1 + \nu)T_{12} = -\frac{1}{E_Y} (1 + \nu) \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} \quad (5.16.6c)$$

$$E_{13} = E_{23} = E_{33} = 0 \quad (5.16.d)$$

and substitute them into the compatibility equations, Eqs. (3.16.7) to (3.16.12). For plane strain problems, the only compatibility equation that is not automatically satisfied is

$$\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2} = 2 \frac{\partial^2 E_{12}}{\partial x_1 \partial x_2} \quad (5.16.7)$$

Thus, we obtain the following equation governing the scalar function  $\varphi$ :

$$(1 - \nu) \left( \frac{\partial^4 \varphi}{\partial x_1^4} + 2 \frac{\partial^4 \varphi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \varphi}{\partial x_2^4} \right) = 0$$

i.e.,

$$\frac{\partial^4 \varphi}{\partial x_1^4} + 2 \frac{\partial^4 \varphi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \varphi}{\partial x_2^4} = 0 \quad (5.16.8)$$

Any function  $\varphi$  which satisfies Eq. (5.16.8) generates a possible elastic solution. In particular, any third degree polynomial (generating a linear stress and strain field) may be utilized. The stress function  $\varphi$  defined by Eqs. (5.16.5) and satisfying Eq. (5.16.8), is called the **Airy Stress Function**.

We can also obtain from the Hooke's law [Eq. (5.16.6)], the compatibility equation [Eq. (5.16.7)] and the equations of equilibrium [Eqs. (5.16.4)] the following: [See Prob. 5.77]

$$\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) (T_{11} + T_{22}) = 0 \quad (5.16.9)$$

which may also be written as

$$\nabla^2 (T_{11} + T_{22}) = 0 \quad (5.16.10)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \tag{iiic}$$

Example 5.16.1

Consider the Airy stress function

$$\varphi(x_1, x_2) = \frac{\beta}{6} x_2^3 \tag{i}$$

- (a) Obtain the stresses for the state of plane strain;
- (b) If the stresses of part(a) are those inside a rectangular bar bounded by  $x_1 = 0$ ,  $x_1 = l$ ,  $x_2 = \pm(h/2)$  and  $x_3 = \pm(b/2)$ , find the surface tractions on the boundaries
- (c) If the boundary surfaces  $x_3 = \pm(b/2)$  are traction-free, find the solution.

*Solution.* (a) From Eq. (5.16.5)

$$T_{11} = \beta x_2, \quad T_{22} = 0, \quad T_{33} = \nu \beta x_2 \tag{iiia}$$

$$T_{12} = T_{13} = T_{23} = 0 \tag{iiib}$$

that is,

$$[\mathbf{T}] = \begin{bmatrix} \beta x_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \nu \beta x_2 \end{bmatrix} \tag{iic}$$

(b) On the face  $x_1 = 0$ ,  $\mathbf{t} = \mathbf{T}(-\mathbf{e}_1) = -\beta x_2 \mathbf{e}_1$  (iiia)

On the face  $x_1 = l$ ,  $\mathbf{t} = \mathbf{T}(\mathbf{e}_1) = \beta x_2 \mathbf{e}_1$  (iiib)

On the faces  $x_2 = \pm(h/2)$ ,  $\mathbf{t} = \mathbf{T}(\pm \mathbf{e}_2) = 0$  (iiic)

On the faces  $x_3 = \pm(b/2)$ ,  $\mathbf{t} = \mathbf{T}(\pm \mathbf{e}_3) = \pm \nu \beta x_2 \mathbf{e}_3$  (iiid)

We note that the surface normal stress on the side faces  $x_3 = \pm(b/2)$  are required to prevent them from moving in the  $x_3$  direction.

(c) In order to obtain the solution for the case where the side faces  $x_3 = \pm(b/2)$  are traction-free (and therefore have non zero  $u_3$ ), it is necessary to remove the normal stresses from these side faces. Let us consider the following state of stress

$$[\mathbf{T}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\nu \beta x_2 \end{bmatrix} \quad (\text{iv})$$

This state of stress is obviously a possible state of stress because it clearly satisfies the equations of equilibrium in the absence of body forces and the stress components, being linear in  $x_2$ , give rise to strain components that are also linear in  $x_2$  so that the compatibility conditions are also satisfied. Superposing this state of stress to that of part (a), that is, adding Eq. (iic) and Eq. (iv) we obtain

$$[\mathbf{T}] = \begin{bmatrix} \beta x_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{v})$$

We note that this is the exact solution for pure bending of the bar with couple vectors parallel to the direction of  $\mathbf{e}_3$ .

In this example, we have easily obtained, from the plane strain solution where the side faces  $x_3 = \pm (b/2)$  of the rectangular bar are prevented from moving normally, the state of stress for the same rectangular bar where the side faces are traction-free, by simply removing the component  $T_{33}$  of the plane strain solution. This is possible for this problem because the  $T_{33}$  obtained in the plane strain solution of part (a) happens to be a linear function of the coordinates.

### Example 5.16.2

Consider the state of stress given by

$$[\mathbf{T}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & G(x_1, x_2) \end{bmatrix} \quad (\text{i})$$

Show that the most general form of  $G(x_1, x_2)$  which gives rise to a possible state of stress in the absence of body force is

$$G(x_1, x_2) = \alpha x_1 + \beta x_2 + \gamma. \quad (\text{ii})$$

*Solution.* The strain components are

$$E_{11} = -\frac{\nu}{E_Y} G(x_1, x_2) = E_{22} \quad (\text{iiia})$$

$$E_{33} = \frac{1}{E_Y} G(x_1, x_2) \quad (\text{iiib})$$

$$E_{12} = E_{13} = E_{23} = 0 \tag{iiic}$$

From the compatibility equations, Eqs. (3.16.8), (3.16.9) and (3.16.7), we have

$$\frac{\partial^2 G}{\partial x_2^2} = 0, \quad \frac{\partial^2 G}{\partial x_1^2} = 0, \quad \frac{\partial^2 G}{\partial x_1 \partial x_2} = 0 \tag{iv}$$

Thus,  $G(x_1, x_2) = \alpha x_1 + \beta x_2 + \gamma$ . In the absence of body forces, the equations of equilibrium are obviously satisfied.

Example 5.16.3

Consider the stress function  $\varphi = \alpha x_1 x_2^3 + \beta x_1 x_2$

- (a) Is this an allowable stress function?
- (b) Determine the associated stresses for the plane strain case.
- (c) Determine  $\alpha$  and  $\beta$  in order to solve the plane strain problem of a cantilever beam with end load  $P$  (Fig. 5.16).

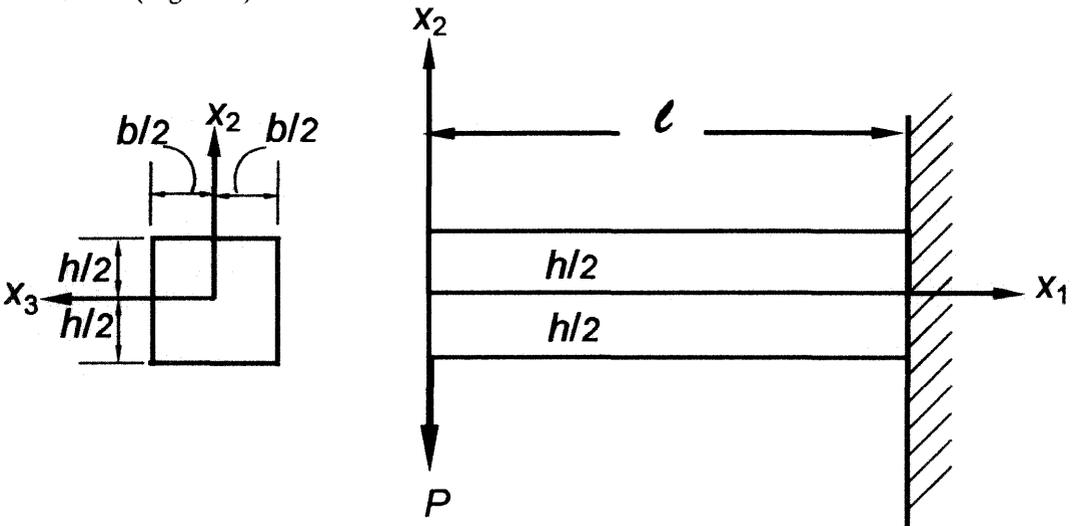


Fig. 5.16

- (d) If the faces  $x_3 = \pm b/2$  are traction-free, are the stress components given in (b) still valid for this case if we simply remove  $T_{33}$  from them?

*Solution.* (a) Yes, because the stress function satisfies Eq. (5.16.8) exactly.

(b) The stress components are

$$T_{11} = \frac{\partial^2 \varphi}{\partial x_2^2} = 6\alpha x_1 x_2, T_{22} = \frac{\partial^2 \varphi}{\partial x_1^2} = 0, T_{12} = -\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = -\beta - 3\alpha x_2^2 \quad (i)$$

i.e., for the plane strain problem

$$[\mathbf{T}] = \begin{bmatrix} 6\alpha x_1 x_2 & -\beta - 3\alpha x_2^2 & 0 \\ -\beta - 3\alpha x_2^2 & 0 & 0 \\ 0 & 0 & -6\nu\alpha x_1 x_2 \end{bmatrix} \quad (ii)$$

(c) On the boundaries,  $x_2 = \pm h/2$ , the tractions are

$$\mathbf{t} = \pm(T_{22}\mathbf{e}_2 + T_{12}\mathbf{e}_1) = \pm\left(-\beta - \frac{3\alpha h^2}{4}\right)\mathbf{e}_1 \quad (iii)$$

But, we wish the lateral surface ( $x_2 = \pm h/2$ ) to be traction-free, therefore

$$\beta = -\frac{3h^2}{4}\alpha \quad (iv)$$

On the boundary  $x_1 = 0$ ,

$$\mathbf{t} = -\mathbf{T}\mathbf{e}_1 = (\beta + 3\alpha x_2^2)\mathbf{e}_2 \quad (v)$$

This shearing traction can be made equipollent to an applied load  $P\mathbf{e}_2$  by setting

$$-P = \beta \int dA + 3\alpha \int x_2^2 dA = \beta A + 3\alpha I$$

where  $A = bh$  and  $I = bh^3/12$ . Substituting for  $\beta$ , we have

$$P = \alpha \left( \frac{3}{4}bh^3 - \frac{bh^3}{4} \right) = \left( \frac{bh^3}{2} \right) \alpha$$

Therefore,  $\alpha = 2P/bh^3, \beta = -3P/2bh$  and the stresses are

$$T_{11} = \frac{12P}{bh^3}x_1 x_2 = \frac{P}{I}x_1 x_2$$

$$T_{12} = \frac{3P}{2A} - \frac{P}{2I}x_2^2$$

In order that the state of plane strain is achieved, it is necessary to have normal tractions acting on the side faces  $x_3 = \pm b/2$ . The tractions are in fact  $\mathbf{t} = \pm T_{33}\mathbf{e}_3 = \pm 6\nu\alpha x_1 x_2 \mathbf{e}_2$ .

(d) Since  $T_{33}$  is not a linear function of the coordinates  $x_1$  and  $x_2$ , from example 5.16.2, we see that we cannot simply remove  $T_{33}$  from the plane strain solution to arrive at a the stress

state for the beam where the side faces  $x_3 = \pm b/2$  are traction free. However, if  $b$  is very very small, then it seems reasonable to expect that the application of  $-T_{33}$  on these side face alone will result in a state of stress inside the body which is essentially given by

$$[\mathbf{T}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -T_{33} \end{bmatrix} \quad (\text{iv})$$

(Indeed it can be proved that the errors incurred in this equation approach zero with the second power of  $b$  as  $b$  approaches zero). Thus, the state of stress obtained in part (b), with  $T_{33}$  taken to be zero, is the state of stress inside a thin beam under the same external loading as that in the plane strain case. Such a state of stress is known as the state of **plane stress** where the stress matrix given by

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (5.16.11)$$

The strain field corresponding to the plane stress state is given by

$$E_{11} = \frac{1}{E_Y} [T_{11} - \nu T_{22}], \quad E_{22} = \frac{1}{E_Y} [T_{22} - \nu T_{11}], \quad E_{33} = -\frac{\nu}{E_Y} (T_{11} + T_{22})$$

$$E_{12} = \frac{1}{E_Y} (1 + \nu) T_{12}, \quad E_{13} = E_{23} = 0 \quad (5.16.12)$$

### 5.17 Plane Strain Problem in Polar Coordinates

In Polar coordinates, the strain components in plane strain problem are, [with  $T_{zz} = \nu (T_{rr} + T_{\theta\theta})$ ],

$$E_{rr} = \frac{1}{E_Y} [(1 - \nu^2) T_{rr} - \nu (1 + \nu) T_{\theta\theta}]$$

$$E_{\theta\theta} = \frac{1}{E_Y} [(1 - \nu^2) T_{\theta\theta} - \nu (1 + \nu) T_{rr}]$$

$$E_{r\theta} = \frac{(1 + \nu)}{E_Y} T_{r\theta}$$

$$E_{rz} = E_{\theta z} = E_{zz} = 0 \quad (5.17.1)$$

The equations of equilibrium are [see Eqs. (4.8.1)], (noting that there is no  $z$  dependence).

$$\frac{1}{r} \frac{\partial(r T_{rr})}{\partial r} + \frac{1}{r} \frac{\partial T_{r\theta}}{\partial \theta} - \frac{T_{\theta\theta}}{r} = 0 \quad (5.17.2a)$$

$$\frac{1}{r^2} \frac{\partial(r^2 T_{\theta r})}{\partial r} + \frac{1}{r} \frac{\partial T_{\theta\theta}}{\partial \theta} = 0 \quad (5.17.2b)$$

The third equation is automatically satisfied, because  $T_{z\theta} = T_{rz} = 0$  and  $T_{zz}$  is not a function of  $z$ .

It can be easily verified that the equations of equilibrium Eq. (5.17.2a) are identically satisfied if

$$T_{rr} = \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} \quad (5.17.3a)$$

$$T_{\theta\theta} = \frac{\partial^2 \varphi}{\partial r^2} \quad (5.17.3b)$$

$$T_{r\theta} = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \varphi}{\partial \theta} \right) \quad (5.17.3c)$$

where  $\varphi$  is the Airy stress function. In Section 5.16, we see that in order to satisfy the compatibility conditions, the Cartesian stress components  $T_{11} + T_{22}$  must satisfy Eq. (5.16.9), i.e.,

$$\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) (T_{11} + T_{22}) = 0 \quad (5.17.4)$$

To derive the equivalent expression in cylindrical coordinates, we note that  $T_{11} + T_{22}$  is the first scalar invariant of the stress tensor. Therefore

$$T_{11} + T_{22} = T_{rr} + T_{\theta\theta} = \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial r^2} \quad (5.17.5)$$

Also, the Laplacian operator  $\nabla^2 = (\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2)$  takes the following form in polar coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

Thus, the function  $\varphi$  must satisfy the biharmonic equation

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r} \frac{\partial \varphi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \varphi}{\partial \theta^2} \right) = 0 \quad (5.17.6)$$

If  $\varphi$  is a function of  $r$  only, we have,

$$T_{rr} = \frac{1}{r} \frac{d\varphi}{dr}, \quad T_{\theta\theta} = \frac{d^2\varphi}{dr^2}, \quad T_{r\theta} = 0 \quad (5.17.7)$$

and

$$\frac{d^4\varphi}{dr^4} + \frac{2d^3\varphi}{r dr^3} - \frac{1d^2\varphi}{r^2 dr^2} + \frac{1d\varphi}{r^3 dr} = 0 \quad (5.17.8)$$

The general solution of this equation is [See Prob. 5.78]

$$\varphi = A \ln r + Br^2 \ln r + Cr^2 + D \quad (5.17.9)$$

The stress field corresponding to this stress function is

$$T_{rr} = \frac{A}{r^2} + B(1 + 2 \ln r) + 2C \quad (5.17.10a)$$

$$T_{\theta\theta} = -\frac{A}{r^2} + B(3 + 2 \ln r) + 2C \quad (5.17.10b)$$

$$T_{r\theta} = 0 \quad (5.17.10c)$$

and the strain components are:

$$E_{rr} = \frac{1}{E_Y} \left[ \frac{(1+\nu)A}{r^2} + (1-3\nu-4\nu^2)B + 2(1-\nu-2\nu^2)B \ln r + 2(1-\nu-2\nu^2)C \right] \quad (5.17.11a)$$

$$E_{\theta\theta} = \frac{1}{E_Y} \left[ -\frac{(1+\nu)A}{r^2} + (3-\nu-4\nu^2)B + 2(1-\nu-2\nu^2)B \ln r + 2(1-\nu-2\nu^2)C \right] \quad (5.17.11b)$$

$$E_{r\theta} = 0 \quad (5.17.11c)$$

Since

$$E_{rr} = \frac{\partial u_r}{\partial r}, \quad E_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \quad (5.17.12a)$$

$$E_{r\theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \quad (5.17.12b)$$

the displacement components can be obtained by integrating the above equations. They are [See Prob. 5.79], (ignoring the terms that represent rigid body displacements)

$$u_r = \frac{1}{E_Y} \left[ -\frac{(1+\nu)A}{r} - B(1+\nu)r + 2B(1-\nu-2\nu^2)r \ln r + 2C(1-\nu-2\nu^2)r \right] \quad (5.17.13a)$$

$$u_{\theta} = \frac{4Br\theta}{E_Y} (1 - \nu^2) \quad (5.17.13b)$$

### 5.18 Thick-walled Circular Cylinder under Internal and External Pressure

Consider a circular cylinder subjected to the action of an internal pressure  $p_i$  and an external pressure  $p_o$ . The boundary conditions for the plane strain problem are:

$$T_{rr} = -p_i \quad \text{at } r = a \quad (5.18.1a)$$

$$T_{rr} = -p_o \quad \text{at } r = b \quad (5.18.1b)$$

These boundary conditions can be easily shown to be satisfied by the following stress field

$$T_{rr} = \frac{A}{r^2} + 2C, \quad T_{\theta\theta} = -\frac{A}{r^2} + 2C, \quad T_{r\theta} = 0 \quad (5.18.2)$$

These components of stress are taken from Eq. (5.17.10) with  $B = 0$  and represent therefore, a possible state of stress for the plane strain problem, where  $T_{zz} = \nu (T_{rr} + T_{\theta\theta})$ . We note that if  $B$  is not taken to be zero, then  $u_{\theta} = \frac{4Br\theta}{E_Y} (1 - \nu^2)$  which is not acceptable because if we start from a point at  $\theta = 0$ , trace a circuit around the origin and return to the same point,  $\theta$  becomes  $2\pi$  and the displacement at the point takes on a different value. Now applying the boundary conditions given in Eqs. (5.18.1), we find that

$$T_{rr} = -p_i \frac{(b^2/r^2) - 1}{(b^2/a^2) - 1} - p_o \frac{1 - (a^2/r^2)}{1 - (a^2/b^2)} \quad (5.18.3a)$$

$$T_{\theta\theta} = p_i \frac{(b^2/r^2) + 1}{(b^2/a^2) - 1} - p_o \frac{1 + (a^2/r^2)}{1 - (a^2/b^2)} \quad (5.18.3b)$$

$$T_{r\theta} = 0 \quad (5.18.3c)$$

We note that if only the internal pressure  $p_i$  is acting,  $T_{rr}$  is always a compressive stress and  $T_{\theta\theta}$  is always a tensile stress.

The above stress components together with  $T_{zz} = \nu (T_{rr} + T_{\theta\theta})$  constitute the exact plane strain solution for the cylinder whose axial end faces are fixed.

As discussed in the last section, the state of stress given by Eqs. (5.18.3) above and with  $T_{zz} = 0$ , can also be regarded as an approximation to the problem of a cylinder which is very thin in the axial direction, under the action of internal and external pressure with traction-free end faces. However, the strain field is not given by Eq. (5.17.11), which is for the plane strain case. For the plane stress case,

$$E_{rr} = \frac{1}{E_Y}(T_{rr} - \nu T_{\theta\theta}), \quad E_{\theta\theta} = \frac{1}{E_Y}(T_{\theta\theta} - \nu T_{rr}) \quad (5.18.4b)$$

$$E_{zz} = \frac{\nu}{E_Y}(T_{rr} + T_{\theta\theta}), \quad E_{r\theta} = \frac{(1 + \nu)}{E_Y} T_{r\theta} \quad (5.18.4c)$$

### Example 5.18.1

Consider a thick-wall cylinder subjected to the action of external pressure  $p_o$  only. If the outer radius is much much larger than the inner radius. What is the stress field?

*Solution.* From Eqs. (5.18.3), we have

$$T_{rr} = -p_o \frac{1 - (a^2/r^2)}{1 - (a^2/b^2)}$$

$$T_{\theta\theta} = -p_o \frac{1 + (a^2/r^2)}{1 - (a^2/b^2)}$$

$$T_{r\theta} = 0$$

When  $b$  is much much larger than  $a$ , these become

$$T_{rr} = -p_o[1 - (a^2/r^2)] \quad (5.18.5a)$$

$$T_{\theta\theta} = -p_o[1 + (a^2/r^2)] \quad (5.18.5b)$$

$$T_{r\theta} = 0 \quad (5.18.5c)$$

## 5.19 Pure Bending of a Curved Beam

Fig. 5.17 shows a curved beam whose boundary surfaces are given by  $r = a$ ,  $r = b$ ,  $\theta = \pm\alpha$  and  $z = \pm h/2$ . The boundary surface  $r = a$ ,  $r = b$  and  $z = \pm h/2$  are traction-free. Assuming the dimension  $h$  is very small compared with the other dimensions, we wish to obtain a plane stress solution for this curved beam under the action of equal and opposite bending couples acting on the faces  $\theta = \pm\alpha$ .

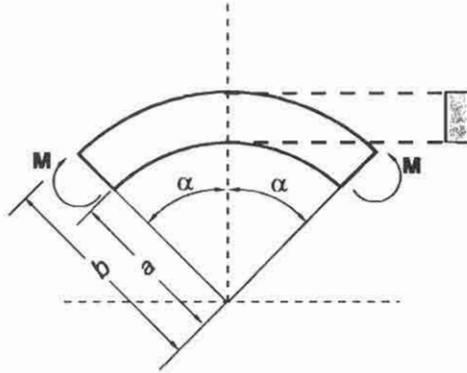


Fig. 5.17

In the following we shall show that the state of stresses given in Eqs. (5.17.10) together with  $T_{zz} = 0$  can be used to give the desired solution. The stress components are:

$$T_{rr} = \frac{A}{r^2} + B(1 + 2 \ln r) + 2C \quad (5.19.1a)$$

$$T_{\theta\theta} = \frac{-A}{r^2} + B(3 + 2 \ln r) + 2C \quad (5.19.1b)$$

$$T_{r\theta} = 0 \quad (5.19.1c)$$

Since the surfaces  $r = a$  and  $r = b$  are traction-free, the constants  $A$ ,  $B$  and  $C$  must satisfy

$$0 = \frac{A}{a^2} + B(1 + 2 \ln a) + 2C \quad (5.19.2a)$$

$$0 = \frac{A}{b^2} + B(1 + 2 \ln b) + 2C \quad (5.19.2b)$$

On the face  $\theta = \alpha$ , there is a distribution of normal stress  $T_{\theta\theta}$  given by Eq. (5.19.1b). Let us compute the resultant of this distribution of the normal stresses:

$$R = \int_a^b T_{\theta\theta} h \, dr = h \left[ \frac{A}{r} + B(r + 2r \ln r) + 2Cr \right]_a^b \quad (5.19.3)$$

In view of Eqs. (5.19.2), we have

$$R = 0$$

That is, the resultant of the distribution of normal stresses must be a couple. Let the moment of this couple per unit width be  $M$  as shown in Fig. 5.17, then

$$-M = \int_a^b T_{\theta\theta} r \, dr$$

i.e.,

$$-M = \left[ -A \ln \frac{b}{a} + B(b^2 - a^2) + B(b^2 \ln b - a^2 \ln a) + C(b^2 - a^2) \right] \quad (5.19.4a)$$

In view of Eqs. (5.19.2), Eq. (5.19.4) can also be written as [see Prob. 5.80]

$$-M = \left[ -A \ln \frac{b}{a} - B(b^2 \ln b - a^2 \ln a) - C(b^2 - a^2) \right] \quad (5.19.4b)$$

Equations (5.19.2a) (5.19.2b) and (5.19.4) are three equations for the three constants  $A, B$  and  $C$ . We obtain,

$$A = -\frac{4M}{N} a^2 b^2 \ln \frac{b}{a} \quad B = -\frac{2M}{N} (b^2 - a^2) \quad (5.19.5a)$$

$$C = \frac{M}{N} [b^2 - a^2 + 2(b^2 \ln b - a^2 \ln a)] \quad (5.19.5b)$$

where

$$N = (b^2 - a^2)^2 - 4a^2 b^2 \left( \ln \frac{b}{a} \right)^2 \quad (5.19.5c)$$

Thus

$$T_{rr} = -\frac{4M}{N} \left( \frac{a^2 b^2}{r^2} \ln \frac{b}{a} + b^2 \ln \frac{r}{b} + a^2 \ln \frac{a}{r} \right) \quad (5.19.6a)$$

$$T_{\theta\theta} = -\frac{4M}{N} \left( \frac{-a^2 b^2}{r^2} \ln \frac{b}{a} + b^2 \ln \frac{r}{b} + a^2 \ln \frac{a}{r} + b^2 - a^2 \right) \quad (5.19.6b)$$

$$T_{r\theta} = 0 \quad (5.19.6c)$$

### 5.20 Stress Concentration due to a Small Circular Hole in a Plate under Tension

Fig. 5.18 shows a plate with a small circular hole of radius  $a$  subjected to the actions of uniform tensile stress of magnitude  $\sigma$  on the faces perpendicular to the  $x$  direction. Let us

consider the region between two concentric circles:  $r = a$  and  $r = b$ . The surface  $r = a$  is traction-free, i.e.,

$$T_{rr} = 0, T_{r\theta} = 0 \quad \text{at } r = a. \tag{i}$$

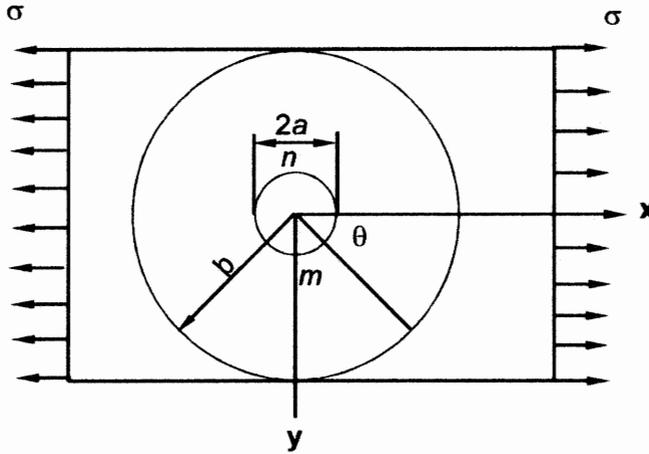


Fig. 5.18

If  $b$  is much larger than  $a$ , then the effect of the small hole will be negligible on points lying on the surface  $r = b$  so that the state of stress at  $r = b$  as  $a/b \rightarrow 0$  will be that due to the uniaxial tensile stress  $\sigma$  in the absence of the hole. In Cartesian coordinates, this state of stress is  $T_{11} = \sigma$  with all other stress components zero. In cylindrical coordinates this same state of stress has the following nonzero stress components

$$T_{rr} = \frac{\sigma}{2} + \frac{\sigma}{2} \cos 2\theta, \quad T_{\theta\theta} = \frac{\sigma}{2} - \frac{\sigma}{2} \cos 2\theta, \quad T_{r\theta} = -\frac{\sigma}{2} \sin 2\theta \tag{5.20.1}$$

Thus, the stress vector acting on the surface  $r = b$  has the  $r$ -component and  $\theta$ -component given by

$$T_{rr} = \frac{\sigma}{2} + \frac{\sigma}{2} \cos 2\theta \tag{5.20.2a}$$

and

$$T_{r\theta} = -\frac{\sigma}{2} \sin 2\theta \tag{5.20.2b}$$

Therefore, the solution to the problem at hand can be obtained as follows: Find the elastically possible equilibrium plane stress field which satisfies the boundary conditions: (i) at

$r = b (>>a)$  ,  $T_{rr}$  is given by Eq. (5.20.2a) and  $T_{r\theta}$  is given by Eq. (5.20.2b) and (ii) at  $r = a$   $T_{rr} = T_{r\theta} = 0$ .

First, we shall demonstrate that the stress field generated from the Airy stress function in the form of  $\varphi = f(r) \cos 2\theta$ , can be used to give a stress field which satisfies the boundary conditions

$$\text{at } r = b, \quad T_{rr} = \frac{\sigma}{2} \cos 2\theta, \quad T_{r\theta} = -\frac{\sigma}{2} \sin 2\theta \tag{5.20.3a}$$

$$\text{at } r = a, \quad T_{rr} = 0, \quad T_{r\theta} = 0 \tag{5.20.3b}$$

Then, to this stress field, we will superpose the stress field

$$T_{rr} = \frac{\sigma}{2} \left(1 - \frac{a^2}{r^2}\right) \quad T_{\theta\theta} = \frac{\sigma}{2} \left(1 + \frac{a^2}{r^2}\right) \quad T_{r\theta} = 0 \tag{5.20.4}$$

which is the solution for a hollow cylinder with a very thick wall (i.e.,  $b/a \rightarrow \infty$ ), acted on by a uniform radial traction  $\frac{\sigma}{2}$  on the outer surface  $r=b$  only [see Eqs. (5.18.5) in Example 5.18.1].

In this way, the boundary conditions Eqs. (5.20.2) can be satisfied..

Substituting

$$\varphi = f(r) \cos 2\theta \tag{5.20.5}$$

into the equation governing the Airy's stress function, Eq. (5.17.6), i.e.,

$$\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \varphi = 0$$

we obtain that the function  $f(r)$  must satisfy the following equation

$$\left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{4}{r^2} \right) \left( \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \frac{4f}{r^2} \right) = 0 \tag{5.20.6}$$

The general solution for this equation is [see Prob. 5.81]

$$f(r) = A r^2 + B r^4 + C \frac{1}{r^2} + D \tag{5.20.7}$$

Thus,

$$\varphi = \left( A r^2 + B r^4 + C \frac{1}{r^2} + D \right) \cos 2\theta \tag{5.20.8}$$

and the corresponding stress components are [see Eqs. (5.17.3)]

$$T_{rr} = - \left( 2A + \frac{6C}{r^4} + \frac{4D}{r^2} \right) \cos 2\theta \tag{5.20.9a}$$

$$T_{\theta\theta} = \left(2A + 12Br^2 + \frac{6C}{r^4}\right) \cos 2\theta \quad (5.20.9b)$$

$$T_{r\theta} = \left(2A + 6Br^2 - \frac{6C}{r^4} + \frac{2D}{r^2}\right) \sin 2\theta \quad (5.20.9c)$$

Using Eqs. (5.20.9), the boundary conditions (5.20.3) become

$$2A + \frac{6C}{b^4} + \frac{4D}{b^2} = -\frac{\sigma}{2} \quad (5.20.10a)$$

$$2A + 6Bb^2 - \frac{6C}{b^4} - \frac{2D}{b^2} = -\frac{\sigma}{2} \quad (5.20.10b)$$

$$2A + \frac{6C}{a^4} + \frac{4D}{a^2} = 0 \quad (5.20.10c)$$

$$2A + 6Ba^2 - \frac{6C}{a^4} - \frac{2D}{a^2} = 0 \quad (5.20.10d)$$

As  $b \rightarrow \infty$ , Eq. (5.20.10a) becomes  $2A = -\frac{\sigma}{2}$ , so that  $A = -\frac{\sigma}{4}$ , Eq. (5.20.10b) becomes  $6Bb^2 = 0$  so that  $B = 0$  and Eqs. (5.20.10c) and (d) become

$$-\frac{\sigma}{2} + \frac{6C}{a^4} + \frac{4D}{a^2} = 0$$

$$-\frac{\sigma}{2} - \frac{6C}{a^4} - \frac{2D}{a^2} = 0$$

Thus,

$$A = -\frac{\sigma}{4}, \quad B = 0, \quad C = -\frac{a^4}{4}\sigma, \quad D = \frac{a^2}{2}\sigma \quad (5.20.11)$$

Substituting these values into Eqs. (5.20.9) and superpose them onto the stress field given in Eq. (5.20.4), we obtain

$$T_{rr} = \frac{\sigma}{2} \left(1 - \frac{a^2}{r^2}\right) + \frac{\sigma}{2} \left(1 + \frac{3a^4}{r^4} - \frac{4a^2}{r^2}\right) \cos 2\theta \quad (5.20.12a)$$

$$T_{\theta\theta} = \frac{\sigma}{2} \left(1 + \frac{a^2}{r^2}\right) - \frac{\sigma}{2} \left(1 + \frac{3a^4}{r^4}\right) \cos 2\theta \quad (5.20.12b)$$

$$T_{r\theta} = -\frac{\sigma}{2} \left(1 - \frac{3a^4}{r^4} + \frac{2a^2}{r^2}\right) \sin 2\theta \quad (5.20.12c)$$

Putting  $r = a$  in these equations, we find that

$$T_{rr} = 0, \quad T_{r\theta} = 0, \quad T_{\theta\theta} = \sigma - 2\sigma \cos 2\theta \tag{5.20.13}$$

We see therefore, at  $\theta = \frac{\pi}{2}$  (point m in Fig. 5. 18) and at  $\theta = \frac{3\pi}{2}$  (point n in Fig. 5.18),  $T_{\theta\theta} = 3\sigma$ . This is the maximum tensile stress which is three times the uniform stress  $\sigma$  applied at the ends of the plate. This is referred to as a stress concentration.

**5.21 Hollow Sphere Subjected to Internal and External Pressure**

Let the internal and external radii of the hollow sphere be denoted by  $a_i$  and  $a_o$  respectively and let the internal pressure be  $p_i$  and the external pressure be  $p_o$ , both pressures are assumed to be uniform. With respect to the spherical coordinates  $(r, \theta, \varphi)$ , it is clear that due to the spherical symmetry of the geometry and the loading that each particle of the elastic sphere will experience only a radial displacement whose magnitude depends only on  $r$ , that is,

$$u_r = u(r), \quad u_\theta = 0, \quad u_\varphi = 0 \tag{5.21.1}$$

substituting Eq. (5.21.1) into the Navier equation of equilibrium in spherical coordinates, Eqs. (5.6.4) in the absence of body forces, we obtain

$$(\lambda + \mu) \frac{de}{dr} + \mu \left[ \frac{d}{dr} \left( \frac{1}{r^2} \frac{d}{dr} (r^2 u) \right) \right] = 0 \tag{5.21.2a}$$

where, see Eq. (5.6.3g)

$$e = \frac{du}{dr} + \frac{2u}{r} \tag{5.21.2b}$$

Thus,

$$(\lambda + 2\mu) \frac{d}{dr} \left( \frac{du}{dr} + \frac{2u}{r} \right) = 0 \tag{5.21.3}$$

The general solution of the above equation is

$$u = A r + \frac{B}{r^2} \tag{5.21.4}$$

The stress components corresponding to this displacement field can be obtained from Eqs. (5.6.3), with  $e = 3A$  :

$$T_{rr} = \lambda e + 2\mu \frac{du}{dr} = (3\lambda + 2\mu)A - \frac{4\mu B}{r^3} \tag{5.21.5a}$$

$$T_{\theta\theta} = T_{\varphi\varphi} = \lambda e + \frac{2\mu u}{r} = (3\lambda + 2\mu)A + \frac{2\mu B}{r^3} \tag{5.21.5b}$$

$$T_{r\phi} = T_{\phi\theta} = T_{r\theta} = 0 \tag{5.21.5c}$$

To determine the constants A and B, we use the boundary conditions:

$$T_{rr} = -p_i \quad \text{at } r = a_i \tag{5.21.6a}$$

$$T_{rr} = -p_o \quad \text{at } r = a_o \tag{5.21.6b}$$

i.e.,

$$(3\lambda + 2\mu)A - \frac{4\mu B}{a_i^3} = -p_i \tag{5.21.7a}$$

$$(3\lambda + 2\mu)A - \frac{4\mu B}{a_o^3} = -p_o \tag{5.21.7b}$$

Thus,

$$A = \frac{p_i a_i^3 - p_o a_o^3}{(3\lambda + 2\mu)(a_o^3 - a_i^3)}, \quad B = \frac{a_i^3 a_o^3 (p_i - p_o)}{4\mu (a_o^3 - a_i^3)} \tag{5.21.8}$$

and the stress components become

$$T_{rr} = \frac{p_i a_i^3 - p_o a_o^3}{(a_o^3 - a_i^3)} - \frac{a_i^3 a_o^3}{r^3} \frac{p_i - p_o}{(a_o^3 - a_i^3)} \tag{5.21.9a}$$

$$T_{\theta\theta} = T_{\phi\phi} = \frac{p_i a_i^3 - p_o a_o^3}{a_o^3 - a_i^3} + \frac{a_i^3 a_o^3}{2r^3} \frac{(p_i - p_o)}{(a_o^3 - a_i^3)} \tag{5.21.9b}$$

We note that the stresses are not dependent on the elastic properties.

## Part B Linear Anisotropic Elastic Solid

### 5.22 Constitutive Equations for Linearly Anisotropic Elastic Solid

In Section 5. 2, we concluded that due to the symmetry of the strain and the stress tensors  $E_{ij}$  and  $T_{ij}$  respectively, and the assumption that there exists a strain energy function  $U$  given by  $U = 1/2 C_{ijkl} E_{ij} E_{kl}$ , the most general anisotropic elastic solid requires 21 elastic constants for its description. We can write the stress-strain relation for this general case in the following matrix notation:

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1113} & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{2213} & C_{1222} \\ C_{1133} & C_{2233} & C_{3333} & C_{2333} & C_{1333} & C_{1233} \\ C_{1123} & C_{2223} & C_{2333} & C_{2323} & C_{2313} & C_{1223} \\ C_{1113} & C_{2213} & C_{1333} & C_{2313} & C_{1313} & C_{1213} \\ C_{1112} & C_{1222} & C_{1233} & C_{1223} & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} \quad (5.22.1)$$

The indices in Eq. (5.22.1) are quite cumbersome, but they emphasize the tensorial character of the tensors  $\mathbf{T}$ ,  $\mathbf{E}$  and  $\mathbf{C}$ . Equation (5.22.1) is often written in the following “contracted form” in which the indices are simplified or “contracted.”

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} \quad (5.22.2)$$

or

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix} \quad (5.22.3a)$$

We note that Eq. (5.22.3a) can also be written in indicial notation

$$T_i = C_{ij} E_j \quad (5.22.3b)$$

However, it must be emphasized that  $C_{ij}$  are not components of a second order tensor and  $T_i$  are not those of a vector.

The matrix  $C$  is known as the **stiffness matrix** for the elastic solid. In the notation of Eq. (5.22.3), the strain energy  $U$  is given by

$$U = \frac{1}{2} [E_1, E_2, E_3, E_4, E_5, E_6] \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ E_5 \\ E_6 \end{bmatrix} \quad (5.22.4)$$

We require that the strain energy  $U$  be a positive definite function of the strain components. That is, it is zero if and only if all strain components are zero and otherwise it is positive. Thus, the stiffness matrix is said to be a positive definite matrix which has among its properties : (1) All diagonal elements are positive, i.e.,  $C_{ii} > 0$  (no sum on  $i$ )<sup>†</sup> (2) the determinant of  $C$  is positive, i.e.  $\det C > 0$ , and (3) its inverse  $S = C^{-1}$  exists and is also symmetric and positive definite. (See Example 5.22.1). The matrix  $S$  ( the inverse of  $C$  ) is known as the **compliance matrix**.

As already mentioned in the beginning of this chapter, the assumption of the existence of a strain energy function is motivated by the concept of elasticity which implies that all strain states of an elastic body requires positive work to be done on it and the work is completely used to increase the strain energy of the body.

Example 5.22.1

Show that (a)  $C_{ii} > 0$  (no sum on  $i$ ) (b) the determinant of  $C$  is positive (c) the inverse of  $C$  is symmetric and (d) the inverse is positive definite, (e) the submatrices

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix}, \begin{bmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{bmatrix}, \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

etc. are positive definite.

*Solution.* (a) Consider the case where only  $E_1$  is nonzero, all other  $E_i = 0$ , then the strain energy is  $U = \frac{1}{2} C_{11} E_1^2$ . Since  $U > 0$ , therefore  $C_{11} > 0$ . Similarly if we consider the case where  $E_2$  is nonzero, all other  $E_i = 0$ , then  $U = \frac{1}{2} C_{22} E_2^2$  and  $C_{22} > 0$  etc.

---

<sup>†</sup> An obvious consequence of these restrictions is that in uniaxial loading, a positive strain gives rise to a positive stress and vice versa.

(b) Since the diagonal elements are positive, the eigenvalues of  $\mathbf{C}$  are all positive. Thus, the determinant of  $\mathbf{C}$  is positive (and nonzero) and the inverse of  $\mathbf{C}$  exists.

(c) From  $\mathbf{C} \mathbf{C}^{-1} = \mathbf{I}$ ,  $(\mathbf{C} \mathbf{C}^{-1})^T = \mathbf{I}$ , thus,  $(\mathbf{C}^{-1})^T \mathbf{C}^T = \mathbf{I}$ , i.e.,  $(\mathbf{C}^{-1})^T = (\mathbf{C}^T)^{-1}$ .

Now,  $\mathbf{C} = \mathbf{C}^T$ , therefore  $\mathbf{C}^{-1} = (\mathbf{C}^T)^{-1} = (\mathbf{C}^{-1})^T$  and  $\mathbf{C}^{-1}$  is symmetric.

(d) Since  $\mathbf{C}$  is positive definite, therefore,  $\mathbf{a} \cdot \mathbf{C} \mathbf{a} > 0$  for any nonzero  $\mathbf{a}$ . Let  $\mathbf{b} = \mathbf{C} \mathbf{a}$  and consider  $\mathbf{b} \cdot \mathbf{C}^{-1} \mathbf{b}$ . We have

$$\mathbf{b} \cdot \mathbf{C}^{-1} \mathbf{b} = \mathbf{C} \mathbf{a} \cdot \mathbf{C}^{-1} \mathbf{C} \mathbf{a} = \mathbf{C} \mathbf{a} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{C} \mathbf{a} > 0$$

(e) Consider the case where only  $E_1$  and  $E_2$  are not zero, then from Eq. (5.22.4)

$$2U = [E_1 \ E_2] \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} > 0$$

That is, the sub-matrix is indeed positive definite. We note that since the inverse of this submatrix is positive definite therefore, the submatrix  $\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix}$  is also positive definite. Now

$$S_{11} = \frac{C_{11}}{\Delta_1}$$

where

$$\Delta_1 = C_{11} C_{22} - C_{12}^2$$

Since both  $C_{11}$  and  $S_{11}$  are positive, therefore  $C_{11} C_{22} - C_{12}^2 > 0$ .

Similarly, the positive definiteness of the submatrix

$$\begin{bmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{bmatrix}$$

can be proved by considering the case where only  $E_2$  and  $E_3$  are nonzero and the positive definiteness of the matrix

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

can be proved by considering the case where only  $E_1, E_2$  and  $E_3$  are nonzero, etc.

Thus, we see that the determinant of  $\mathbf{C}$  and of all submatrices whose diagonal elements are diagonal elements of  $\mathbf{C}$  are all positive definite, and similarly the determinant of  $\mathbf{S}$  and of all submatrices whose diagonal elements are diagonal elements of  $\mathbf{S}$  are all positive definite.

### 5.23 Plane of Material Symmetry

Let  $S_1$  be a plane whose normal is in the direction of  $\mathbf{e}_1$ . The transformation

$$\mathbf{e}'_1 = -\mathbf{e}_1, \quad \mathbf{e}'_2 = \mathbf{e}_2, \quad \mathbf{e}'_3 = \mathbf{e}_3 \quad (5.23.1a)$$

describes a reflection with respect to the plane  $S_1$ . This transformation can be more conveniently represented by the tensor  $\mathbf{Q}$  in the equation

$$\mathbf{e}'_i = \mathbf{Q}\mathbf{e}_i \quad (5.23.1b)$$

where

$$[\mathbf{Q}] = [\mathbf{Q}_1] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5.23.1c)$$

If the constitutive relations for a material, written with respect to the  $\{\mathbf{e}_i\}$  basis, remain the same under the transformation  $[\mathbf{Q}_1]$ , then we say that the plane  $S_1$  is a **plane of material symmetry** for that material. For a linearly elastic material, material symmetry with respect to the  $S_1$  plane requires that the components of  $C_{ijkl}$  in the equation

$$T_{ij} = C_{ijkl} E_{kl} \quad (5.23.2)$$

be exactly the same as  $C'_{ijkl}$  in the equation

$$T'_{ij} = C'_{ijkl} E'_{kl} \quad (5.23.3)$$

under the transformation Eq. (5.23.1). When this is the case, restrictions are imposed on the components of the elasticity tensor, thereby reducing the number of independent components. Let us first demonstrate this kind of reduction with a simpler example, relating the thermal strain with the rise in temperature.

#### Example 5.23.1

Consider a homogeneous continuum undergoing a uniform temperature change  $\Delta\theta = \theta - \theta_0$ . Let the relation between the thermal strain  $e_{ij}$  and  $\Delta\theta$  be given by

$$e_{ij} = -\alpha_{ij}(\Delta\theta) \quad (i)$$

where  $\alpha_{ij}$  is the thermal expansion coefficient tensor.

(a) If the plane  $S_1$  defined in Eq. (5.23.1) is a plane of symmetry for the thermal expansion property of the material, what restrictions must be placed on the components of  $\alpha_{ij}$ ?

(b) If the planes  $S_2$  and  $S_3$  whose normals are in the direction of  $\mathbf{e}_2$  and  $\mathbf{e}_3$  respectively are also planes of symmetry, what are the additional restrictions? In this case, the material is said to be **orthotropic** with respect to thermal expansion.

(c) If every plane perpendicular to the  $S_3$  plane is a plane of symmetry, what are the additional restrictions? In this case, the material is said to be **transversely isotropic** with respect to thermal expansion.

*Solution.* (a) Using the transformation law [See Eq. (2B.13.1c)]

$$[\alpha]' = [\mathbf{Q}]^T [\alpha] [\mathbf{Q}] \quad (\text{ii})$$

we obtain, with  $\mathbf{Q}_1$  from Eq. (5.23.1c)

$$\begin{aligned} [\alpha]' &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \alpha_{11} & -\alpha_{12} & -\alpha_{13} \\ -\alpha_{21} & \alpha_{22} & \alpha_{23} \\ -\alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \quad (\text{iii}) \end{aligned}$$

The requirement that  $[\alpha]' = [\alpha]$  results in the restriction that

$$\alpha_{12} = -\alpha_{12} = 0, \quad \alpha_{21} = -\alpha_{21} = 0, \quad \alpha_{13} = -\alpha_{13} = 0, \quad \alpha_{31} = -\alpha_{31} = 0 \quad (\text{iv})$$

Thus, only five coefficients are needed to describe the thermo-expansion behavior if there is one plane of symmetry:

$$[\alpha] = [\alpha]' = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & \alpha_{23} \\ 0 & \alpha_{32} & \alpha_{33} \end{bmatrix} \quad (\text{v})$$

(b) Corresponding to the  $S_2$  plane,  $[\mathbf{Q}_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$  (vi)

Thus, from Eq. (ii) and (vi)

$$[\alpha]' = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & -\alpha_{23} \\ 0 & -\alpha_{32} & \alpha_{33} \end{bmatrix} \quad (\text{vii})$$

The requirements that  $[\alpha]' = [\alpha]$  results in

$$\alpha_{23} = \alpha_{32} = 0 \quad (\text{viii})$$

Thus, only three coefficients are needed to describe the thermal expansion behavior if there are two mutually orthogonal planes of symmetry, i.e.,

$$[\boldsymbol{\alpha}]' = [\boldsymbol{\alpha}] = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix} \quad (\text{ix})$$

If the  $S_3$  plane is also a plane of symmetry, then with

$$[\mathbf{Q}_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (\text{x})$$

one obtains from Eq. (ii) and (x) that

$$[\boldsymbol{\alpha}]' = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix} \quad (\text{xi})$$

so that no further reduction takes place. That is, the symmetry with respect to  $S_1$  and  $S_2$  planes automatically ensures the symmetry with respect to the  $S_3$  plane.

(c) All planes that are perpendicular to the  $S_3$  plane have their normals parallel to the plane formed by  $\mathbf{e}_1$  and  $\mathbf{e}_2$ . Let  $\mathbf{e}_1'$  denote the normal to the  $S_\beta$  plane which makes an angle of  $\beta$  with the  $\mathbf{e}_1$  axis and  $\frac{\pi}{2} - \beta$  with the  $\mathbf{e}_2$  axis, then with respect to the following set of prime basis:

$$\begin{aligned} \mathbf{e}_1' &= \cos\beta \mathbf{e}_1 + \sin\beta \mathbf{e}_2 \\ \mathbf{e}_2' &= -\sin\beta \mathbf{e}_1 + \cos\beta \mathbf{e}_2 \\ \mathbf{e}_3' &= \mathbf{e}_3 \end{aligned} \quad (\text{xii})$$

the transformation law Eq. (ii) gives

$$\alpha'_{11} = \cos^2\beta \alpha_{11} + \sin^2\beta \alpha_{22} \quad (\text{xiii a})$$

$$\alpha'_{12} = (\alpha_{22} - \alpha_{11}) \sin\beta \cos\beta \quad (\text{xiii b})$$

$$\alpha'_{13} = 0 \quad (\text{xiii c})$$

$$\alpha'_{22} = \alpha_{11} \sin^2\beta + \alpha_{22} \cos^2\beta \quad (\text{xiii d})$$

$$\alpha'_{23} = 0 \quad (\text{xiii e})$$

$$\alpha'_{33} = \alpha_{33} \quad (\text{xiii f})$$

In obtaining the above equations, we have made use of the fact that  $e_1, e_2, e_3$  are planes of symmetry so that  $\alpha_{12} = \alpha_{21} = \alpha_{13} = \alpha_{31} = \alpha_{23} = \alpha_{32} = 0$ . Now, in addition, since any  $S_\beta$  plane is a plane of symmetry, therefore, [see part (a)]

$$\alpha_{12}' = 0 \tag{xiv}$$

so that from Eq. (xiiib)

$$\alpha_{11} = \alpha_{22} \tag{xv}$$

Thus, only two coefficients are needed to describe the thermal expansion behavior of the a transversely isotropic material.

Finally, if the material is also transversely isotropic with  $e_1$  as its axis of symmetry, then

$$\alpha_{22} = \alpha_{33} \tag{xvi}$$

so that

$$\alpha_{11} = \alpha_{22} = \alpha_{33} \tag{xvii}$$

and the material is isotropic with respect to thermal expansion with only one coefficient for its description.

### 5.24 Constitutive Equation for a Monoclinic Anisotropic Linearly Elastic Solid.

If a linearly elastic solid has one plane of material symmetry, it is called a **monoclinic material**. We shall demonstrate that for such a material there are 13 independent elasticity coefficients.

Let  $e_1$  be normal to the plane of material symmetry  $S_1$ . Then by definition, under the change of basis

$$e_1' = -e_1, \quad e_2' = e_2, \quad e_3' = e_3 \tag{5.24.1a}$$

the components of the fourth order elasticity tensor remain unchanged, i.e.,

$$C_{ijkl}' = C_{ijkl} \tag{i}$$

Now,  $C_{ijkl}' = Q_{mi} Q_{nj} Q_{rk} Q_{sl} C_{mnr}$  [Sect. 2B14], therefore

$$C_{ijkl} = Q_{mi} Q_{nj} Q_{rk} Q_{sl} C_{mnr} \tag{ii}$$

where

$$[Q] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.24.1b}$$

i.e.,  $Q_{11} = -1, Q_{22} = Q_{33} = 1$ , and all other  $Q_{ij} = 0$ . Thus,

$$C_{1112} = Q_{11} Q_{11} Q_{11} Q_{22} C_{1112} + 0 + 0 \dots = (-1)(-1)(-1)(+1) C_{1112} = -C_{1112} \quad (\text{iii})$$

so that

$$C_{1112} = 0 \quad (5.24.2a)$$

Indeed, one can easily see that all  $C_{ijkl}$  with an odd number of the subscript 1 are all zero. That is, among the 21 independent coefficients, the following eight (8) are zero

$$C_{1112} = C_{1113} = C_{1222} = C_{1223} = C_{1233} = C_{1322} = C_{1323} = C_{1333} = 0 \quad (5.24.3)$$

and the constitutive equations involve 13 nonzero independent coefficients. Thus, the stress strain laws for a **monoclinic elastic solid** having the  $x_2 x_3$  plane as the plane of symmetry, are:

$$T_{11} = C_{1111} E_{11} + C_{1122} E_{22} + C_{1133} E_{33} + 2C_{1123} E_{23} \quad (5.24.4a)$$

$$T_{22} = C_{1122} E_{11} + C_{2222} E_{22} + C_{2233} E_{33} + 2C_{2223} E_{23} \quad (5.24.4b)$$

$$T_{33} = C_{1133} E_{11} + C_{2233} E_{22} + C_{3333} E_{33} + 2C_{2333} E_{23}. \quad (5.24.4c)$$

$$T_{23} = C_{1123} E_{11} + C_{2223} E_{22} + C_{2333} E_{33} + 2C_{2323} E_{23} \quad (5.24.4d)$$

$$T_{31} = 2C_{1213} E_{12} + 2C_{1313} E_{13} \quad (5.24.4e)$$

$$T_{12} = 2C_{1212} E_{12} + 2C_{1213} E_{13} \quad (5.24.4f)$$

or

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & C_{2333} & 0 & 0 \\ C_{1123} & C_{2223} & C_{2333} & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & C_{1213} \\ 0 & 0 & 0 & 0 & C_{1213} & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} \quad (5.24.5)$$

Or, in contracted notation, the stiffness matrix is given by

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix} \quad (5.24.6)$$

The coefficients in the stiffness matrix  $C$  must satisfy the conditions [See Sect. 5.22] that each diagonal element  $C_{ii} > 0$  (no sum on  $i$ ) for  $i = 1, 2, \dots, 6$  and the determinant of every submatrix whose diagonal elements are diagonal elements of the matrix  $C$  is positive definite [See Example 5.22.1].

### 5.25 Constitutive Equations for an Orthotropic Linearly Elastic Solid.

If a linearly elastic solid has two mutually perpendicular planes of symmetry, say  $S_1$  plane with unit normal  $e_1$  and  $S_2$  plane with unit normal  $e_2$ , then automatically, the  $S_3$  plane with a normal in the direction of  $e_3$ , is also a plane of material symmetry [see Example 5.25.1 below]. The material is called an **orthotropic material**.

For this solid, the coefficient  $C_{ijkl}$  now must be invariant with respect to the transformation given by Eq. (5.24.1) above as well as the following transformation

$$e_1' = e_1, \quad e_2' = -e_2, \quad e_3' = e_3. \quad (5.25.1)$$

Thus, all those  $C_{ijkl}$  which appear in Eq. (5.24.5) and which have an odd number of the subscript 2 must also be zero. For example

$$C_{1123} = Q_{11} Q_{11} Q_{22} Q_{33} C_{1123} + 0 + 0 \dots = (-1)(-1)(-1)(+1)C_{1123} = -C_{1123} \quad (i)$$

That is, in addition to Eqs. (5.24.3), we also have

$$C_{1123} = C_{2223} = C_{2333} = C_{1213} = 0 \quad (5.25.2)$$

Therefore, there are now only 9 independent coefficients and the constitutive equations become:

$$T_{11} = C_{1111} E_{11} + C_{1122} E_{22} + C_{1133} E_{33} \quad (5.25.3a)$$

$$T_{22} = C_{1122} E_{11} + C_{2222} E_{22} + C_{2233} E_{33} \quad (5.25.3b)$$

$$T_{33} = C_{1133} E_{11} + C_{2233} E_{22} + C_{3333} E_{33}. \quad (5.25.3c)$$

$$T_{12} = 2C_{1212} E_{12} \quad (5.25.3d)$$

$$T_{31} = 2C_{1313} E_{31} \quad (5.25.3e)$$

$$T_{23} = 2C_{2323} E_{23} \quad (5.25.3f)$$

or,

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} \quad (5.25.4)$$

and in contracted notation, the stiffness matrix is

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \quad (5.25.5)$$

where again each diagonal element  $C_{ii} > 0$  (no sum on  $i$ ) for  $i = 1, 2, \dots, 6$  and

$$\det \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} > 0, \quad \det \begin{bmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{bmatrix} > 0$$

and

$$\det \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} > 0$$

#### Example 5.25.1

(a) Show that all the components  $C_{ijkl}$  remain the same under the transformation

$$[Q] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -[I]$$

(b) Let

$$[Q_1] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [Q_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [Q_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Verify that  $[\mathbf{Q}_1][\mathbf{Q}_2] = -[\mathbf{I}][\mathbf{Q}_3]$ .

*Solution.* (a) With

$$Q_{ij} = -\delta_{ij}$$

the equation

$$C'_{ijkl} = Q_{mi} Q_{nj} Q_{rk} Q_{sl} C_{mnrsl}$$

becomes

$$C'_{ijkl} = (-\delta_{mi})(-\delta_{nj})(-\delta_{rk})(-\delta_{sl}) C_{mnrsl} = C_{ijkl}$$

(b)

$$[\mathbf{Q}_1][\mathbf{Q}_2] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

That is

$$[\mathbf{Q}_1][\mathbf{Q}_2] = -[\mathbf{I}][\mathbf{Q}_3]$$

From the results of (a) and (b), we see that if the  $x$ -plane and the  $y$ -plane are planes of material symmetry, then the  $z$ -plane is also a plane of symmetry.

## 5.26 Constitutive Equation for a Transversely Isotropic Linearly Elastic Material

If there exists a plane, say  $S_3$  plane, such that every plane perpendicular to it, is a plane of material symmetry, then the material is called a **transversely isotropic material**. The  $S_3$  plane is called the plane of isotropy and its normal direction  $\mathbf{e}_3$  is the axis of transverse isotropy. Clearly, a transversely isotropic material is also orthotropic.

Let  $S_\beta$  represent a plane whose normal  $\mathbf{e}_1'$  is parallel to the  $S_3$  plane and which makes an angle of  $\beta$  with the  $\mathbf{e}_1$  axis which lies in the  $S_3$  plane. Then, for every angle  $\beta$ , the plane  $S_\beta$  is, by definition, a plane of symmetry. Thus, if  $C'_{ijkl}$  are components of the tensor  $\mathbf{C}$  with respect to the basis  $\mathbf{e}_i'$  given below:

$$\begin{aligned} \mathbf{e}_1' &= \cos\beta \mathbf{e}_1 + \sin\beta \mathbf{e}_2 \\ \mathbf{e}_2' &= -\sin\beta \mathbf{e}_1 + \cos\beta \mathbf{e}_2 \end{aligned} \quad (5.26.1)$$

$$\mathbf{e}_3' = \mathbf{e}_3$$

then, from Eq. (5.24.3), we must have

$$C'_{1113} = C'_{1223} = C'_{1322} = C'_{1333} = 0 \quad (\text{ia})$$

$$C'_{1112} = C'_{1222} = C'_{1233} = C'_{1323} = 0 \quad (\text{ib})$$

We now show that the condition given in Eq. (ia) are automatically satisfied for every  $\beta$  and therefore do not lead to any further restrictions on  $C_{ijkl}$  whereas the conditions given in Eq. (ib) do lead to additional restrictions, in addition to those restricted by orthotropy.

Since

$$Q_{13} = Q_{31} = Q_{23} = Q_{32} = 0 \quad (\text{ii})$$

therefore,

$$\begin{aligned} C'_{1113} = & Q_{11}^3 Q_{13} C_{1111} + Q_{11}^2 Q_{21} Q_{23} C_{1122} + Q_{21}^2 Q_{11} Q_{13} C_{2211} \\ & + Q_{11}^2 Q_{31} Q_{33} C_{1133} + Q_{31}^2 Q_{11} Q_{13} C_{3311} + Q_{11}^2 Q_{21} Q_{23} C_{1212} + Q_{11} Q_{21}^2 Q_{13} C_{1221} \\ & + Q_{21}^2 Q_{11} Q_{13} C_{2121} + Q_{21} Q_{11}^2 Q_{23} C_{2112} + \dots = 0 + 0 + \dots = 0 \end{aligned} \quad (\text{iii})$$

That is,  $C'_{1113} = 0$  is automatically satisfied together with

$$C'_{1223} = C'_{1322} = C'_{1333} = 0 \quad (\text{iv})$$

On the other hand, since  $Q_{33} = 1$ , we have

$$C'_{1323} = Q_{11} Q_{12} C_{1313} + Q_{21} Q_{22} C_{2323} = 0 \quad (\text{v})$$

This requirement leads to

$$\cos \beta \sin \beta (C_{1313} - C_{2323}) = 0 \quad (\text{vi})$$

That is,

$$C_{1313} = C_{2323} \quad (5.26.2)$$

Similarly, the equation  $C'_{1233} = 0$  leads to [See Prob. 5.85]

$$C_{1133} = C_{2233} \quad (5.26.3)$$

Also, from  $C'_{1112} = 0$ , we obtain

$$\begin{aligned} C'_{1112} = & Q_{11}^3 Q_{12} C_{1111} + Q_{21}^3 Q_{22} C_{2222} + Q_{11}^2 Q_{21} Q_{22} C_{1122} + Q_{21}^2 Q_{11} Q_{12} C_{2211} \\ & + Q_{11}^2 Q_{21} Q_{22} C_{1212} + Q_{11} Q_{21}^2 Q_{12} C_{1221} + Q_{21}^2 Q_{11} Q_{12} C_{2121} + Q_{21} Q_{11}^2 Q_{22} C_{2112} \\ = & \cos \beta \sin \beta (-\cos^2 \beta C_{1111} + \sin^2 \beta C_{2222} + \cos^2 \beta C_{1122} - \sin^2 \beta C_{2211} \\ & + \cos^2 \beta C_{1212} - \sin^2 \beta C_{1221} - \sin^2 \beta C_{2121} + \cos^2 \beta C_{2112}) = 0 \end{aligned} \quad (\text{vii})$$

i.e.,

$$-\cos^2\beta C_{1111} + \sin^2\beta C_{2222} + (\cos^2\beta - \sin^2\beta)C_{1122} + 2(\cos^2\beta - \sin^2\beta)C_{1212} = 0 \quad (\text{viii})$$

Similarly, we can obtain from the equation [see Prob. 5.86]  $C_{1222} = 0$

$$-\sin^2\beta C_{1111} + \cos^2\beta C_{2222} - (\cos^2\beta - \sin^2\beta)C_{1122} - 2(\cos^2\beta - \sin^2\beta)C_{1212} = 0 \quad (\text{ix})$$

The addition of Eq. (viii) and Eq. (ix) gives

$$C_{1111} = C_{2222} \quad (5.26.4)$$

and Eq. (ix) then gives

$$C_{1212} = \frac{1}{2}(C_{1111} - C_{1122}) \quad (5.26.5)$$

Thus, the number of independent coefficients reduces to 5 and we have for a transversely isotropic elastic solid with the axis of symmetry in the  $e_3$  direction the following stress strain laws

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{1111} & C_{1133} & 0 & 0 & 0 \\ C_{1133} & C_{1133} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{1313} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1313} & 0 \\ 0 & 0 & 0 & 0 & 0 & (1/2)(C_{1111} - C_{1122}) \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} \quad (5.26.6)$$

and in contracted notation, the stiffness matrix is

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (1/2)(C_{11} - C_{12}) \end{bmatrix} \quad (5.26.7)$$

In the above reduction of the elastic coefficients, we demanded that every  $S_\beta$  plane be a plane of material symmetry so that Eqs. (i) must be satisfied for all  $\beta$ . Equivalently, we can demand that the elastic coefficients  $C_{ijkl}$  be the same as  $C_{ij'kl}$  for all  $\beta$  and achieve the same reductions.

The elements of the stiffness matrix satisfy the conditions:

$$C_{11} > 0, C_{33} > 0, C_{44} > 0, C_{11} - C_{12} > 0 \quad (5.26.7a)$$

$$\det \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{11} \end{bmatrix} = C_{11}^2 - C_{12}^2 > 0 \tag{5.26.7b}$$

and

$$\det \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{11} & C_{13} \\ C_{13} & C_{13} & C_{33} \end{bmatrix} = C_{11}^2 C_{33} - 2C_{12} C_{13}^2 - 2C_{11} C_{13}^2 - C_{33} C_{12}^2 > 0. \tag{5.26.7c}$$

We note also that the stiffness matrix for transverse isotropy has also been written in the following form:

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu_T & \lambda & \lambda + \alpha & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu_T & \lambda + \alpha & 0 & 0 & 0 \\ \lambda + \alpha & \lambda + \alpha & \lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu_T & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_T & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu_L \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} \tag{5.26.8}$$

where we note that there are five constants  $\lambda, \mu_T, \mu_L, \alpha$  and  $\beta$ .

### 5.27 Constitutive Equation for Isotropic Linearly Elastic Solids

The stress strain equations given in the last section is for a transversely isotropic elastic solid whose axis of transverse isotropy is in the  $e_3$  direction. If, in addition,  $e_1$  is also an axis of transverse isotropy, then clearly we have

$$C_{2222} = C_{3333} \quad (C_{22} = C_{33}) \tag{i}$$

$$C_{1122} = C_{1133} \quad (C_{12} = C_{13}) \tag{ii}$$

$$C_{1313} = C_{1212} \quad \left( C_{44} = \frac{(C_{11} - C_{12})}{2} \right) \tag{iii}$$

and the stress strain law is

$$\begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{(C_{11}-C_{12})}{2} \end{bmatrix} \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} \quad (5.27.1)$$

where

$$C_{11} > 0, C_{11} - C_{12} > 0, C_{11}^2 - C_{12}^2 > 0, C_{11}^3 + 2C_{12}^3 - 3C_{11}C_{12}^2 > 0 \quad (5.27.2)$$

The elements  $C_{ij}$  are related to the Lames constants  $\lambda$  and  $\mu$  as follows

$$C_{11} = \lambda + 2\mu, \quad C_{12} = \lambda \quad (5.27.3)$$

## 5.28 Engineering Constants for Isotropic Elastic Solids.

Since the stiffness matrix is positive definite, the stress-strain law given in Eq. (5.27.1) can be inverted to give the strain components in terms of the stress components. They can be written in the following form

$$\begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} \quad (5.28.1)$$

where as we already know from Section 5.4,  $E$  is Young's modulus<sup>†</sup>,  $\nu$  is the Poisson's ratio and  $G$  is the shear modulus and

$$G = \frac{E}{2(1 + \nu)} \quad (5.28.2)$$

The compliance matrix is positive definite, therefore the diagonal elements are all positive, thus

$$E > 0, \quad G > 0 \quad (5.28.3a)$$

$$\det \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} \\ -\frac{\nu}{E} & \frac{1}{E} \end{bmatrix} = \frac{1}{E^2} (1 - \nu^2) > 0, \quad \text{i.e., } \nu^2 < 1 \quad (5.28.3b)$$

and

$$\det \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} \end{bmatrix} = \frac{1}{E^3} (1 - 2\nu^3 - 3\nu^2) = \frac{1}{E^3} (1 - 2\nu)(1 + \nu)^2 > 0 \quad (5.28.3c)$$

i.e.,

$$\nu < \frac{1}{2} \quad (5.28.3d)$$

Thus,

$$-1 < \nu < \frac{1}{2} \quad (5.28.4)$$

## 5.29 Engineering Constants for Transversely Isotropic Elastic Solid

For a transversely isotropic elastic solid, the symmetric stiffness matrix with five independent coefficients can be inverted to give a symmetric compliance matrix with also five independent constants. The compliance matrix is

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<sup>†</sup> To simplify the notation, we drop the subscript  $Y$  from  $E$ .

$$\begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{13}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} \quad (5.29.1)$$

The relations between  $C_{ij}$  and the engineering constants can be obtained to be [See Prob. 5.88]

$$C_{11} = \frac{E_1}{1 + \nu_{21}} \frac{1 - \nu_{31}^2(E_1/E_3)}{D} \quad C_{12} = \frac{E_1}{1 + \nu_{21}} \frac{\nu_{21} + \nu_{31}^2(E_1/E_3)}{D} \quad (5.29.2a)$$

$$C_{13} = \frac{E_1 \nu_{31}}{D}, \quad C_{22} = C_{11} \quad C_{23} = C_{13}, \quad C_{44} = G_{13} \quad (5.29.2b)$$

and

$$\frac{C_{11} - C_{12}}{2} = G_{12} \quad (5.29.2c)$$

where

$$D = 1 - \nu_{21} - 2\nu_{31}^2 \left( \frac{E_1}{E_3} \right) > 0 \quad (5.29.2d)$$

From Eq. (5.29.2), it can be obtained easily (See Prob. 5.89)

$$G_{12} = \frac{E_1}{2(1 + \nu_{21})} \quad (5.29.3)$$

According to this Eq. (5.29.1), if  $T_{33}$  is the only nonzero stress component, then

$$E_{33} = \frac{T_{33}}{E_3}, \quad E_{11} = E_{22} = -\frac{\nu_{31}T_{33}}{E_3} = -\nu_{31}E_{33} \quad (i)$$

Thus,  $E_3$  is the Young's modulus in the  $\mathbf{e}_3$  direction (the direction of the axis of transverse isotropy),  $\nu_{31}$  is the Poisson's ratio for the transverse strain in the  $x_1$  or  $x_2$  direction when stressed in the  $x_3$  direction.

If  $T_{11}$  is the only nonzero stress component, then

$$E_{11} = \frac{T_{11}}{E_1}, \quad E_{22} = -\nu_{21} \frac{T_{11}}{E_1} = -\nu_{21} E_{11}, \quad E_{33} = -\frac{\nu_{13} T_{11}}{E_1} = -\nu_{13} E_{11} \quad (\text{ii})$$

and if  $T_{22}$  is the only nonzero stress component, then

$$E_{22} = \frac{T_{22}}{E_1}, \quad E_{11} = -\nu_{21} \frac{T_{22}}{E_1} = -\nu_{21} E_{22}, \quad E_{33} = -\nu_{13} \frac{T_{22}}{E_1} = -\nu_{13} E_{22} \quad (\text{iii})$$

Thus,  $E_1$  is the Young's modulus in the  $\mathbf{e}_1$  and  $\mathbf{e}_2$  directions (i.e., in the plane of isotropy),  $\nu_{21}$  is the Poisson's ratio for the transverse strain in the  $x_2$  direction when stressed in the  $x_1$  direction or transverse strain in the  $x_1$  direction when stressed in the  $x_2$  direction (i.e., Poisson's ratio in the plane of isotropy,  $\nu_{12} = \nu_{21}$ ) and  $\nu_{13}$  is the Poisson's ratio for the transverse strain in the  $\mathbf{e}_3$  direction (the axis of transverse isotropy) when stressed in a direction in the plane of isotropy. We note that since the compliance matrix is symmetric, therefore

$$\frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3} \quad (5.29.4)$$

From  $2E_{23} = \frac{T_{23}}{G_{13}}$ ,  $2E_{31} = \frac{T_{13}}{G_{13}}$  and  $2E_{12} = \frac{T_{12}}{G_{12}}$ , it is clear that  $G_{12}$  is the shear modulus in the plane of transverse isotropy and  $G_{13}$  is the shear modulus in planes perpendicular to the plane of transverse isotropy.

Since the compliance matrix is positive definite, therefore, the diagonal elements are positive definite. That is,

$$E_1 > 0, \quad E_3 > 0, \quad G_{12} > 0, \quad G_{13} > 0 \quad (5.29.5)$$

Also,

$$\det \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_1} \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_1} \end{bmatrix} = \frac{1}{E_1^2} (1 - \nu_{21}^2) > 0 \quad (\text{iv})$$

i.e.,

$$-1 < \nu_{21}^2 < 1 \quad (5.29.6)$$

$$\det \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} \\ -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} \\ -\frac{\nu_{31}}{E_3} & \frac{1}{E_3} \end{bmatrix} = \frac{1}{E_1 E_3} \left( 1 - \nu_{31}^2 \frac{E_1}{E_3} \right) > 0 \quad (\text{v})$$

i.e.,

$$\nu_{31}^2 < \frac{E_3}{E_1} \quad \text{or} \quad \nu_{13}\nu_{31} < 1 \quad (5.29.7)$$

Also,

$$\begin{aligned} \det \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_1} & -\frac{\nu_{31}}{E_3} \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} \end{bmatrix} &= \frac{1}{E_1^2 E_3} \left[ 1 - 2 \left( \frac{E_1}{E_3} \right) \nu_{21} \nu_{31}^2 - 2 \left( \frac{E_1}{E_3} \right) \nu_{31}^2 - \nu_{21}^2 \right] \\ &= \left[ 1 - 2 \left( \frac{E_1}{E_3} \right) \nu_{31}^2 - \nu_{21} \right] (1 + \nu_{21}) > 0 \end{aligned} \quad (vi)$$

Since  $1 + \nu_{21} > 0$ , therefore,

$$1 - 2\nu_{31}^2 \left( \frac{E_1}{E_3} \right) > \nu_{21} \quad \text{or} \quad 1 - 2\nu_{31}\nu_{13} > \nu_{21} \quad (5.28.9)$$

### 5.30 Engineering Constants for Orthotropic Elastic Solid

For an orthotropic elastic solid, the symmetric stiffness matrix with nine independent coefficients can be inverted to give a symmetric compliance matrix with also nine independent constants. The compliance matrix is

$$\begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} \quad (5.30.1)$$

The meanings of the constants in the compliance matrix can be obtained in the same way as in the previous section for the transversely isotropic solid. We have,  $E_1, E_2$  and  $E_3$  are Young's moduli in the  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  directions respectively,  $G_{23}, G_{31}$  and  $G_{12}$  are shear moduli

in the  $x_2x_3$ ,  $x_1x_3$  and  $x_1x_2$  plane respectively and  $\nu_{ij}$  is Poisson's ratio for transverse strain in the  $j$ -direction when stressed in the  $i$ -th direction.

The relationships between  $C_{ij}$  and the engineering constants are given by

$$C_{11} = \frac{1 - \nu_{23} \nu_{32}}{E_2 E_3 \Delta}, \quad C_{12} = \frac{\nu_{21} + \nu_{31} \nu_{23}}{E_2 E_3 \Delta}, \quad C_{23} = \frac{\nu_{31} + \nu_{21} \nu_{32}}{E_2 E_3 \Delta} \quad (5.30.2a)$$

$$C_{22} = \frac{1 - \nu_{13} \nu_{31}}{E_1 E_3 \Delta}, \quad C_{23} = \frac{\nu_{32} + \nu_{12} \nu_{31}}{E_1 E_3 \Delta}, \quad C_{33} = \frac{1 - \nu_{12} \nu_{21}}{E_1 E_2 \Delta} \quad (5.30.2b)$$

$$C_{44} = G_{23}, \quad C_{55} = G_{31}, \quad C_{66} = G_{12} \quad (5.30.2c)$$

where

$$\Delta = \frac{1 - \nu_{12} \nu_{21} - \nu_{23} \nu_{32} - \nu_{31} \nu_{13} - 2\nu_{21} \nu_{32} \nu_{13}}{E_1 E_2 E_3} \quad (5.30.2d)$$

We note also that the compliance matrix is symmetric so that

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \quad \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}, \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3} \quad (5.30.3)$$

Using the same procedures as in the previous sections we can establish the restrictions for the engineering constants:

$$E_1 > 0, \quad E_2 > 0, \quad E_3 > 0, \quad G_{23} > 0, \quad G_{31} > 0, \quad G_{12} > 0 \quad (5.30.4a)$$

$$\nu_{21}^2 < \left( \frac{E_2}{E_1} \right); \quad \nu_{12}^2 < \left( \frac{E_1}{E_2} \right) \quad (5.30.4b)$$

$$\nu_{32}^2 < \left( \frac{E_3}{E_2} \right); \quad \nu_{23}^2 < \left( \frac{E_2}{E_3} \right) \quad (5.30.4c)$$

$$\nu_{13}^2 < \left( \frac{E_1}{E_3} \right); \quad \nu_{31}^2 < \left( \frac{E_3}{E_1} \right) \quad (5.30.4d)$$

Also,

$$1 - \nu_{12} \nu_{21} - \nu_{23} \nu_{32} - \nu_{31} \nu_{13} - 2 \nu_{21} \nu_{32} \nu_{13} > 0 \quad (5.30.4e)$$

### 5.31 Engineering Constants for a Monoclinic Elastic Solid

For a monoclinic elastic solid, the symmetric stiffness matrix with thirteen independent coefficients can be inverted to give a symmetric compliance matrix with also thirteen independent constants. The compliance matrix for the case where the  $e_1$  plane is the plane of symmetry can be written:

$$\begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & \frac{\eta_{41}}{G_4} & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & \frac{\eta_{42}}{G_4} & 0 & 0 \\ \frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & \frac{\eta_{43}}{G_4} & 0 & 0 \\ \frac{\eta_{14}}{E_1} & \frac{\eta_{24}}{E_2} & \frac{\eta_{34}}{E_3} & \frac{1}{G_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_5} & \frac{\mu_{65}}{G_6} \\ 0 & 0 & 0 & 0 & \frac{\mu_{56}}{G_5} & \frac{1}{G_6} \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix} \quad (5.31.1)$$

The symmetry of the compliance matrix requires that

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \quad \frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}, \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}, \quad (5.31.2a)$$

$$\frac{\eta_{14}}{E_1} = \frac{\eta_{41}}{G_4}, \quad \frac{\eta_{24}}{E_2} = \frac{\eta_{42}}{G_4}, \quad \frac{\eta_{34}}{E_3} = \frac{\eta_{43}}{G_4}, \quad \frac{\mu_{56}}{G_5} = \frac{\mu_{65}}{G_6} \quad (5.31.2b)$$

If only  $T_{11}$  is nonzero, then the strain-stress law gives

$$E_{11} = \frac{T_{11}}{E_1}, \quad \nu_{12} = -\frac{E_{22}}{E_{11}}, \quad \nu_{13} = -\frac{E_{33}}{E_{11}}, \quad 2E_{23} = \eta_{14}E_{11} \quad (i)$$

and if only  $T_{22}$  is nonzero, then

$$E_{22} = \frac{T_{22}}{E_2}, \quad \nu_{21} = -\frac{E_{11}}{E_{22}}, \quad \nu_{23} = -\frac{E_{33}}{E_{22}}, \quad 2E_{23} = \eta_{24}E_{22} \quad (ii)$$

etc. Thus,  $E_1$ ,  $E_2$  and  $E_3$  are Young's modulus in the  $x_1$ ,  $x_2$  and  $x_3$  direction respectively and again,  $\nu_{ij}$  is Poisson's ratio for transverse strain in the  $j$ -direction when stressed in the  $i$ -direction. We note also, for the monoclinic elastic solid with  $e_1$  plane as its plane of symmetry, a uniaxial stress in the  $x_1$  direction, or  $x_2$  direction, produces a shear strain in the  $x_2x_3$  plane also, with  $\eta_{ij}$  as the coupling coefficients.

If only  $T_{12} = T_{21}$  are nonzero, then,

$$T_{12} = 2G_6E_{12} \quad \text{and} \quad 2E_{31} = \mu_{65}\frac{T_{12}}{G_6} \quad (iii)$$

and if only  $T_{13} = T_{31}$  are nonzero, then,

$$T_{13} = 2G_5 E_{13} \quad \text{and} \quad 2E_{12} = \mu_{56} \frac{T_{31}}{G_5} \quad (\text{iv})$$

Thus  $G_6$  is the shear modulus in the plane of  $x_1 x_2$  and  $G_5$  is the shear modulus in the plane of  $x_1 x_3$ . Note also that the shear stresses in the  $x_1 x_2$  plane produce shear strain in the  $x_1 x_3$  plane and vice versa with  $\mu_{ij}$  representing the coupling coefficients.

Finally if only  $T_{23} = T_{32}$  are nonzero,

$$E_{11} = \eta_{41} \frac{T_{23}}{G_4}, \quad E_{22} = \eta_{42} \frac{T_{23}}{G_4}, \quad E_{33} = \eta_{43} \frac{T_{23}}{G_4}, \quad T_{23} = 2G_4 E_{23} \quad (\text{v})$$

We see that  $G_4$  is the shear modulus in the plane of  $x_2 x_3$  plane, and the shear stresses in this plane produces normal strains in the three coordinate directions, with  $\eta_{ij}$  representing normal stress-shear stress coupling.

Obviously, due to the positive definiteness of the compliance matrix, all the Young's moduli and the shear moduli are positive. Other restrictions regarding the engineering constants can be obtained in the same way as in the previous section.

## Part C Constitutive Equation for Isotropic Elastic Solid Under Large Deformation

### 5.32 Change of Frame

In classical mechanics, an observer is defined as a rigid body with a clock. In the theory of continuum mechanics, an observer is often referred to as a **frame**. One then speaks of "a change of frame" to mean the transformation between the pair  $\{\mathbf{x}, t\}$  in one frame to the pair  $\{\mathbf{x}^*, t^*\}$  of a different frame, where  $\mathbf{x}$  is the position vector of a material point as observed by the un-starred frame and  $\mathbf{x}^*$  is that observed by the starred frame and  $t$  and  $t^*$  are times in the two frames. Since the two frames are rigid bodies, the most general change of frame is given by [See Section 3.6]

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_0) \quad (5.32.1a)$$

$$t^* = t - a \quad (5.32.1b)$$

where  $\mathbf{c}(t)$  represents the relative displacement of the base point  $\mathbf{x}_0$ ,  $\mathbf{Q}(t)$  is a time-dependent orthogonal tensor, representing a rotation and possibly reflection also (the reflection is included to allow for the observers to use different handed coordinate systems),  $a$  is a constant.

It is important to note that a change of frame is different from a change of coordinate system. Each frame can perform any number of coordinate transformations within itself, whereas a transformation between two frames is given by Eqs. (5.32).

The distance between two material points is called a **frame-indifferent** (or objective) scalar because it is the same for any two observers. On the other hand, the speed of a material point obviously depends on the observers as the observers in general move relative to each other. The speed is therefore not frame indifferent (non-objective). We see therefore, that while a scalar is by definition coordinate-invariant, it is not necessarily frame-indifferent (or frame-invariant).

The position vector and the velocity vector of a material point are obviously dependent on the observer. They are examples of vectors that are not frame indifferent. On the other hand, the vector connecting two material points, and the relative velocity of two material points are examples of frame indifferent vectors.

Let the position vector of two material points be  $\mathbf{x}_1, \mathbf{x}_2$  in the unstarred frame and  $\mathbf{x}_1^*, \mathbf{x}_2^*$  in the starred frame, then we have from Eq. (5.32.1a)

$$\mathbf{x}_1^* = \mathbf{c} + \mathbf{Q}(t)(\mathbf{x}_1 - \mathbf{x}_0) \quad (i)$$

$$\mathbf{x}_2^* = \mathbf{c} + \mathbf{Q}(t)(\mathbf{x}_2 - \mathbf{x}_0) \quad (ii)$$

Thus,

$$\mathbf{x}_1^* - \mathbf{x}_2^* = \mathbf{Q}(t)(\mathbf{x}_1 - \mathbf{x}_2) \quad (5.32.2)$$

or,

$$\mathbf{b}^* = \mathbf{Q}(t)\mathbf{b} \quad (5.32.3)$$

where  $\mathbf{b}$  and  $\mathbf{b}^*$  denote the same vector connecting the two material points.

Let  $\mathbf{T}$  be a tensor which transforms a frame-indifferent vector  $\mathbf{b}$  into a frame-indifferent vector  $\mathbf{c}$ , i.e.,

$$\mathbf{c} = \mathbf{T}\mathbf{b} \quad (iiia)$$

let  $\mathbf{T}^*$  be the same tensor as observed by the starred- frame, then

$$\mathbf{c}^* = \mathbf{T}^*\mathbf{b}^* \quad (iiib)$$

Now since  $\mathbf{c}^* = \mathbf{Q}\mathbf{c}$ ,  $\mathbf{b}^* = \mathbf{Q}\mathbf{b}$ , therefore,

$$\mathbf{c}^* = \mathbf{Q}\mathbf{c} = \mathbf{Q}\mathbf{T}\mathbf{b} = \mathbf{Q}\mathbf{T}\mathbf{Q}^T\mathbf{b}^* \quad (iv)$$

i.e.,

$$\mathbf{T}^*\mathbf{b}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T\mathbf{b}^*$$

Thus,

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T \quad (5.32.4)$$

Summarizing the above, we define that, in a change of frame,

$$\alpha = \alpha^* \quad \text{for indifferent (or objective) scalar} \quad (5.32.5a)$$

$$\mathbf{b}^* = \mathbf{Q}(t)\mathbf{b} \quad \text{for indifferent (or objective) vector} \quad (5.32.5b)$$

$$\mathbf{T}^* = \mathbf{Q}(t)\mathbf{T}\mathbf{Q}^T(t) \quad \text{for indifferent (or objective) tensor} \quad (5.32.5c)$$

Example 5.32.1

Show (a)  $d\mathbf{x}$  is an objective vector (b)  $ds$  is an objective scalar

*Solution.* From Eq. (5.32.1)

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_0) \quad (i)$$

we have

$$\mathbf{x}^* + d\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)(\mathbf{x} + d\mathbf{x} - \mathbf{x}_0) \quad (ii)$$

therefore

$$d\mathbf{x}^* = \mathbf{Q}(t)d\mathbf{x} \quad (5.32.6)$$

so that  $d\mathbf{x}$  is an objective vector

(b) From Eq. (5.32.6),

$$ds^{*2} = d\mathbf{x}^* \cdot d\mathbf{x}^* = \mathbf{Q}(t)d\mathbf{x} \cdot \mathbf{Q}(t)d\mathbf{x} = d\mathbf{x} \cdot \mathbf{Q}^T\mathbf{Q}d\mathbf{x} = d\mathbf{x} \cdot d\mathbf{x} = ds^2 \quad (iii)$$

that is,  $ds$  is an objective scalar.

Example 5.32.2

Show that in a change of frame, (a) the velocity vector  $\mathbf{v}$  transforms in accordance with the following equation and is therefore not objective

$$\mathbf{v}^* = \mathbf{Q}(t)\mathbf{v} + \dot{\mathbf{Q}}(t)(\mathbf{x} - \mathbf{x}_0) + \dot{\mathbf{c}}(t) \quad (5.32.7)$$

(b) the velocity gradient transform in accordance with the following equation and is also not objective

$$\nabla^* \mathbf{v}^* = \mathbf{Q}(\nabla \mathbf{v})\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T \quad (5.32.8)$$

*Solution.* (a) From Eqs. (5.32.1)

$$\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_0) \quad (i)$$

$$t^* = t - a \quad (\text{ii})$$

therefore,

$$\frac{dx^*}{dt^*} = \frac{dx^*}{dt} = \dot{c}(t) + \dot{Q}(t)(x - x_0) + Qv \quad (\text{iii})$$

That is

$$v^* = Q(t)v + \dot{c}(t) + \dot{Q}(t)(x - x_0) \quad (\text{iv})$$

This is not the transformation law for an objective vector. Therefore the velocity vector is non-objective as expected.

(b) From the result of part (a), we have

$$v^*(x^* + dx^*) = Q(t)v(x + dx) + \dot{c}(t) + \dot{Q}(t)(x + dx - x_0) \quad (\text{v})$$

and

$$v^*(x^*) = Q(t)v(x) + \dot{c}(t) + \dot{Q}(t)(x - x_0) \quad (\text{vi})$$

Subtraction of the above two equations then gives

$$(\nabla^* v^*) dx^* = Q(t)(\nabla v) dx + \dot{Q}(t) dx \quad (\text{vii})$$

But  $dx^* = Q dx$ , therefore

$$[(\nabla^* v^*)Q - Q(\nabla v) - \dot{Q}(t)] dx = 0 \quad (\text{viii})$$

Thus,

$$\nabla^* v^* = Q(\nabla v)Q^T + \dot{Q}Q^T \quad (\text{ix})$$

### Example 5.32.3

Show that in a change of frame, the deformation gradient  $F$  transforms according to the equation

$$F^* = Q(t)F \quad (5.32.9)$$

*Solution.* We have, for the starred frame

$$dx^* = F^* dX^* \quad (\text{i})$$

and for the unstarred frame

$$dx = F dX \quad (\text{ii})$$

In a change of frame,  $dx$  and  $dx^*$  are related by Eq. (5.32.6), i.e.,

$$dx^* = Q(t)dx \quad (\text{iii})$$

therefore, using Eqs. (i) and (iii), we have

$$Q(t)dx = F^*dX^* \quad (\text{iv})$$

Using Eq. (ii), the above equation becomes

$$Q(t)FdX = F^*dX^* \quad (\text{v})$$

Now, both  $dX$  and  $dX^*$  denote the same material element at the fixed reference time  $t_0$ , therefore, without loss of generality, we can take  $Q(t_0) = I$ , so that

$$dX = dX^* \quad (\text{vi})$$

Thus,

$$Q(t)F = F^* \quad (\text{vii})$$

which is Eq. (5.32.9).

#### Example 5.32.4

Derive the transformation law for (a) the right Cauchy-Green deformation tensor and (b) the left Cauchy-Green deformation tensor

*Solution.*

(a) The right Cauchy-Green tensor  $C$  is related to the deformation gradient  $F$  by the equation

$$C = F^T F \quad (5.32.10)$$

Thus, from the result of the last example, we have

$$C^* = F^*{}^T F^* = [Q(t)F]^T Q(t)F = F^T Q^T Q F = F^T F \quad (\text{i})$$

i.e, in a change of frame

$$C^* = C \quad (5.32.11)$$

That is, the right Cauchy-Green deformation tensor is not frame-indifferent (or, it is non-objective).

(b) The left Cauchy-Green tensor  $B$  is related to the deformation gradient  $F$  by the equation

$$B = FF^T \quad (5.32.12)$$

Thus, from the result of the last example, we have

$$\mathbf{B}^* = \mathbf{F}^* \mathbf{F}^{T*} = \mathbf{Q}(t) \mathbf{F} [\mathbf{Q}(t) \mathbf{F}]^T = \mathbf{Q}(t) \mathbf{F} \mathbf{F}^T \mathbf{Q}^T(t) \tag{ii}$$

i.e, in a change of frame

$$\mathbf{B}^* = \mathbf{Q}(t) \mathbf{B} \mathbf{Q}^T(t) \tag{5.32.13}$$

Thus, the left Cauchy-Green deformation tensor is frame-indifferent (i.e., it is objective).

We note that it can be easily proved that the inverse of an objective tensor is also objective and that the identity tensor is obviously objective. Thus both the left Cauchy Green deformation tensor  $\mathbf{B}$  and the Eulerian strain tensor  $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$  are objective, while the right Cauchy Green deformation tensor  $\mathbf{C}$  and the Lagrangian strain tensor  $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$  are non-objective.

We note also that the material time derivative of an objective tensor is in general non-objective.

### 5.33 Constitutive Equation for an Elastic Medium under Large Deformation.

As in the case of infinitesimal theory for an elastic body, the constitutive equation relates the state of stress to the state of deformation. However, in the case of finite deformation, there are different finite deformation tensors (left Cauchy-Green tensor  $\mathbf{B}$ , right Cauchy-Green tensor  $\mathbf{C}$ , Lagrangian strain tensor  $\mathbf{E}$ , etc. .) and different stress tensors (Cauchy stress tensor and the two Piola-Kirchhoff stress tensors) defined in Chapter 3 and Chapter 4 respectively. It is not immediately clear what stress tensor is to be related to what deformation tensor. For example, if one assumes that

$$\mathbf{T} = \mathbf{T}(\mathbf{C}) \tag{i}$$

where  $\mathbf{T}$  is the Cauchy stress tensor, and  $\mathbf{C}$  is the right Cauchy-Green tensor, then it can be shown [see Example 5.33.1 below] that this is not an acceptable form of constitutive equation because the law will not be frame-indifferent. On the other hand if one assumes

$$\mathbf{T} = \mathbf{T}(\mathbf{B}) \tag{ii}$$

then, this law is acceptable in that it is independent of observers, but it is limited to isotropic material only (See Example 5.33.3).

The requirement that a constitutive equation must be invariant under the transformation Eq. (5.32.1) (i.e., in a change of frame), is known as the **principle of material frame indifference**. In applying this principle, we shall insist that force and therefore, the Cauchy stress tensor be frame-indifferent. That is in a change of frame

$$\mathbf{T}^* = \mathbf{Q} \mathbf{T} \mathbf{Q}^T \tag{5.33.1}$$

## Example 5.33.1

Assume that for some elastic medium, the Cauchy stress  $\mathbf{T}$  is proportional to the right Cauchy-Green tensor  $\mathbf{C}$ . Show that this assumption does not result in a frame-indifferent constitutive equation and is therefore not acceptable.

*Solution.* The assumption states that ,

$$\text{for the starred frame:} \quad \mathbf{T}^* = \alpha \mathbf{C}^* \quad (\text{i})$$

$$\text{and for the un-starred frame:} \quad \mathbf{T} = \alpha \mathbf{C} \quad (\text{ii})$$

where we note that since the *same* material is considered by the two frames, therefore the proportional constant must be the same. Now,

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T \text{ [See Eq. (5.33.1)] and } \mathbf{C}^* = \mathbf{C} \text{ [See Eq. (5.32.11)]}$$

therefore, from Eq. (i)

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \alpha \mathbf{C} \quad (\text{iii})$$

so that from Eq. (ii) for all  $\mathbf{Q}(t)$

$$\mathbf{T} = \mathbf{Q}\mathbf{T}\mathbf{Q}^T. \quad (\text{iv})$$

The only  $\mathbf{T}$  for the above equation to be true is  $\mathbf{T} = \mathbf{I}$ . Thus, the law is not acceptable.

More generally, if we assume the Cauchy stress to be a function of the right Cauchy Green tensor, then for the starred frame  $\mathbf{T}^* = \mathbf{f}(\mathbf{C}^*)$ , and for the un-starred frame,  $\mathbf{T} = \mathbf{f}(\mathbf{C})$ , where again,  $\mathbf{f}$  is the same function for both frames because it is for the same material. In a change of frame,

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{f}(\mathbf{C}) \quad (\text{v})$$

That is, again

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{T} \quad (\text{vi})$$

So that Eq. (i) is not acceptable.

## Example 5.33.2

If we assume that the second Piola-Kirchhoff stress tensor  $\tilde{\mathbf{T}}$  is a function of the right Cauchy-Green deformation tensor  $\mathbf{C}$ . Show that it is an acceptable constitutive equation.

*Solution.* We have, according to the assumption

$$\tilde{\mathbf{T}} = \mathbf{f}(\mathbf{C}) \quad (\text{5.33.2a})$$

and

$$\tilde{\mathbf{T}}^* = \mathbf{f}(\mathbf{C}^*) \quad (5.33.2b)$$

where we demand that both frames (the unstarred and the starred) have the same function  $\mathbf{f}$  for the same material. Now, in a change of frame, the deformation gradient  $\mathbf{F}$  and the Cauchy stress tensor  $\mathbf{T}$  transform in accordance with the following equation:

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F} \text{ and } \mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T \quad (5.33.3)$$

Thus, the second Piola-Kirchhoff stress tensor transforms as [See Prob.5.98]

$$\tilde{\mathbf{T}}^* = \tilde{\mathbf{T}} \quad (5.33.4)$$

Therefore, in a change of frame, the equation

$$\tilde{\mathbf{T}}^* = \mathbf{f}(\mathbf{C}^*) \quad (5.33.5a)$$

transforms into

$$\tilde{\mathbf{T}} = \mathbf{f}(\mathbf{C}) \quad (5.33.5b)$$

which shows that the assumption is acceptable. In fact, it can be shown that Eq. (5.33.5) is the most general constitutive equation for an anisotropic elastic solid [See Prob. 5.100].

### Example 5.32.3

If we assume that the Cauchy stress  $\mathbf{T}$  is a function of the left Cauchy Green tensor  $\mathbf{B}$ , is it an acceptable constitutive law?

*Solution.* For the starred frame,

$$\mathbf{T}^* = \mathbf{f}(\mathbf{B}^*) \quad (5.33.6a)$$

and for the un-starred frame,

$$\mathbf{T} = \mathbf{f}(\mathbf{B}) \quad (5.33.6b)$$

where we note both frames have the same function  $\mathbf{f}$ . In a change of frame, (see Example 5.32.4, Eq. (5.32.13)),

$$\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^T \text{ and } \mathbf{B}^* = \mathbf{Q}\mathbf{B}\mathbf{Q}^T$$

Thus,

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{f}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T) \quad (5.33.7)$$

That is

$$\mathbf{Q}\mathbf{f}(\mathbf{B})\mathbf{Q}^T = \mathbf{f}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T) \quad (5.33.8)$$

Thus, in order that Eq. (5.32.6) be acceptable as a constitutive law, it must satisfy the condition given by Eq. (5.32.8). Now, in matrix form, the equation

$$\mathbf{T} = \mathbf{f}(\mathbf{B}) \quad (5.33.6b)$$

becomes

$$[\mathbf{T}] = [\mathbf{f}([\mathbf{B}])] \quad (5.33.9)$$

and the equation

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{f}(\mathbf{Q}\mathbf{B}\mathbf{Q}^T) \quad (5.33.10)$$

becomes

$$[\mathbf{Q}][\mathbf{T}][\mathbf{Q}]^T = [\mathbf{f}([\mathbf{Q}][\mathbf{B}][\mathbf{Q}]^T)] \quad (5.33.11)$$

Now, if we view the above two matrix equations, Eqs. (5.33.9) and (5.33.11), as those corresponding to a change of rectangular Cartesian basis, then we come to the conclusion that the constitutive equation given by Eq. (5.33.6) describes an isotropic material because both Eqs. (5.33.9) and (5.33.11) have the same function  $\mathbf{f}$ .

We note that the special case

$$\mathbf{T} = \alpha \mathbf{B} \quad (5.32.12)$$

where  $\alpha$  is a constant, is called a Hookean Solid.

### 5.34 Constitutive Equation for an Isotropic Elastic Medium

From the above example, we see that the assumption that  $\mathbf{T}$  is a function of  $\mathbf{B}$  alone leads to the constitutive equation for an isotropic elastic medium under large deformation.

A function such as the function  $\mathbf{f}$ , which satisfies the condition Eq. (5.33.8) is called an **isotropic function**. Thus for an isotropic elastic solid, the Cauchy stress tensor is an isotropic function of the left Cauchy-Green tensor  $\mathbf{B}$ .

It can be proved that in three dimensional space, the most general isotropic function  $\mathbf{f}(\mathbf{B})$  can be represented by the following equation

$$\mathbf{f}(\mathbf{B}) = a_0 \mathbf{I} + a_1 \mathbf{B} + a_2 \mathbf{B}^2 \quad (5.34.1)$$

where  $a_0$ ,  $a_1$  and  $a_2$  are scalar functions of the scalar invariants of the tensor  $\mathbf{B}$ , so that the general constitutive equation for an isotropic elastic solid under large deformation is given by

$$\mathbf{T} = a_0 \mathbf{I} + a_1 \mathbf{B} + a_2 \mathbf{B}^2 \quad (5.34.2)$$

Since a tensor satisfies its own characteristic equation [See Example 5.34.1 below], therefore we have

$$\mathbf{B}^3 - I_1 \mathbf{B}^2 + I_2 \mathbf{B} - I_3 \mathbf{I} = 0 \quad (5.34.3)$$

or,

$$\mathbf{B}^2 = I_1 \mathbf{B} - I_2 \mathbf{I} + I_3 \mathbf{B}^{-1} \quad (5.34.4)$$

Substituting Eq. (5.34.4) into Eq. (5.34.2), we obtain

$$\mathbf{T} = \varphi_0 \mathbf{I} + \varphi_1 \mathbf{B} + \varphi_2 \mathbf{B}^{-1} \quad (5.34.5)$$

where  $\varphi_0$ ,  $\varphi_1$  and  $\varphi_2$  and  $\varphi_2$  are scalar functions of the scalar invariants of  $\mathbf{B}$ . This is the alternate form of the constitutive equation for an isotropic elastic solid under large deformations.

#### Example 5.34.1

Derive the Cayley-Hamilton Theorem, Eq. (5.34.3).

*Solution.* Since  $\mathbf{B}$  is real and symmetric, there always exists three eigenvalues corresponding to three mutually perpendicular eigenvector directions.[See Section 2B18]. The eigenvalues  $\lambda_i$  satisfies the characteristic equation

$$\lambda_i^3 - I_1 \lambda_i^2 + I_2 \lambda_i - I_3 = 0 \quad i = 1, 2, 3 \quad (5.34.6)$$

The above three equations can be written in a matrix form as

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}^3 - I_1 \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}^2 + I_2 \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} - I_3 = 0 \quad (5.34.7)$$

Now, the matrix in this equation is the matrix for the tensor  $\mathbf{B}$  using its eigenvectors as the Cartesian rectangular basis. Thus, Eq. (5.34.7) has the invariant form

$$\mathbf{B}^3 - I_1 \mathbf{B}^2 + I_2 \mathbf{B} - I_3 \mathbf{I} = \mathbf{0} \quad (5.34.8)$$

Equation (5.34.2) or equivalently, Eq. (5.34.5) is the most general constitutive equation for an isotropic elastic solid under large deformation.

If the material is incompressible, then the constitutive equation is indeterminate to an arbitrary hydrostatic pressure and the constitutive equation becomes

$$\mathbf{T} = -p \mathbf{I} + \varphi_1 \mathbf{B} + \varphi_2 \mathbf{B}^{-1} \quad (5.34.9)$$

If the functions  $\varphi_1$  and  $\varphi_2$  are derived from a potential function  $\mathcal{A}$  of the invariants  $I_1$  and  $I_2$  such that

$$\varphi_1 = 2 \frac{\partial \mathcal{A}}{\partial I_1} \quad \text{and} \quad \varphi_2 = -2 \frac{\partial \mathcal{A}}{\partial I_2}, \quad (5.34.10)$$

then the constitutive equation becomes

$$\mathbf{T} = -p \mathbf{I} + 2 \frac{\partial \mathcal{A}}{\partial I_1} \mathbf{B} - 2 \frac{\partial \mathcal{A}}{\partial I_2} \mathbf{B}^{-1} \quad (5.34.11)$$

and the solid is known as an incompressible **hyperelastic isotropic solid**.

### 5.35 Simple Extension of an Incompressible Isotropic Elastic Solid

A rectangular bar is pulled in the  $x_1$  direction. At equilibrium, the ratio of the deformed length to the undeformed length (i.e., the stretch) is  $\lambda_1$  in the  $x_1$  direction and  $\lambda_2$  in the transverse direction. Thus, the equilibrium configuration is given by

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_2 X_3, \quad \lambda_1 \lambda_2^2 = 1 \quad (5.35.1)$$

where the condition  $\lambda_1 \lambda_2^2 = 1$  describes the **isochoric condition** (i.e., no change in volume).

The left Cauchy-Green deformation tensor  $\mathbf{B}$  and its inverse are given by

$$[\mathbf{B}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_2^2 \end{bmatrix} \quad [\mathbf{B}^{-1}] = \begin{bmatrix} \frac{1}{\lambda_1^2} & 0 & 0 \\ 0 & \frac{1}{\lambda_2^2} & 0 \\ 0 & 0 & \frac{1}{\lambda_2^2} \end{bmatrix} \quad (5.35.2)$$

From the constitutive equation

$$\mathbf{T} = -p \mathbf{I} + \varphi_1 \mathbf{B} + \varphi_2 \mathbf{B}^{-1} \quad (5.35.3)$$

we have

$$T_{11} = -p + \varphi_1 \lambda_1^2 + \varphi_2 \frac{1}{\lambda_1^2} \quad (i)$$

$$T_{22} = T_{33} = -p + \varphi_1 \lambda_2^2 + \varphi_2 \frac{1}{\lambda_2^2} \quad (ii)$$

Since these stress components are constants, therefore the equations of equilibrium are clearly satisfied. Also, from the boundary conditions that on the surface  $x_2 = b$ ,  $T_{22} = 0$  and on the surface  $x_3 = c$ ,  $T_{33} = 0$ , we obtain

$$T_{22} = T_{33} = 0 \tag{5.35.4}$$

everywhere in the bar. From these equations, we obtain ( noting that  $\lambda_1 \lambda_2^2 = 1$  )

$$p = \varphi_1 \lambda_2^2 + \frac{\varphi_2}{\lambda_2^2} = \frac{\varphi_1}{\lambda_1} + \varphi_2 \lambda_1 \tag{5.35.5}$$

Thus, the normal stress  $T_{11}$  needed to stretch the bar (which is laterally unconfined) in the  $x_1$  direction is given by

$$T_{11} = \left( \lambda_1^2 - \frac{1}{\lambda_1} \right) \left( \varphi_1 - \frac{\varphi_2}{\lambda_1} \right) \tag{5.35.6}$$

### 5.36 Simple Shear of an Incompressible Isotropic Elastic Rectangular Block

The state of simple shear deformation is defined by the following equations relating the spatial coordinates  $x_i$  to the material coordinates  $X_i$ :

$$x_1 = X_1 + K X_2, \quad x_2 = X_2, \quad x_3 = X_3 \tag{5.36.1}$$

The deformed configuration of the rectangular block is shown in plane view in Fig. 5.19, where one sees that the constant  $K$  is the amount of shear

The left Cauchy-Green tensor  $\mathbf{B}$  and its inverse are given by

$$[\mathbf{B}] = [\mathbf{F}][\mathbf{F}]^T = \begin{bmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.36.2}$$

$$[\mathbf{B}^{-1}] = \begin{bmatrix} 1 & -K & 0 \\ -K & 1 + K^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.36.3}$$

The scalar invariants are

$$I_1 = 3 + K^2, \quad I_2 = 3 + K^2, \quad I_3 = 1 \tag{5.36.4}$$

Thus, from Eq. (5.34.9), we have

$$T_{11} = -p + \varphi_1(1 + K^2) + \varphi_2, \quad T_{22} = -p + \varphi_1 + \varphi_2(1 + K^2) \tag{i}$$

$$T_{33} = -p + \varphi_1 + \varphi_2, \quad T_{12} = K(\varphi_1 - \varphi_2), \quad T_{13} = T_{23} = 0 \tag{ii}$$

Let

$$P = -p + \varphi_1 + \varphi_2 \tag{iii}$$

then

$$T_{11} = -P + \varphi_1 K^2 \quad T_{22} = -P + \varphi_2 K^2 \quad T_{33} = -P \tag{5.36.5a}$$

$$T_{12} = K(\varphi_1 - \varphi_2) \quad T_{13} = T_{23} = 0 \tag{5.36.5b}$$

where  $\varphi_1$  and  $\varphi_2$  are function of  $K^2$ .

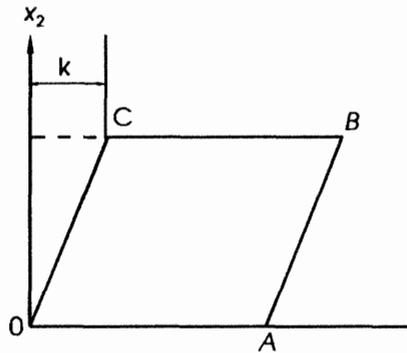


Fig.5.19

The stress components are constants, therefore, the equations of equilibrium are clearly satisfied.

If  $X_3 = \text{constant}$  plane is free of stress, then

$$P = 0 \tag{iv}$$

so that

$$T_{11} = \varphi_1 K^2, \quad T_{22} = \varphi_2 K^2, \quad T_{33} = 0 \quad T_{12} = K(\varphi_1 - \varphi_2) \tag{5.36.6}$$

where  $(\varphi_1 - \varphi_2)$  is sometimes called the generalized shear modulus in the particular undistorted state used as reference. It is an even function of  $K$ , the amount of shear. The surface traction needed to maintain this simple shear state of deformation are as follows:

On the top face in the Fig. 5.19, there is a normal stress ( $\varphi_2 K^2$ ) and a shear stress, ( $K(\varphi_1 - \varphi_2)$ ). On the bottom face, an equal and opposite surface traction to that on the top face is acting. On the right face, which at equilibrium is no longer perpendicular to the  $x_1$  axis, but has a unit normal given by

$$\mathbf{n} = \frac{(\mathbf{e}_1 - K\mathbf{e}_2)}{(1+K^2)^{1/2}} \quad (5.36.7)$$

therefore, the surface traction on this deformed surface is given by

$$\mathbf{t} = \mathbf{T}\mathbf{n} = \frac{K}{\sqrt{1+K^2}} [\varphi_2 K\mathbf{e}_1 + (\varphi_1 - (1+K^2)\varphi_2)\mathbf{e}_2] \quad (5.36.8)$$

Thus, the normal stress on this surface is

$$T_n = \mathbf{t} \cdot \mathbf{n} = -\frac{K^2}{1+K^2} [\varphi_1 - (2+K^2)\varphi_2] \quad (5.36.9)$$

and the shear stress on this same surface is, with  $\mathbf{e}_T = \frac{(K\mathbf{e}_1 + \mathbf{e}_2)}{\sqrt{1+K^2}}$

$$T_s = \mathbf{t} \cdot \mathbf{e}_T = \frac{K}{1+K^2} (\varphi_1 - \varphi_2) \quad (5.36.10)$$

We see from the above equation that, in addition to shear stresses, normal stresses are needed to maintain the simple shear state of deformation.

We also note that

$$T_{11} - T_{22} = KT_{12} \quad (5.36.11)$$

This is a universal relation, independent of the coefficients  $\varphi_i$  of the material.

### 5.37 Bending of an Incompressible Rectangular Bar.

It is easy to see that the deformation of a rectangular bar into a curved bar shown in Fig. 5.20 can be described by the following equations

$$r = (2\alpha X + \beta)^{1/2}, \quad \theta = cY, \quad z = Z, \quad \alpha = \frac{1}{c} \quad (5.37.1)$$

where  $(X, Y, Z)$  are Cartesian material coordinates and  $(r, \theta, z)$  are cylindrical spatial coordinates. Indeed, the boundary plane  $X = -a$  and  $X = a$  deform into cylindrical surfaces  $r_1 = \sqrt{-2\alpha a + \beta}$  and  $r_2 = \sqrt{2\alpha a + \beta}$  and the boundary planes  $Y = \pm b$  deform into the planes  $\theta = \pm cb$ .

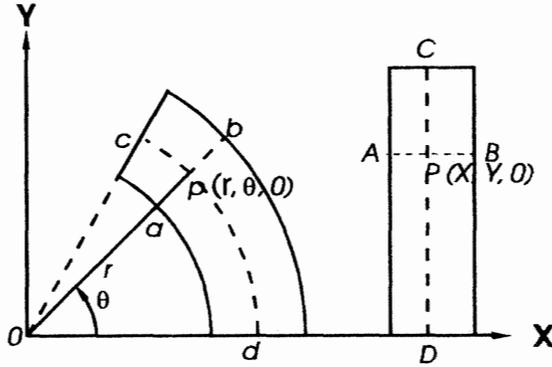


Fig.5.20

The left Cauchy-Green tensor  $\mathbf{B}$  corresponding to this deformation field can be calculated using Eqs. (3.30.12): [Note  $I_3 = \alpha c = 1$ ]

$$[\mathbf{B}] = \begin{bmatrix} \alpha^2/r^2 & 0 & 0 \\ 0 & c^2 r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha^2/r^2 & 0 & 0 \\ 0 & r^2/\alpha^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5.37.2)$$

The inverse of  $\mathbf{B}$  can be obtained to be

$$[\mathbf{B}^{-1}] = \begin{bmatrix} r^2/\alpha^2 & 0 & 0 \\ 0 & 1/(c^2 r^2) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} r^2/\alpha^2 & 0 & 0 \\ 0 & \alpha^2/r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (5.37.3)$$

The scalar invariants of  $\mathbf{B}$  are

$$I_1 = \frac{\alpha^2}{r^2} + \frac{r^2}{\alpha^2} + 1 = I_2, \quad I_3 = 1 \quad (5.37.4)$$

We shall use the constitutive equation for a hyperelastic solid for this problem. Thus, from Eq. (5.34.11), we have

$$T_{rr} = -p + \frac{2\alpha^2 \partial A}{r^2 \partial I_1} - \frac{2r^2 \partial A}{\alpha^2 \partial I_2} \quad (5.37.5a)$$

$$T_{\theta\theta} = -p + 2c^2 r^2 \frac{\partial A}{\partial I_1} - \frac{2}{c^2 r^2} \frac{\partial A}{\partial I_2} \quad (5.37.5b)$$

$$T_{zz} = -p + 2 \frac{\partial A}{\partial I_1} - 2 \frac{\partial A}{\partial I_2} \quad (5.37.5c)$$

$$T_{r\theta} = T_{rz} = T_{z\theta} = 0 \quad (5.37.5d)$$

The equations of equilibrium are

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0 \quad (5.37.6a)$$

$$\frac{\partial T_{\theta\theta}}{\partial \theta} = 0 \quad (5.37.6b)$$

$$\frac{\partial T_{zz}}{\partial z} = 0 \quad (5.37.6c)$$

From Eq. (5.37.6b) and (5.37.6c), we have, since  $A(I_1(r), I_2(r))$  is function of  $r$  only,

$$p = p(r) \quad (5.37.7)$$

Since

$$\frac{dA}{dr} = \frac{\partial A}{\partial I_1} \frac{dI_1}{dr} + \frac{\partial A}{\partial I_2} \frac{dI_2}{dr} = 2 \left( \frac{-\alpha^2}{r^3} + \frac{r}{\alpha^2} \right) \left( \frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2} \right) = -\frac{T_{rr} - T_{\theta\theta}}{r} \quad (i)$$

Thus, from Eq. (5.37.6a), we have

$$\frac{dT_{rr}}{dr} - \frac{dA}{dr} = 0 \quad (ii)$$

and

$$T_{rr} = A(r) + K \quad (5.37.8)$$

Furthermore, Eq. (5.37.6a) and Eq. (5.37.8) give

$$T_{\theta\theta} = r \frac{dT_{rr}}{dr} + T_{rr} = \frac{d}{dr}(rT_{rr}) = \frac{d}{dr}[rA(r)] + K \quad (5.37.9)$$

The boundary conditions are :

$$\text{At } r = r_1, T_{rr} = 0 \quad \text{and at } r = r_2, T_{rr} = 0 \quad (5.37.10)$$

Thus,

$$A(r_1) + K = 0, \quad A(r_2) + K = 0 \quad (iii)$$

so that

$$A(r_1) = A(r_2) \tag{iv}$$

But,

$$A = A(I_1(r), I_2(r)) \tag{v}$$

where

$$I_1 = I_2 = \frac{\alpha^2}{r^2} + \frac{r^2}{\alpha^2} + 1 \tag{vi}$$

therefore

$$\frac{\alpha^2}{r_1^2} + \frac{r_1^2}{\alpha^2} + 1 = \frac{\alpha^2}{r_2^2} + \frac{r_2^2}{\alpha^2} + 1 \tag{vii}$$

or,

$$\alpha^2 \left( \frac{1}{r_1^2} - \frac{1}{r_2^2} \right) = \frac{1}{\alpha^2} (r_2^2 - r_1^2) \tag{viii}$$

which leads to

$$\alpha^2 = r_1 r_2 \tag{5.37.11}$$

The normal force on the end plane  $\theta = \pm cb$  is given by (see Eq. (5.37.9) and (i))

$$\int_{r_1}^{r_2} T_{\theta\theta} dr = [r A(r) + Kr]_{r_1}^{r_2} = r_1[A(r_1) + K] - r_2[A(r_2) + K] = 0 \tag{5.37.12}$$

Thus, at the end plane, there is a flexural couple. Let  $M$  denotes the flexural couple per unit width, then

$$\begin{aligned} M &= \int_{r_1}^{r_2} r T_{\theta\theta} dr = \int_{r_1}^{r_2} \left( r \frac{d(rA)}{dr} + Kr \right) dr \\ &= r^2 A(r) \Big|_{r_1}^{r_2} - \int_{r_1}^{r_2} r A(r) dr + \frac{Kr^2}{2} \Big|_{r_1}^{r_2} = -Kr_2^2 + Kr_1^2 - \int_{r_1}^{r_2} r A(r) dr + \frac{Kr_2^2}{2} - \frac{Kr_1^2}{2} \end{aligned} \tag{5.37.13}$$

i.e.

$$M = \frac{1}{2}K(r_1^2 - r_2^2) - \int_{r_1}^{r_2} r A(r) dr \tag{5.37.14}$$

We note that with  $z = Z$ , the bar is in a plane strain state.

### 5.38 Torsion and Tension of an Incompressible Solid Cylinder

Consider the following equilibrium configuration for a circular cylinder

$$r = \lambda_1 R, \quad \theta = \Theta + KZ, \quad z = \lambda_3 Z \quad \lambda_1^2 \lambda_3 = 1 \quad (5.38.1)$$

where  $(r, \theta, z)$  are the spatial coordinates and  $(R, \Theta, Z)$  are the material coordinates for a material point,  $\lambda_1$  and  $\lambda_3$  are stretches for elements which were in the radial and axial directions.

The left Cauchy-Green tensor  $\mathbf{B}$  and its inverse can be obtained from Eq. (3.30.8) as

$$[\mathbf{B}] = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_1^2 + r^2 K^2 & rK\lambda_3 \\ 0 & rK\lambda_3 & \lambda_3^2 \end{bmatrix}, \quad [\mathbf{B}^{-1}] = \begin{bmatrix} \frac{1}{\lambda_1^2} & 0 & 0 \\ 0 & \frac{1}{\lambda_1^2} & -Kr \\ 0 & -Kr & \lambda_1^4 + \lambda_1^2 K^2 r^2 \end{bmatrix} \quad (5.38.2)$$

The scalar invariants of  $\mathbf{B}$  are

$$I_1 = 2\lambda_1^2 + r^2 K^2 + \lambda_3^2 = \frac{2}{\lambda_3} + r^2 K^2 + \lambda_3^2, \quad (5.38.3a)$$

$$I_2 = \frac{2}{\lambda_1^2} + \lambda_1^4 \left(1 + \frac{K^2 r^2}{\lambda_1^2}\right) = 2\lambda_3 + \frac{1}{\lambda_3^2} (1 + K^2 r^2 \lambda_3), \quad I_3 = 1 \quad (5.38.3b)$$

Now, from the constitutive equation  $\mathbf{T} = -p\mathbf{I} + \varphi_1 \mathbf{B} + \varphi_2 \mathbf{B}^{-1}$ , we obtain

$$T_{rr} = -p + \varphi_1 \lambda_1^2 + \frac{\varphi_2}{\lambda_1^2} = -p + \frac{\varphi_1}{\lambda_3} + \varphi_2 \lambda_3 \quad (5.38.4a)$$

$$T_{\theta\theta} = -p + \varphi_1 (\lambda_1^2 + r^2 K^2) + \frac{\varphi_2}{\lambda_1^2} = -p + \varphi_1 \left( \frac{1}{\lambda_3} + r^2 K^2 \right) + \varphi_2 \lambda_3 \quad (5.38.4b)$$

$$T_{zz} = -p + \varphi_1 \lambda_3^2 + \varphi_2 (\lambda_1^4 + \lambda_1^2 K^2 r^2) = -p + \varphi_1 \lambda_3^2 + \frac{\varphi_2}{\lambda_3} \left( \frac{1}{\lambda_3} + K^2 r^2 \right), \quad (5.38.4c)$$

$$T_{r\theta} = T_{rz} = 0 \quad (5.38.4d)$$

$$T_{\theta z} = K \lambda_3 \left( \varphi_1 - \frac{\varphi_2}{\lambda_3} \right) r \quad (5.38.4e)$$

The equations of equilibrium are:

$$\frac{\partial T_{rr}}{\partial r} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0 \quad (5.38.5a)$$

$$\frac{\partial T_{\theta\theta}}{\partial \theta} = 0 \quad (5.38.5b)$$

$$\frac{\partial T_{zz}}{\partial z} = 0 \quad (5.38.5c)$$

Noting that  $I_1$  and  $I_2$  ( and therefore  $\varphi_1$  and  $\varphi_2$ ), are functions of  $r$  only, we obtain, from the second and the third equations of equilibrium,

$$\frac{\partial p}{\partial \theta} = \frac{\partial p}{\partial z} = 0 \quad (i)$$

That is,  $p$  is a function of  $r$  only. Thus,

$$p = p(r) \quad (5.38.6)$$

From the first equation of equilibrium, we have

$$r \frac{dT_{rr}}{dr} = -(T_{rr} - T_{\theta\theta}) \quad (ii)$$

The total normal force  $N$  on a cross sectional plane is given by

$$N = \int_0^{r_0} T_{zz} 2\pi r dr \quad (5.38.7)$$

To evaluate the integral, we first need to eliminate  $p$  from the equation for  $T_{zz}$ . This can be done in the following way:

With

$$T_{zz} = -p + \tau_{zz}, \quad T_{rr} = -p + \tau_{rr}, \quad T_{\theta\theta} = -p + \tau_{\theta\theta} \quad (5.38.8)$$

we have

$$\begin{aligned} 2T_{zz} &= -2p + 2\tau_{zz} = (T_{rr} - \tau_{rr}) + (T_{\theta\theta} - \tau_{\theta\theta}) + 2\tau_{zz} \\ &= T_{rr} + T_{\theta\theta} + 2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta} \end{aligned} \quad (iii)$$

Now, in view of Eq. (ii), we have

$$2T_{zz} = 2T_{rr} + r \frac{dT_{rr}}{dr} + 2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta} = \frac{1}{r} \frac{d}{dr} (r^2 T_{rr}) + 2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta} \quad (iv)$$

Thus, from Eq. (5.38.7),

$$N = \pi r^2 T_{rr} \Big|_0^{r_0} + \pi \int_0^{r_0} (2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta}) r dr \quad (v)$$

With  $T_{rr}(r_0) = 0$ , we have

$$N = \pi \int_0^{r_0} (2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta}) r dr \quad (5.38.9)$$

From Eqs. (5.38.4) and (5.38.8), we have

$$2\tau_{zz} - \tau_{rr} - \tau_{\theta\theta} = 2 \left( \lambda_3^2 - \frac{1}{\lambda_3} \right) \left( \varphi_1 - \frac{\varphi_2}{\lambda_3} \right) - K^2 r^2 \left( \varphi_1 - \frac{2\varphi_2}{\lambda_3} \right) \quad (vi)$$

Thus,

$$\begin{aligned} N &= 2\pi\lambda_3 \left( \lambda_3 - \frac{1}{\lambda_3^2} \right) \int_0^{r_0} \left( \varphi_1 - \frac{\varphi_2}{\lambda_3} \right) r dr \\ &\quad - \pi K^2 \int_0^{r_0} \left( \varphi_1 - \frac{2\varphi_2}{\lambda_3} \right) r^3 dr \end{aligned} \quad (5.38.10)$$

Since  $r = \lambda_1 R$ , therefore,

$$r dr = \lambda_1^2 R dR = \frac{1}{\lambda_3} R dR \quad (vii)$$

Thus,

$$\begin{aligned} N &= 2\pi \left( \lambda_3 - \frac{1}{\lambda_3^2} \right) \int_0^{R_0} \left( \varphi_1 - \frac{\varphi_2}{\lambda_3} \right) R dR \\ &\quad - \frac{\pi K^2}{\lambda_3^2} \int_0^{R_0} \left( \varphi_1 - \frac{2\varphi_2}{\lambda_3} \right) R^3 dR \end{aligned} \quad (5.38.11)$$

Similarly, the twisting moment can be obtained to be

$$M = \int_0^{r_0} r T_{\theta z} 2\pi r dr = \frac{2\pi K}{\lambda_3} \int_0^{R_0} \left( \varphi_1 - \frac{\varphi_2}{\lambda_3} \right) R^3 dR \quad (5.38.12)$$

In Eqs. (5.38.11) and (5.38.12)  $\varphi_1$  and  $\varphi_2$  are functions of  $I_1$  and  $I_2$  and are therefore functions of  $R$  (see Eq. (5.38.3)).

If the angle of twist  $K$  is very small, then  $I_1$  and  $I_2$  and therefore  $\varphi_1$  and  $\varphi_2$  may be regarded as independent of  $R$  and the integrals can be integrated to give

$$N = \pi R_o^2 \left( \lambda_3 - \frac{1}{\lambda_3} \right) \left( \varphi_1 - \frac{\varphi_2}{\lambda_3} \right) + O(K^2) \quad (5.38.18)$$

and

$$M = \frac{K \pi R_o^4}{\lambda_3} \frac{1}{2} \left( \varphi_1 - \frac{\varphi_2}{\lambda_3} \right) + O(K^2) \quad (5.38.19)$$

We see therefore, that if the bar is prevented from extension or contraction (i.e.,  $\lambda_3 = 1$ ), then twisting of the bar with a  $K$  approaching zero, gives rise to a small axial force  $N$  which approaches zero with  $K^2$ . On the other hand if the bar is free from axial force (i.e.,  $N = 0$ ), then as  $K$  approaches zero, there is an axial stretch  $\lambda_3$  such that  $(\lambda_3 - 1)$  approaches zero with  $K^2$ . Thus, when a circular bar is twisted with an infinitesimal angle of twist, the axial stretch is negligible as was assumed earlier in the infinitesimal theory.

From Eqs. (5.38.18) and (5.38.19), we can obtain

$$\frac{M}{K} = \frac{R_o^2}{2} \frac{N}{\left( \lambda_3^2 - \frac{1}{\lambda_3} \right)} \quad (5.38.20)$$

Equation (5.38.20) is known as "Rivlin's Universal relation". This equation gives, for small twisting angle, the torsional stiffness as a function of  $\lambda_3$ , the stretch in the axial direction. We see, therefore, that the torsional stiffness can be obtained from a simple-extension experiment which measures  $N$  as a function of the axial stretch  $\lambda_3$ .

## PROBLEMS

5.1. Show that the null vector is the only isotropic vector.

(Hint: Assume that  $\mathbf{a}$  is an isotropic vector, and use a simple change of basis to equate the primed and the unprimed components)

5.2. Show that the most general isotropic second-order tensor is of the form  $\alpha \mathbf{I}$ , where  $\alpha$  is a scalar and  $\mathbf{I}$  is the identity tensor.

5.3. Show that for an anisotropic linear elastic material, the principal directions of stress and strain are usually not coincident.

5.4. If the Lamé constants for a material are

$$\lambda = 119.2 \text{ GPa } (17.3 \times 10^6 \text{ psi}), \quad \mu = 79.2 \text{ GPa } (11.5 \times 10^6 \text{ psi}),$$

find Young's modulus, Poisson's ratio, and the bulk modulus.

5.5. Given Young's modulus  $E_Y = 103 \text{ GPa}$  and Poisson's ratio  $\nu = 0.34$ , find the Lamé constants  $\lambda$  and  $\mu$ . Also find the bulk modulus.

5.6. Given Young's modulus  $E_Y = 193 \text{ GPa}$  and shear modulus  $\mu = 76 \text{ GPa}$ , find Poisson's ratio  $\nu$ , Lamé's constant  $\lambda$  and the bulk modulus  $k$

5.7. If the components of strain at a point of structural steel are

$$E_{11} = 36 \times 10^{-6}, \quad E_{22} = 40 \times 10^{-6}, \quad E_{33} = 25 \times 10^{-6}$$

$$E_{12} = 12 \times 10^{-6}, \quad E_{23} = 0, \quad E_{13} = 30 \times 10^{-6}$$

find the stress components,  $\lambda = 119.2 \text{ GPa } (17.3 \times 10^6 \text{ psi})$ ,  $\mu = 79.2 \text{ GPa } (11.5 \times 10^6 \text{ psi})$ .

5.8. Do Problem 5.7 if the strain components are

$$E_{11} = 100 \times 10^{-6}, \quad E_{22} = -50 \times 10^{-6}, \quad E_{33} = 200 \times 10^{-6}$$

$$E_{12} = -100 \times 10^{-6}, \quad E_{23} = 0, \quad E_{13} = 0$$

5.9. (a) If the state of stress at a point of structural steel is

$$[\mathbf{T}] = \begin{bmatrix} 100 & 42 & 6 \\ 42 & -2 & 0 \\ 6 & 0 & 15 \end{bmatrix} \text{ MPa}$$

what are the strain components?  $E_Y = 207 \text{ GPa}$ ,  $\mu = 79.2 \text{ GPa}$ ,  $\nu = 0.30$

(b) Suppose that a five centimeter cube of structural steel has a constant state of stress given in part (a). Determine the total change in volume induced by this stress field.

5.10. (a) For the constant stress field below, find the strain components

$$[\mathbf{T}] = \begin{bmatrix} 6 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ MPa}$$

(b) Suppose that a sphere of 5 cm radius is under the influence of this stress field, what will be the change in volume of the sphere? Use the elastic constants of Prob.5.9.

5.11. Show that for an incompressible material ( $\nu = 1/2$ ) that

(a)

$$\mu = \frac{E_Y}{3}, \quad \lambda = \infty, \quad k = \infty$$

(b) Hooke's law becomes

$$\mathbf{T} = 2\mu \mathbf{E} + \frac{1}{3}(T_{kk}) \mathbf{I}$$

5.12. Given a function  $f(a, b) = ab$  and a motion

$$x_1 = X_1 + k(X_1 + X_2)$$

$$x_2 = X_2 + k(X_1 - X_2),$$

where  $k = 10^{-4}$

(a) Show that  $f(X_1, X_2) \approx f(x_1, x_2)$ .

(b) Show that

$$\frac{\partial f(x_1, x_2)}{\partial x_1} \approx \frac{\partial f(X_1, X_2)}{\partial X_1}$$

and

$$\frac{\partial f(x_1, x_2)}{\partial x_2} \approx \frac{\partial f(X_1, X_2)}{\partial X_2}$$

5.13. Do the previous problem for  $f(a, b) = a^2 + b^2$

5.14. Given the following displacement field

$$u_1 = kX_3X_2, \quad u_2 = kX_3X_1, \quad u_3 = k(X_1^2 - X_2^2), \quad k = 10^{-4}$$

(a) Find the corresponding stress components.

(b) In the absence of body forces, is the state of stress a possible equilibrium stress field?

5.15. Repeat Problem 5.14, except that the displacement components are

$$u_1 = kX_2X_3, \quad u_2 = kX_1X_3, \quad u_3 = kX_1X_2, \quad k = 10^{-4}$$

5.16. Repeat Problem 5.14, except that the displacement components are:

$$u_1 = -kX_3X_2, \quad u_2 = kX_1X_3, \quad u_3 = k \sin X_2, \quad k = 10^{-4}$$

5.17. Calculate the ratio of  $c_L/c_T$  for Poisson's ratio equal to  $\frac{1}{3}$ , 0.49, 0.499

5.18. Assume an arbitrary displacement field that depends only on the field variable  $x_2$  and time  $t$ , determine what differential equations the displacement field must satisfy in order to be a possible motion (with zero body force).

5.19. Consider a linear elastic medium. Assume the following form for the displacement field

$$u_1 = \varepsilon [\sin \beta(x_3 - ct) + \alpha \sin \beta(x_3 + ct)], \quad u_2 = u_3 = 0$$

(a) What is the nature of this elastic wave (longitudinal, transverse, direction of propagation)?

(b) Find the associated strains, stresses and determine under what conditions the equations of motion are satisfied with zero body force.

(c) Suppose that there is a boundary at  $x_3 = 0$  that is traction-free. Under what conditions will the above motion satisfy this boundary condition for all time?

(d) Suppose that there is a boundary at  $x_3 = l$  that is also traction-free. What further conditions will be imposed on the above motion to satisfy this boundary condition for all time?

5.20. Do the previous problem if the boundary  $x_3 = 0$  is fixed (no motion) and  $x_3 = l$  is still traction-free.

5.21. Do problem 5.19 if the boundaries  $x_3 = 0$  and  $x_3 = l$  are both rigidly fixed.

5.22. Do Problem 5.19 if the assumed displacement field is of the form

$$u_3 = \sin \beta(x_3 - ct) + \alpha \sin \beta(x_3 + ct), \\ u_1 = u_2 = 0.$$

5.23. Do Problem 5.22 if the boundary  $x_3 = 0$  is fixed (no motion) and  $x_3 = l$  is traction-free ( $t = 0$ ).

5.24. Do Problem 5.22 if the boundary  $x_3 = 0$  and  $x_3 = l$  are both rigidly fixed.

5.25. Consider an arbitrary displacement field  $\mathbf{u} = \mathbf{u}(x_1, t)$ .

(a) Show that if the motion is equivoluminal ( $\frac{\partial u_i}{\partial x_i} = 0$ ) that  $\mathbf{u}$  must satisfy the equation

$$\mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \rho_0 \frac{\partial^2 u_i}{\partial t^2}.$$

(b) Show that if the motion is irrotational ( $\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}$ ) that the dilatation  $e = \frac{\partial u_i}{\partial x_i}$  must satisfy the equation

$$(2\mu + \lambda) \frac{\partial^2 e}{\partial x_i \partial x_i} = \rho_0 \frac{\partial^2 e}{\partial t^2}.$$

5.26. (a) Write a displacement field for an infinite train of longitudinal waves propagating in the direction  $3 \mathbf{e}_1 + 4\mathbf{e}_2$ .

(b) Write a displacement field for an infinite train of transverse waves propagating in the direction  $3 \mathbf{e}_1 + 4\mathbf{e}_2$  and polarized in the  $x_1 x_2$  plane.

5.27. Consider a material with Poisson's ratio equal  $1/3$  and a transverse elastic wave (as in Section 5.10) of amplitude  $\varepsilon_1$  and incident on a plane boundary at an angle  $\alpha_1$ . Determine the amplitudes and angles of reflection of the reflected waves if

(a)  $\alpha_1 = 0$

(b)  $\alpha_1 = 15^\circ$ .

5.28. Consider an incident transverse wave on a free boundary as in Section 5.10. For what particular angles of incidence will the only reflected wave be transverse? ( Take  $\nu = 1/3$  ).

5.29. Consider a transverse elastic wave incident on a traction-free plane surface and polarized normal to the plane of incidence. Show that the boundary condition can be satisfied with only a reflected transverse wave that is similarly polarized. what is the relation of the amplitudes, wavelengths, and direction of propagation of the incident and reflected wave?

5.30. Consider the problem of Section 5.10 and determine the characteristics of the reflected waves if the boundary  $x_2 = 0$  is fixed (no motion). How are the results different from the case of a free boundary.

5.31. A longitudinal elastic wave is incident on a fixed boundary

(a) Show that in general there are two reflected waves, one longitudinal and the other transverse (polarized in plane normal to incident plane).

(b) Find, as in Section 5.10, the amplitude ratio of reflected to incident elastic waves.

5.32. Do the previous problem for a free boundary.

5.33. Verify that the thickness stretch vibration given by Eq. (5.11.3) does satisfy the longitudinal wave equation.

5.34. Do Example 5.11.1 if the right face  $x_1 = l$  is free.

5.35. (a) Find the thickness stretch vibration if the  $x_1 = 0$  face is being forced by a traction  $\mathbf{t} = (\beta \cos \omega t) \mathbf{e}_1$  and the right-hand face  $x_1 = l$  is fixed.

(b) Find the resonant frequencies.

5.36. (a) Find the thickness-shear vibration if the left-hand face  $x_1 = 0$  has a forced displacement  $\mathbf{u} = (\alpha \cos \omega t) \mathbf{e}_3$  and the right-hand face  $x_1 = l$  is fixed.

(b) Find the resonant frequencies.

5.37. Do the previous problem if the forced displacement is given by  $\mathbf{u} = \alpha (\cos \omega t \mathbf{e}_2 + \sin \omega t \mathbf{e}_3)$ . Describe the particle motion throughout the plate.

5.38. Determine the total elongation of a steel bar 76 cm long if the tensile stress is 0.1 GPa and  $E_Y = 207$  GPa.

5.39. A cast iron bar, 4 ft (122 cm) long and 1.5 in.(3.81 cm) in diameter is pulled by equal and opposite axial forces  $P$  at its ends.

(a) Find the maximum normal and shearing stresses if  $P = 20,000$  lb (89000 N).

(b) Find the total elongation and lateral contraction ( $E_Y = 15 \times 10^6$  psi (103 GPa),  $\nu = 0.25$ ).

5.40. A steel bar ( $E_Y = 207$ GPa) of  $6 \text{ cm}^2$  cross-section and 6 m length is acted on by the indicated (Fig.P5.1) axially applied forces. Find the total elongation of the bar.

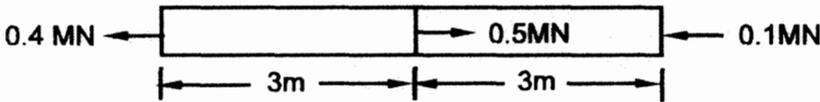


Fig. P5.1

5.41. A steel bar of 10 ft (3.05 m) length is to be designed to carry a tensile load of 100,000 lb (444.8 kN). What should the minimum cross-sectional area be if the maximum shearing stress should not exceed 15,000 psi (103 MPa) and the maximum normal stress should not exceed 20,000 psi (138 MPa)? If it is further required that the elongation should not exceed 0.05 in.(0.127 cm), what should the area be?

5.42. Consider a bar of cross-sectional area  $A$  that is stretched by a tensile force  $P$  at each end.

(a) Determine the normal and shearing stresses on a plane with a normal vector that makes an angle  $\alpha$  with the cylindrical axis. For what values of  $\alpha$  are the normal and shearing stresses equal?

(b) If the load carrying capacity of the bar is based on the shearing stress on the plane defined by  $\alpha = \alpha_o$  remaining less than  $\tau_o$ , sketch how the maximum load will depend on the angle  $\alpha_o$ .

5.43. Consider a cylindrical bar that is acted upon by an axial stress  $T_{11} = \sigma$ ,

(a) What will the state of stress in the bar be if the lateral surface is constrained so that there is no contraction or expansion?

(b) Show that the effective Young's modulus  $E_Y' \equiv T_{11}/E_{11}$  is given by

$$E_Y' = \frac{(1 - \nu)}{(1 - 2\nu)(1 + \nu)} E_Y.$$

(c) Evaluate the effective modulus for Poisson's ratio equal to  $1/3$  and  $1/2$ .

5.44. Let the state of stress in a tension specimen be given by  $T_{11} = \sigma$ , all other  $T_{ij} = 0$ .

(a) Find the components of the deviatoric stress  $\mathbf{T}^o = \mathbf{T} - \frac{1}{3} T_{kk} \mathbf{I}$ .

(b) Find the scalar invariants of  $\mathbf{T}^o$

5.45. Three identical steel rods support the load  $P$ , as shown in Fig.P5.2. How much load does each rod carry? Neglect the weights of the rod and the rigid bar.

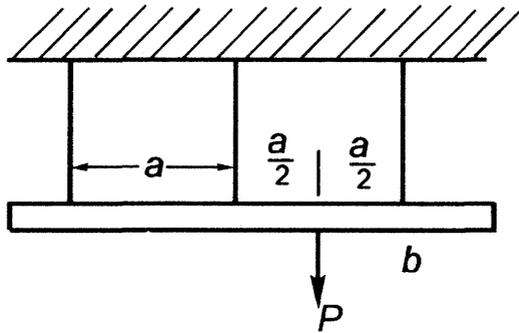


Fig. P5.2

5.46. Solve the previous problem if the cross-sectional area of the middle bar is twice that of the left- and right-hand bars.

5.47. Let the axis of a cylindrical bar be vertical and initially coincide with the  $x_1$  axis. If  $x_1 = 0$  corresponds to the lower face, then the body force is given by  $\rho \mathbf{B} = -\rho g \mathbf{e}_1$ . Assume that the stress distribution induced by the body force alone is of the form

$$T_{11} = \rho g x_1$$

and all other  $T_{ij} = 0$ .

(a) Show that the stress tensor is a possible state of stress in the presence of the body force mentioned above.

(b) If this possible state of stress is the actual distribution of stress in the cylindrical bar, what surface tractions should act on the lateral face and the pair of end faces in order to produce this state of stress.

5.48. A circular steel shaft is subjected to twisting couples of  $2700 \text{ N} \cdot \text{m}$ . The allowable tensile stress is  $0.124 \text{ GPa}$ . If the allowable shearing stress is  $0.6$  times the allowable tensile stress, what is the minimum allowable diameter?

**5.49.** A circular steel shaft is subjected to twisting couples of 5000 ft·lb (6780 N·m). Determine the shaft diameter if the maximum shear stress is not to exceed 10,000 psi (69 MPa) and the angle of twist is not to exceed  $1.5^\circ$  in 20 diameters of length.  $\mu = 12 \times 10^6$  psi (82.7 GPa).

**5.50.** Demonstrate that the elastic solution for the solid circular bar in torsion is also valid for a circular cylindrical tube in torsion. If  $a$  is the outside radius and  $b$  is the inside radius, how must Eq. (5.13.10) for the twist per unit length be altered?

**5.51.** In Example 5.13.2, if the radius of the left portion is  $a_1$  and the radius of the right portion is  $a_2$ , what is the twisting moment produced in each portion of the shaft? Both shafts are of the same material.

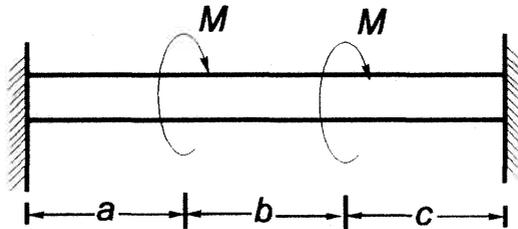


Fig. P5.3

**5.52.** Solve the previous problem if  $a_1 = 3.0$  cm,  $a_2 = 2.5$  cm,  $l_1 = l_2 = 75$  cm, and  $M_t = 700$  N·m

**5.53.** For the circular shaft shown in Fig. P5.3, determine the twisting moment produced in each part of the shaft.

**5.54.** A circular bar of one-inch (2.54 cm) radius is under the action of an axial tensile load of 30,000 lb (133 kN) and a twisting couple of 25,000 in·lbs (2830 N·m).

(a) Determine the stress throughout the bar.

(b) Find the maximum normal and shearing stress that occurs over all locations and all cross-sectional planes throughout the bar.

**5.55.** Show that for any cylindrical bar of non-circular cross-section in torsion that the stress vector at all points along the lateral boundary acting on any of the normal cross-sectional planes must be tangent to the boundary.

**5.56.** Demonstrate that the displacement and stress for the elliptic bar in torsion may also be used for an elliptic tube, if the inside boundary is defined by

$$\frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = k^2$$

where  $k < 1$ .

5.57. Compare the twisting torque which can be transmitted by a shaft with an elliptical cross-section having a major axis equal to twice the minor axis with a shaft of circular cross-section having a diameter equal to the major axis of the elliptical shaft. Both shafts are of the same material. Also compare the unit twist under the same twisting moment.

5.58. Repeat the previous problem, except that the circular shaft has a diameter equal to the minor axis of the elliptical shaft.

5.59. (a) For an elliptic bar in torsion, show that the magnitude of the maximum shearing stress varies linearly along radial lines  $x_2 = kx_3$  and reaches a maximum on the outer boundary.

(b) Show that on the boundary the maximum shearing stress is given by

$$(T_s)_{\max} = \frac{2M_t}{\pi a^2 b^3} \sqrt{b^2 + x_3^2(a^2 - b^2)}$$

so that the greatest shearing stress does occur at the end of the minor axis.

5.60. Consider the torsion of a cylindrical bar with an equilateral triangular cross-section as in Fig.P5.4.

(a) Show that a warping function  $\varphi = \alpha (3x_2^2 x_3 - x_3^3)$  generates an equilibrium stress field.

(b) Determine the constant  $\alpha$  in order to satisfy the traction-free lateral boundary condition. Demonstrate that the entire lateral surface is traction-free.

(c) Write out explicitly the stress distribution generated by this warping function. Evaluate the maximum shearing stress at the triangular corners and along the line  $x_3 = 0$  in a cross-section. Along the line  $x_3 = 0$  where does the greatest shearing stress occur?

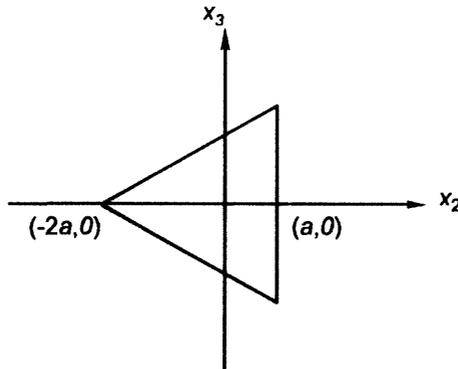


Fig. P5.4

5.61. An alternate manner of formulating the problem of the torsion of a cylinder of noncircular cross-section employs a stress function  $\psi(x_2, x_3)$  such that the stresses are given by

$$T_{12} = \frac{\partial \psi}{\partial x_3}, \quad T_{13} = -\frac{\partial \psi}{\partial x_2}$$

and all other  $T_{ij} = 0$

(a) Demonstrate that the equilibrium equations are identically satisfied for any choice of  $\psi$ .

(b) Show that if  $\psi$  satisfies the equation

$$\frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial x_3^2} = \text{constant}$$

then the stress will correspond to a compatible strain field for simply-connected cross-sectional areas.

(c) Show that the lateral boundary condition requires that  $\nabla \psi$  be in the same direction as the outward normal. In other words, the values of  $\psi$  on the outer boundary is a constant.

**5.62.** A beam of circular cross-section is subjected to pure bending. The magnitude of each end couple is  $14,000 \text{ N}\cdot\text{m}$ . If the maximum normal stress is not to exceed  $0.124 \text{ GPa}$ , what should be the diameter?

**5.63.** The rectangular beam of Example 5.15.1 has a width  $b$  and a height  $1.2b$ . If the right-hand couple is given by  $\mathbf{M} = 24,000\mathbf{e}_2 \text{ ft}\cdot\text{lb}$  ( $32,500 \text{ N}\cdot\text{m}$ ), determine the dimension  $b$  in order that the maximum shearing stress does not exceed  $600 \text{ psi}$  ( $4.14 \text{ MPa}$ ).

**5.64.** Let the beam of Example 5.15.1 be loaded by both the indicated bending moment and a centroidally applied tensile force  $P$ . Determine the magnitude of  $P$  in order that  $T_{11} \geq 0$ .

**5.65.** Verify that if  $\varphi(x_1, x_2)$  satisfy Eq. (5.16.7), then it does correspond to a compatible strain field.

**5.66.** Show that if the bending moment applied to a bar in pure bending is not referred to principal axes, then the flexural stress will be

$$T_{11} = \frac{M_2 I_{zz} + M_3 I_{zy}}{I_{zz} I_{yy} - I_{zy}^2} x_3 - \frac{M_3 I_{yy} + M_2 I_{zy}}{I_{zz} I_{yy} - I_{zy}^2} x_2$$

**5.67.** Figure P5.5 shows the cross-section of a beam subjected to pure bending. If the end couples are given by  $\pm 10^4 \text{ N}\cdot\text{m}$ , find the maximum normal stress.

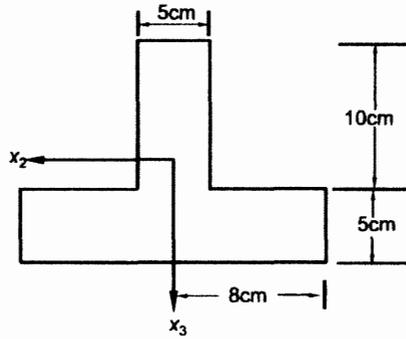


Fig. P5.5

5.68. Consider the stress function

$$\varphi = \alpha_1 x_1^2 + \alpha_2 x_1 x_2 + \alpha_3 x_2^2$$

(a) Verify that this stress function is a possible one for plane strain.

(b) Determine the stresses and sketch the boundary tractions on the rectangular boundary,  $x_1 = 0$ ,  $x_1 = a$ ,  $x_2 = 0$ ,  $x_2 = b$ .

5.69. Consider the stress function  $\varphi = \alpha x_1^2 x_2$

(a) Is this a possible stress function for plane strain?

(b) Determine the stresses.

(c) Determine and sketch the boundary traction on the boundary defined by

$$x_1 = 0, \quad x_1 = a, \quad x_2 = 0, \quad x_2 = b.$$

5.70. Consider the stress function  $\varphi = \alpha x_1^4 + \beta x_2^4$ .

(a) Is this a possible stress function for plane strain?

(b) Determine and sketch the boundary tractions on the rectangular boundary of the previous problem.

5.71. Consider the stress function  $\varphi = \alpha x_1 x_2^2 + \beta x_1 x_2^3$

(a) Is this a possible stress function for plane strain?

(b) Determine the stresses.

(c) Find the condition necessary for the traction on  $x_2 = b$  to vanish and sketch the stress traction on the remaining boundaries  $x_2 = 0$ ,  $x_1 = 0$ ,  $x_1 = a$ .

5.72. By integration, obtain Eq. (5.17.13)

5.73. From Eqs. (5.19.2), show that Eq. (5.19.4) can also be written as:

$$\begin{aligned}
 -M &= \int_a^b T_{\theta\theta} r \, dr \\
 &= \left[ -A \ln \frac{b}{a} - B(b^2 \ln b - a^2 \ln a) - C(b^2 - a^2) \right]
 \end{aligned}$$

5.74. Obtain the general solution of Eq. (5.20.6) as

$$f(r) = A r^2 + B r^4 + C \frac{1}{r^2} + D$$

5.75. A hollow sphere is subjected to an internal pressure  $p_i$  only.

(a) Show that  $T_{rr}$  is always negative (i.e., compressive) and  $T_{\theta\theta}$  is always positive (tensile).

(b) Find the maximum  $T_{\theta\theta}$ .

(c) If the thickness  $t \equiv a_o - a_i$  is small, show that the equation obtained in (b) reduces to  $\frac{p_i a_o}{2t}$

5.76. Using Eq. (5.16.6) in Eq. (5.16.7) to obtain Eq. (5.16.8).

5.77. Derive Eq. (5.16.9).

5.78. Obtain the solution for the differential equation, Eq. (5.17.8).

5.79. Obtain  $u_r$  and  $u_\theta$  from Eqs. (5.17.11) and (5.17.12).

5.80. Verify Eq. (5.19.4)

5.81. Find the general solution for Eq. (5.20.6)

5.82. Write stress strain laws for a monoclinic elastic solid whose plane of symmetry is the  $x_1 x_2$  plane in contracted notation.

5.83. Write stress strain laws for a monoclinic elastic solid whose plane of symmetry is the  $x_3 x_1$  plane in contracted notation.

5.84. Verify any one of the equations in Eqs(iv) of Section 5.26 on transversely isotropic elastic solid.

5.85. Show from the equation  $C'_{1233} = 0$  that  $C_{1133} = C_{2233}$  for a transversely isotropic material [See Section 5.26]

5.86. Referring to Section 5.26, for a transversely isotropic elastic solid, obtain Eq. (ix)

5.87. In Section 5.26 we obtained the reduction in the elastic coefficients for a transversely isotropic elastic solid by demanding that each  $S_\beta$  plane is a plane of material symmetry. We can also obtain the same reduction by demanding that  $C'_{ijkl}$  be the same for all  $\beta$ . Verify that the two procedures lead to the same elastic coefficients.

- 5.88. Verify the relations between  $C_{ij}$  and the engineering constants given in Eqs. (5.29.2a)
- 5.89. Obtain Eq. (5.29.3) from Eq. (5.29.2)
- 5.90. Derive the inequalities expressed in Eq. (5.30.4)
- 5.91. Write down all the restrictions for the engineering constants for a monoclinic elastic solid.
- 5.92. Show that if a tensor is objective, then its inverse is also objective.
- 5.93. Show that the rate of deformation tensor  $\mathbf{D} = \frac{1}{2}[(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T]$  is objective
- 5.94. Show that in a change of frame, the spin tensor  $\mathbf{W}$  transforms in accordance with the equation  $\mathbf{W}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^T + \dot{\mathbf{Q}}\mathbf{Q}^T$
- 5.95. Show that the material derivative of an objective tensor  $\mathbf{T}$  is in general non-objective
- 5.96. The second Rivlin-Ericksen tensor is defined by

$$\mathbf{A}_2 = \dot{\mathbf{A}}_1 + \mathbf{A}_1(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{A}_1$$

where  $\mathbf{A}_1 = 2\mathbf{D}$  [See Prob. 5.93]. Show that  $\mathbf{A}_2$  is objective.

- 5.97. The Jaumann derivative of a second order tensor  $\mathbf{T}$  is

$$\dot{\mathbf{T}} + \mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T}$$

where  $\mathbf{W}$  is the spin tensor [see Prob. 5.94]. Show that the Jaumann derivative of  $\mathbf{T}$  is objective.

- 5.98. In a change of frame, how does the first Piola-Kirchhoff stress tensor transform ?

- 5.99. In a change of frame, how does the second Piola-Kirchhoff tensor transform?

- 5.100. (a) Starting from the assumption that

$$\mathbf{T} = \mathbf{H}(\mathbf{F})$$

and

$$\mathbf{T}^* = \mathbf{H}^*(\mathbf{F}^*),$$

show that in order that the constitutive equation be independent of observers, we must have

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{H}(\mathbf{Q}\mathbf{F})$$

- (b) Choose  $\mathbf{Q} = \mathbf{R}^T$  to obtain

$$\mathbf{T} = \mathbf{R}\mathbf{H}(\mathbf{U})\mathbf{R}^T$$

where  $\mathbf{R}$  is the rotation tensor associated with the deformation gradient  $\mathbf{F}$  and  $\mathbf{U}$  is the right stretch tensor.

- (c) Show that

$$\tilde{\mathbf{T}} = \mathbf{h}(\mathbf{U})$$

where

$$\mathbf{h} = \mathbf{U}\mathbf{H}(\mathbf{U})\mathbf{U}^T$$

Since  $\mathbf{C} = \mathbf{U}^2$ , therefore we may write

$$\tilde{\mathbf{T}} = \mathbf{f}(\mathbf{C})$$

## 6

### Newtonian Viscous Fluid

Substances such as water and air are examples of a fluid. Mechanically speaking they are different from a piece of steel or concrete in that they are unable to sustain shearing stresses without continuously deforming. For example, if water or air is placed between two parallel plates with say one of the plates fixed and the other plate applying a shearing stress, it will deform indefinitely with time if the shearing stress is not removed. Also, in the presence of gravity, the fact that water at rest always conforms to the shape of its container is a demonstration of its inability to sustain shearing stress at rest. Based on this notion of fluidity, we define a fluid to be a class of idealized materials which, when in rigid body motion (including the state of rest), cannot sustain any shearing stress. Water is also an example of a fluid that is referred to as a liquid which undergoes negligible density changes under a wide range of loads, whereas air is a fluid that is referred to as a gas which does otherwise. This aspect of behavior is generalized into the concept of incompressible and compressible fluids. However, under certain conditions (low Mach number flow) air can be treated as incompressible and under other conditions (e.g. the propagation of the acoustic waves) water has to be treated as compressible.

In this chapter, we study a special model of fluid, which has the property that the stress associated with the motion depends linearly on the instantaneous value of the rate of deformation. This model of fluid is known as a **Newtonian fluid** or **linearly viscous fluid** which has been found to describe adequately the mechanical behavior of many real fluids under a wide range of situations. However, some fluids, such as polymeric solutions, require a more general model (**Non-Newtonian Fluids**) for an adequate description. Non-Newtonian fluid models will be discussed in Chapter 8.

#### 6.1 Fluids

Based on the notion of fluidity discussed in the previous paragraphs, we define a **fluid** to be a class of idealized materials which when in rigid body motions (including the state of rest) cannot sustain any shearing stresses. In other words, when a fluid is in a rigid body motion, the stress vector on *any* plane at any point is normal to the plane. That is for any  $\mathbf{n}$ ,

$$\mathbf{T}\mathbf{n} = \lambda\mathbf{n} \tag{i}$$

It is easy to show from Eq. (i), that the magnitude of the stress vector  $\lambda$  is the same for every plane passing through a given point. In fact, let  $\mathbf{n}_1$  and  $\mathbf{n}_2$  be normals to any two such planes, then we have

$$\mathbf{T}\mathbf{n}_1 = \lambda_1\mathbf{n}_1 \quad (\text{ii})$$

and

$$\mathbf{T}\mathbf{n}_2 = \lambda_2\mathbf{n}_2 \quad (\text{iii})$$

Thus,

$$\mathbf{n}_1 \cdot \mathbf{T}\mathbf{n}_2 - \mathbf{n}_2 \cdot \mathbf{T}\mathbf{n}_1 = (\lambda_2 - \lambda_1)\mathbf{n}_1 \cdot \mathbf{n}_2 \quad (\text{iv})$$

Since  $\mathbf{n}_2 \cdot \mathbf{T}\mathbf{n}_1 = \mathbf{n}_1 \cdot \mathbf{T}^T\mathbf{n}_2$  and  $\mathbf{T}$  is symmetric, therefore, the left side of Eq. (iv) is zero.

Thus,

$$(\lambda_1 - \lambda_2)\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$$

Since  $\mathbf{n}_1$  and  $\mathbf{n}_2$  are any two vectors, therefore,

$$\lambda_1 = \lambda_2$$

In other words, on all planes passing through a point, not only are there no shearing stresses but also the normal stresses are all the same. We shall denote this normal stress by  $-p$ . Thus, for a fluid in rigid body motion or at rest

$$\mathbf{T} = -p\mathbf{I} \quad (6.1.1a)$$

Or, in component form

$$T_{ij} = -p\delta_{ij} \quad (6.1.1b)$$

The scalar  $p$  is the magnitude of the compressive normal stress and is known as the **hydrostatic pressure**.

## 6.2 Compressible and Incompressible Fluids

What one generally calls a "liquid" such as water or mercury has the property that its density essentially remains unchanged under a wide range of pressures. Idealizing this property, we define an **incompressible fluid** to be one for which the density of every particle remains the same at all times regardless of the state of stress. That is for an incompressible fluid

$$\frac{D\rho}{Dt} = 0 \quad (6.2.1)$$

It then follows from the equation of conservation of mass, Eq. (3.15.2b)

$$\frac{D\rho}{Dt} + \rho \frac{\partial v_k}{\partial x_k} = 0 \quad (6.2.2)$$

### 350 Equations Of Hydrostatics

that

$$\frac{\partial v_k}{\partial x_k} = 0 \quad (6.2.3a)$$

or,

$$\text{div } \mathbf{v} = 0 \quad (6.2.3b)$$

All incompressible fluids need not have a spatially uniform density (e.g. salt water with nonuniform salt concentration with depth may be modeled as a nonhomogeneous fluid). If the density is also uniform, it is referred to as a “homogeneous fluid,” for which  $\rho$  is constant everywhere.

Substances such as air and vapors which change their density appreciably with pressure are often treated as compressible fluids. Of course, it is not hard to see that there are situations where water has to be regarded as compressible and air may be regarded as incompressible. However, for theoretical studies, it is convenient to regard the incompressible and compressible fluid as two distinct kinds of fluids.

### 6.3 Equations Of Hydrostatics

The equations of equilibrium are [see Eqs. (4.7.3)]

$$\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = 0 \quad (6.3.1)$$

where  $B_i$  are components of body forces per unit mass.

With

$$T_{ij} = -p\delta_{ij},$$

Eq. (6.3.1) becomes

$$\frac{\partial p}{\partial x_i} = \rho B_i \quad (6.3.2a)$$

or,

$$\nabla p = \rho \mathbf{B} \quad (6.3.2b)$$

In the case where  $B_i$  are components of the weight per unit mass, if we let the positive  $x_3$  axis be pointing vertically downward, we have,

$$B_1 = 0, \quad B_2 = 0, \quad B_3 = g \quad (6.3.3)$$

so that

$$\frac{\partial p}{\partial x_1} = 0 \quad (6.3.4a)$$

$$\frac{\partial p}{\partial x_2} = 0 \quad (6.3.4b)$$

$$\frac{\partial p}{\partial x_3} = \rho g \quad (6.3.4c)$$

Equations (6.3.4a, b) state that  $p$  is a function of  $x_3$  alone and Eq. (6.3.4c) gives the pressure difference between point 2 and point 1 in the liquid as

$$p_2 - p_1 = \rho g h \quad (6.3.5)$$

where  $h$  is the depth of point 2 relative to point 1. Thus, the static pressure in the liquid depends only on the depth. It is the same for all particles that are on the same horizontal plane within the same fluid.

If the fluid is in a state of rigid body motion (rate of deformation = 0), then  $T_{ij}$  is still given by Eq. (6.1.1), but the right hand side of Eq. (6.3.1) is equal to the acceleration  $a_i$ , so that the governing equation is given by

$$-\frac{\partial p}{\partial x_i} + \rho B_i = \rho a_i \quad (6.3.6)$$

#### Example 6.3.1

A cylindrical body of radius  $r$ , length  $l$  and weight  $W$  is tied by a rope to the bottom of a container which is filled with a liquid of density  $\rho$  (Fig. 6.1). If the density of the body  $\rho_b$  is less than that of the liquid, find the tension in the rope.

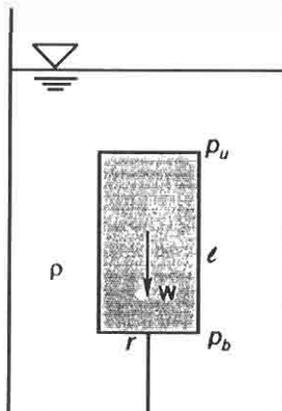


Fig. 6.1

*Solution.* Let  $p_u$  and  $p_b$  be the pressure at the upper and the bottom surface of the cylinder. Let  $T$  be the tension in the rope. Then the equilibrium of the cylindrical body requires that

$$p_b (\pi r^2) - p_u (\pi r^2) - W - T = 0 \tag{i}$$

That is,

$$T = \pi r^2 (p_b - p_u) - W \tag{ii}$$

Now, from Eq. (6.3.5)

$$p_b - p_u = \rho g l \tag{iii}$$

Thus,

$$T = \pi r^2 \rho g l - W \tag{iv}$$

We note that the first term on the right hand side of the above equation is the buoyancy force which is equal to the weight of the liquid displaced by the body.

### Example 6.3.2

In Fig. 6.2, the weight  $W_R$  is supported by the weight  $W_L$ , via the liquids in the container. The area under  $W_R$  is twice that under  $W_L$ . Find  $W_R$  in terms of  $W_L, \rho_1, \rho_2, A_L, h$  ( $\rho_2 < \rho_1$  and assume no mixing).

*Solution.* Using Eq. (6.3.5), we have

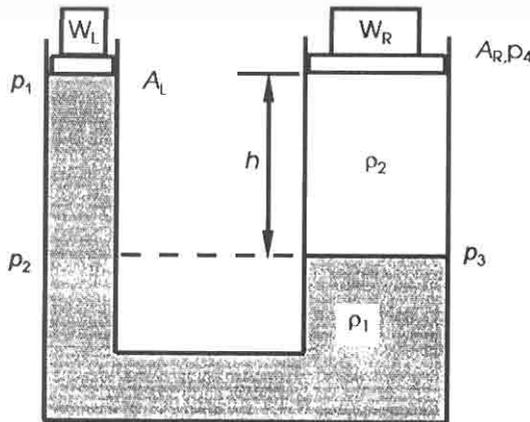


Fig. 6.2

$$p_2 = p_1 + \rho_1 g h \tag{i}$$

$$p_3 = p_2 = p_1 + \rho_1 gh \quad (\text{ii})$$

$$p_4 = p_3 - \rho_2 gh = p_1 + (\rho_1 - \rho_2)gh \quad (\text{iii})$$

Thus,

$$W_R = (p_4)(A_R) = (p_1)(A_R) + (\rho_1 - \rho_2)ghA_R \quad (\text{iv})$$

i.e.,

$$W_R = 2p_1A_L + 2(\rho_1 - \rho_2)ghA_L = 2W_L + 2(\rho_1 - \rho_2)ghA_L \quad (\text{v})$$

### Example 6.3.3

A tank containing a homogeneous fluid moves horizontally to the right with a constant acceleration  $a$  (Fig. 6.3), (a) find the angle  $\theta$  of the inclination of the free surface and (b) find the pressure at any point P inside the fluid.

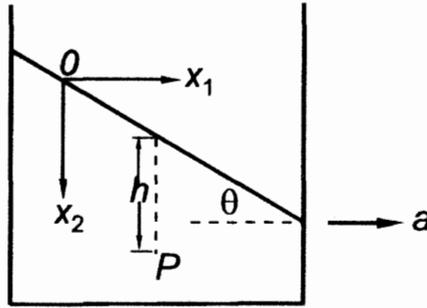


Fig. 6.3

*Solution.* (a) With  $a_1 = a$ ,  $a_2 = a_3 = 0$ ,  $B_1 = B_2 = 0$  and  $B_3 = g$ , the equations of motion, Eqs. (6.3.6) become

$$\rho a = -\frac{\partial p}{\partial x_1} \quad (\text{i})$$

$$0 = -\frac{\partial p}{\partial x_2} \quad (\text{ii})$$

$$0 = -\frac{\partial p}{\partial x_3} + \rho g \quad (\text{iii})$$

From Eq. (ii),  $p$  is independent of  $x_2$ , from Eq. (i)

$$p = -\rho ax_1 + f(x_3) \tag{iv}$$

and from Eqs. (iii) and (iv)

$$\frac{\partial p}{\partial x_3} = \frac{df}{dx_3} = \rho g$$

Thus,

$$f(x_3) = \rho gx_3 + \text{constant.}$$

i.e.,

$$p = -\rho ax_1 + \rho gx_3 + c \tag{v}$$

The integration constant  $c$  can be determined from the fact that on the free surface, the pressure is equal to the ambient pressure  $p_o$ . Let the origin of the coordinate axes (fixed with respect to the earth) be located at a point on the free surface at the instant of interest, then

$$c = p_o$$

Thus, the pressure inside the fluid at any point is given by

$$p = -\rho ax_1 + \rho gx_3 + p_o \tag{vi}$$

To find the equation for the free surface, we substitute  $p = p_o$  in Eq. (vi) and obtain

$$x_3 = \frac{a}{g} x_1 \tag{vii.}$$

Thus, the free surface is a plane with the angle of inclination given by

$$\tan \theta = \frac{dx_3}{dx_1} = \frac{a}{g} \tag{viii}$$

(b) Referring to Fig. 6.3, we have  $(x_3 - h) / x_1 = \tan \theta$ , thus,  $x_3 = h + x_1(a / g)$ , therefore

$$p = -\rho ax_1 + \rho g \left( h + \frac{x_1 a}{g} \right) + p_o = \rho gh + p_o \tag{ix}$$

i.e., the pressure at any point inside the fluid depends only on the depth  $h$  of that point from the free surface directly above it and the pressure at the free surface.

#### Example 6.3.4

For minor altitude differences, the atmosphere can be assumed to have constant temperature. Find the pressure and density distributions for this case.

*Solution.* Let the positive  $x_3$ -axis be pointing vertically upward, then  $\mathbf{B} = -g\mathbf{e}_3$  so that

$$\frac{\partial p}{\partial x_1} = 0 \quad (\text{i})$$

$$\frac{\partial p}{\partial x_2} = 0 \quad (\text{ii})$$

$$\frac{\partial p}{\partial x_3} = -\rho g \quad (\text{iii})$$

From Eqs. (i) and (ii), we see  $p$  is a function of  $x_3$  only, thus Eq. (iii) becomes

$$\frac{dp}{dx_3} = -\rho g \quad (\text{iv})$$

Assuming that  $p$ ,  $\rho$  and  $\Theta$  (absolute temperature) are related by the equation of state for ideal gas, we have

$$p = \rho R \Theta \quad (\text{v})$$

where  $R$  is the gas constant for air. Thus, Eq. (iv) becomes

$$\frac{dp}{p} = -\frac{g}{R\Theta} dx_3. \quad (\text{vi})$$

Integrating, we get

$$\ln p = -\frac{g}{R\Theta} x_3 + \ln p_0, \quad (\text{vii})$$

where  $p_0$  is the pressure at the ground ( $x_3 = 0$ ), thus,

$$p = p_0 e^{(-g/R\Theta)x_3} \quad (\text{viii})$$

and from Eq. (v), if  $\rho_0$  is the density at  $x_3 = 0$ , we have

$$\rho = \rho_0 e^{(-g/R\Theta)x_3} \quad (\text{ix})$$

## 6.4 Newtonian Fluid

When a shear stress is applied to an elastic solid, it deforms from its initial configuration and reaches an equilibrium state with a nonzero shear deformation, the deformation will disappear when the shear stress is removed. When a shear stress is applied to a layer of fluid (such as water, alcohol, mercury, air etc.) it will deform from its initial configuration and eventually reaches a steady state where the fluid continuously deforms with a nonzero rate of shear, as long as the stress is applied. When the shear stress is removed, the fluid will simply remain at the deformed state, obtained prior to the removal of the force. Thus, the state of

shear stress for a fluid in shearing motion is independent of shear deformation, but is dependent on the rate of shear. For such fluids, no shear stress is needed to maintain a given amount of shear deformation, but a definite amount of shear stress is needed to maintain a constant rate of shear of deformation.

Since the state of stress for a fluid under rigid body motion (including rest) is given by an isotropic tensor, therefore in dealing with a fluid in general motion, it is natural to decompose the stress tensor into two parts:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{T}' \tag{6.4.1a}$$

$$T_{ij} = -p\delta_{ij} + T'_{ij} \tag{6.4.1b}$$

where the components of  $\mathbf{T}'$  depend only on the rate of deformation (i.e., not on deformation) in such a way that they are zero when the fluid is under rigid body motion (i.e., zero rate of deformation) and  $p$  is a scalar whose value is not to depend explicitly on the rate of deformation.

We now define a class of idealized materials called **Newtonian** fluids as follows:

I. For every material point, the values of  $T'_{ij}$  at any time  $t$  depend linearly on the components of the rate of deformation tensor

$$D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \tag{6.4.2}$$

at that time and not on any other kinematic quantities (such as higher rates of deformation)

II. The fluid is isotropic with respect to any configuration.

Following the same arguments made in connection with the isotropic linear elastic material, we obtain that for a Newtonian fluid, (also known as **linearly viscous fluid**, the most general form of  $T'_{ij}$  is, with  $\Delta \equiv D_{11} + D_{22} + D_{33} = D_{kk}$ ,

$$T'_{ij} = \lambda \Delta \delta_{ij} + 2\mu D_{ij} \tag{6.4.3}$$

where  $\lambda$  and  $\mu$  are material constants (different from those of an elastic body) having the dimension of (Force)(Time)/(Length)<sup>2</sup>. The stress tensor  $T'_{ij}$  is known as the **viscous stress tensor**. Thus, the total stress tensor is

$$T_{ij} = -p\delta_{ij} + \lambda \Delta \delta_{ij} + 2\mu D_{ij} \tag{6.4.4}$$

i.e.,

$$T_{11} = -p + \lambda \Delta + 2\mu D_{11} \tag{6.4.5a}$$

$$T_{22} = -p + \lambda \Delta + 2\mu D_{22} \tag{6.4.5b}$$

$$T_{33} = -p + \lambda \Delta + 2\mu D_{33} \tag{6.4.5c}$$

$$T_{12} = 2\mu D_{12} \quad (6.4.5d)$$

$$T_{13} = 2\mu D_{13} \quad (6.4.5e)$$

$$T_{23} = 2\mu D_{23} \quad (6.4.5f)$$

The scalar  $p$  in the above equations is called the **pressure**. It is a somewhat ambiguous terminology. As is seen from the above equations, when  $D_{ij}$  are nonzero,  $p$  is only a part of the total compressive normal stress on a plane. It is in general neither the total compressive normal stress on a plane (unless the viscous stress components happen to be zero), nor the mean normal compressive stress, (see next section). As a fluid theory, it is only necessary to remember that the isotropic tensor  $-p\delta_{ij}$  is that part of  $T_{ij}$  which does not depend explicitly on the rate of deformation.

### 6.5 Interpretation of $\lambda$ and $\mu$

Consider the shear flow given by the velocity field:

$$v_1 = v_1(x_2), \quad v_2 = 0, \quad v_3 = 0 \quad (i)$$

For this flow

$$D_{11} = D_{22} = D_{33} = D_{13} = D_{23} = 0 \quad (ii)$$

and

$$D_{12} = \frac{1}{2} \frac{dv_1}{dx_2} \quad (iii)$$

so that

$$T_{11} = T_{22} = T_{33} = -p, \quad T_{13} = T_{23} = 0 \quad (iv)$$

and

$$T_{12} = \mu \frac{dv_1}{dx_2} \quad (6.5.1)$$

Thus,  $\mu$  is the proportionality constant relating the shearing stress to the rate of decrease of angle between two mutually perpendicular material lines (see Sect.3.13). It is called the **first coefficient of viscosity** or simply **viscosity**. From Eq. (6.4.3), we have, for a general velocity field.

$$\frac{1}{3} T_{ii}' = (\lambda + \frac{2}{3}\mu) \Delta \quad (6.5.2)$$

Thus,  $(\lambda + \frac{2}{3}\mu)$  is the proportionality constant relating the viscous mean normal stress to the rate of change of volume. It is known as the **coefficient of bulk viscosity**. The total mean normal stress is given by

$$\frac{1}{3}T_{ii} = -p + (\lambda + \frac{2}{3}\mu)\Delta \tag{6.5.3}$$

and it is clear that the so-called pressure is in general not the mean normal stress, except when either  $\Delta = 0$  or  $(\lambda + \frac{2}{3}\mu)$  is assumed to be zero.

Example 6.5.1

Given the following velocity field:

$$v_1 = -c(x_1 + x_2), v_2 = c(x_2 - x_1), v_3 = 0, \quad c = 1 \text{ s}^{-1} \tag{i}$$

for a Newtonian liquid with viscosity  $\mu = 0.982 \text{ mPa}\cdot\text{s}$  ( $2.05 \times 10^{-5} \text{ lb}\cdot\text{s}/\text{ft}^2$ ). For a plane whose normal is in the  $\mathbf{e}_1$ -direction, (a) find the excess of the total normal compressive stress over the pressure  $p$ , and (b) find the magnitude of the shearing stress.

*Solution.* From

$$T_{11} = -p + 2\mu D_{11} \quad (\Delta = 0) \tag{ii}$$

we have

$$(-T_{11}) - p = -2\mu D_{11} \tag{iii}$$

Now, from Eq. (i),

$$D_{11} = \frac{\partial v_1}{\partial x_1} = -c = -1 \text{ s}^{-1} \tag{iv}$$

Therefore

$$(-T_{11}) - p = -2(0.982)(-1) = 1.96 \text{ mPa} \tag{v}$$

(b)

$$T_{12} = 2\mu D_{12} = \mu \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) = -2c\mu = -1.96 \text{ mPa} \tag{vi}$$

$$T_{13} = 2\mu D_{13} = \mu \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) = 0 \tag{vii}$$

Thus, the magnitude of shearing stress equals 1.96 mPa.

## 6.6 Incompressible Newtonian Fluid

For an incompressible fluid,  $\Delta \equiv D_{ii} = 0$  at all times. Thus, the constitutive equation for such a fluid becomes

$$T_{ij} = -p\delta_{ij} + 2\mu D_{ij} \quad (6.6.1)$$

We see from this equation that

$$T_{ii} = -3p + 2\mu D_{ii} = -3p$$

Thus,

$$p = \frac{-T_{ii}}{3} \quad (6.6.2)$$

Therefore, for an incompressible viscous fluid, the pressure has the meaning of the mean normal compressive stress. The value of  $p$  does not depend explicitly on any kinematic quantities; its value is indeterminate as far as the fluid's mechanical behavior is concerned. In other words, since the fluid is incompressible, one can superpose any pressure to the fluid, without affecting its mechanical behavior. Thus, the pressure in an incompressible fluid is often known constitutively as the "indeterminate pressure". In any given problem with prescribed boundary condition(s) for the pressure, the pressure field is determinate.

Since

$$D_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (6.3)$$

where  $v_i$  are the velocity components, the constitutive equations can be written:

$$T_{ij} = -p\delta_{ij} + \mu \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (6.6.4)$$

i.e.,

$$T_{11} = -p + 2\mu \frac{\partial v_1}{\partial x_1} \quad (6.6.4a)$$

$$T_{22} = -p + 2\mu \frac{\partial v_2}{\partial x_2} \quad (6.6.4b)$$

$$T_{33} = -p + 2\mu \frac{\partial v_3}{\partial x_3} \quad (6.6.4c)$$

$$T_{12} = \mu \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \quad (6.6.4d)$$

$$T_{13} = \mu \left( \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \right) \quad (6.6.4e)$$

$$T_{23} = \mu \left( \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \right) \quad (6.6.4f)$$

## 6.7 Navier-Stokes Equation For Incompressible Fluids

Substituting the constitutive equation [Eq. (6.6.4)] into the equation of motion, Eq. (4.7.2)

$$\rho \left( \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \rho B_i + \frac{\partial T_{ij}}{\partial x_j} \quad (6.7.1)$$

and noting that

$$\begin{aligned} \frac{\partial T_{ij}}{\partial x_j} &= -\frac{\partial p}{\partial x_j} \delta_{ij} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \mu \frac{\partial^2 v_j}{\partial x_j \partial x_i} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + \mu \frac{\partial}{\partial x_i} \left( \frac{\partial v_j}{\partial x_j} \right) \\ &= -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \quad \left( \frac{\partial v_j}{\partial x_j} = 0 \right) \end{aligned}$$

we obtain the following equations of motion in terms of velocity components

$$\rho \left( \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \rho B_i - \frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \quad (6.7.2)$$

i.e.,

$$\rho \left( \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} \right) = \rho B_1 - \frac{\partial p}{\partial x_1} + \mu \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) v_1 \quad (6.7.2a)$$

$$\rho \left( \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3} \right) = \rho B_2 - \frac{\partial p}{\partial x_2} + \mu \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) v_2 \quad (6.7.2b)$$

$$\rho \left( \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3} \right) = \rho B_3 - \frac{\partial p}{\partial x_3} + \mu \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) v_3 \quad (6.7.2c)$$

Or, in invariant form:

$$\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v}) \mathbf{v} \right] = \rho \mathbf{B} - \nabla p + \mu \operatorname{div}(\nabla \mathbf{v}) \quad (6.7.2d)$$

These are known as the **Navier-Stokes Equations** of motion for incompressible Newtonian fluid. There are four unknown functions  $v_1, v_2, v_3$  and  $p$  in the three equations. The fourth equation is supplied by the continuity equation  $\Delta = 0$ , i.e.,

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0 \quad (6.7.3a)$$

or, in invariant form,

$$\text{div } \mathbf{v} = 0 \quad (6.7.3b)$$

#### Example 6.7.1

If all particles have their velocity vectors parallel to a fixed direction, the flow is said to be a **parallel flow** or a **uni-directional flow**. Show that for parallel flows of an incompressible linearly viscous fluid, the total normal compressive stress at any point on any plane parallel to and perpendicular to the direction of flow is the pressure  $p$ .

*Solution.* Let the direction of the flow be the  $x_1$ -axis, then

$$v_2 = 0, v_3 = 0 \quad (i)$$

and from the equation of continuity,

$$\frac{\partial v_1}{\partial x_1} = 0 \quad (ii)$$

Thus, the velocity field for a parallel flow is

$$v_1 = v_1(x_2, x_3, t), v_2 = 0, v_3 = 0 \quad (iii)$$

For this flow,

$$D_{11} = D_{22} = D_{33} = 0 \quad (iv)$$

thus,

$$T_{11} = T_{22} = T_{33} = -p \quad (v)$$

#### Example 6.7.2

Let  $z$ -axis be pointing vertically upward and let

$$h = \frac{p}{\rho g} + z \quad (6.7.4)$$

where  $\rho$  is density and  $g$  is gravitational acceleration. The quantity  $h$  is known as the **piezometric head**. Show that for a uni-direction flow in any direction, the piezometric head is a constant along any direction which is perpendicular to the flow.

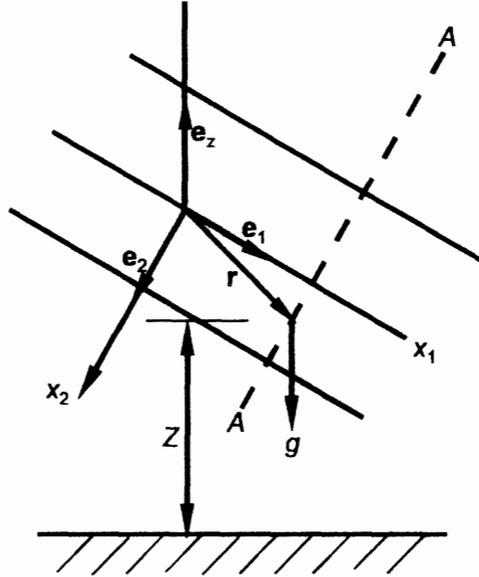


Fig. 6.4

*Solution.* Let  $x_1$ -axis be the direction of flow, then,

$$v_2 = v_3 = 0 \tag{i}$$

Thus, from Eqs. (6.7.2 b and c)

$$\rho B_2 - \frac{\partial p}{\partial x_2} = 0 \tag{ii}$$

$$\rho B_3 - \frac{\partial p}{\partial x_3} = 0 \tag{iii}$$

With  $z$ -axis pointing upward, the body force per unit mass  $\mathbf{B}$  is given by:

$$\mathbf{B} = -g\mathbf{e}_z \tag{iv}$$

where  $\mathbf{e}_z$  is the unit vector in the  $z$ -direction. Thus,

$$B_2 = \mathbf{B} \cdot \mathbf{e}_2 = -g(\mathbf{e}_z \cdot \mathbf{e}_2) \tag{v}$$

Now, Eq. (v) can be written

$$B_2 = -g \frac{\partial}{\partial x_2} [(\mathbf{e}_z \cdot \mathbf{e}_1)x_1 + (\mathbf{e}_z \cdot \mathbf{e}_2)x_2 + (\mathbf{e}_z \cdot \mathbf{e}_3)x_3] \quad (\text{vi})$$

Let  $\mathbf{r}$  be the position vector for a particle at  $\mathbf{x}$ , then

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

and

$$z = \mathbf{e}_z \cdot \mathbf{r} = (\mathbf{e}_z \cdot \mathbf{e}_1)x_1 + (\mathbf{e}_z \cdot \mathbf{e}_2)x_2 + (\mathbf{e}_z \cdot \mathbf{e}_3)x_3 \quad (\text{vii})$$

Thus, Eq. (vi) can be written

$$B_2 = -\frac{\partial}{\partial x_2}(gz) \quad (\text{viii})$$

Using Eqs(ii) and (viii), we obtain

$$\frac{\partial}{\partial x_2}(p + \rho gz) = 0 \quad (\text{ix})$$

or,

$$\frac{\partial}{\partial x_2}\left(\frac{p}{\rho g} + z\right) = 0 \quad (\text{x})$$

Similar derivation will give

$$\frac{\partial}{\partial x_3}\left(\frac{p}{\rho g} + z\right) = 0 \quad (\text{xi})$$

Thus, for all points on the same plane which is perpendicular to the direction of flow (e.g., plane  $A-A$  in Fig. 6.4)

$$\frac{p}{\rho g} + z = \text{constant} \quad (\text{xii})$$

### Example 6.7.3

For the uni-directional flow shown in Fig. 6.5, find the pressure at the point A.

*Solution.* According to the result of the previous example, the piezometric head of the point A and the point B are the same. Thus,

$$\frac{p_A}{\rho g} + z_A = \frac{p_B}{\rho g} + z_B = \frac{p_a}{\rho g} + z_B \quad (\text{i})$$

where  $p_a$  is the atmospheric pressure. Thus,

$$p_A = p_a + \rho g (z_B - z_A) = p_a + \rho g h \cos \theta \tag{ii}$$

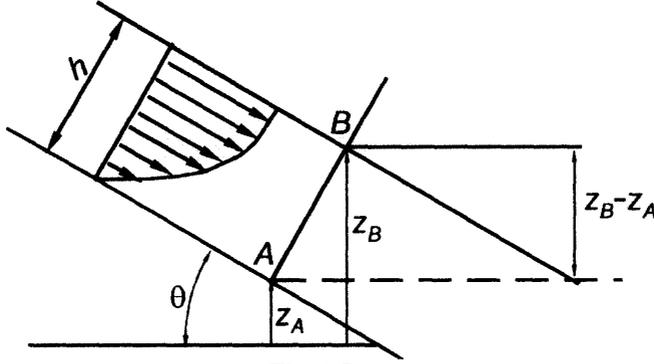


Fig. 6.5

### 6.8 Navier-Stokes Equations for Incompressible Fluids in Cylindrical and Spherical coordinates

#### (A) Cylindrical Coordinates

With  $v_r, v_\theta, v_z$  denoting the velocity components in  $(r, \theta, z)$  direction, the Navier-Stokes equations for an incompressible Newtonian fluid are: [ See Prob. 6.14]

$$\begin{aligned} \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} &= -\frac{1}{\rho} \frac{\partial p}{\partial r} + B_r \\ + \frac{\mu}{\rho} \left[ \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} - \frac{v_r}{r^2} \right] & \end{aligned} \tag{6.8.1a}$$

$$\begin{aligned} \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} &= -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + B_\theta \\ + \frac{\mu}{\rho} \left[ \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2} \right] & \end{aligned} \tag{6.8.1b}$$

$$\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} = -\frac{1}{\rho r} \frac{\partial p}{\partial z} + B_z$$

$$+\frac{\mu}{\rho}\left[\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\right] \quad (6.8.1c)$$

The equation of continuity takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0 \quad (6.8.2)$$

(B)Spherical Coordinates. With  $v_r, v_\theta, v_\phi$  denoting the velocity components in  $(r, \theta, \phi)$  the Navier-Stokes equations for incompressible Newtonian fluid are [see Prob. 6.15]

$$\begin{aligned} \frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^2+v_\phi^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}+B_r+\frac{\mu}{\rho}\left[\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_r)\right)\right. \\ & \left.+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial v_r}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial \phi^2}-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] \end{aligned} \quad (6.8.3a)$$

$$\begin{aligned} \frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_\theta v_r}{r}-\frac{v_\phi^2\cot\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+B_\theta+\frac{\mu}{\rho}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\theta}{\partial r}\right)\right. \\ & \left.+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\theta}{\partial \phi^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right] \end{aligned} \quad (6.8.3b)$$

$$\begin{aligned} \frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi\cot\theta}{r} &= -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi}+B_\phi+\frac{\mu}{\rho}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right)\right. \\ & \left.+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(v_\phi\sin\theta)\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial \phi^2}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cot\theta}{r^2\sin\theta}\frac{\partial v_\theta}{\partial \phi}\right] \end{aligned} \quad (6.8.3c)$$

The equation of continuity takes the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(v_\theta\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}=0 \quad (6.8.4)$$

## 6.9 Boundary Conditions

On a rigid boundary, we shall impose the **non-slip** condition (also known as the adherence condition), i.e., the fluid layer next to a rigid surface moves with that surface, in particular if the surface is at rest, the velocity of the fluid at the surface is zero. The nonslip condition is well supported by experiments for practically all fluids, including those that do not wet the surface (e.g. mercury) and Non-Newtonian fluids (e.g., most polymeric fluids).

### 6.10 Streamline, Pathline, Streakline, Steady, Unsteady, Laminar and Turbulent Flow

(a) *Streamline.*

A **streamline** at time  $t$  is a curve whose tangent at every point has the direction of the instantaneous velocity vector of the particle at the point. Experimentally, streamlines on the surface of a fluid are often obtained by sprinkling it with reflecting particles and making a short-time exposure photograph of the surface. Each reflecting particle produces a short line on the photograph approximating the tangent to a streamline. Mathematically, streamlines can be obtained from the velocity field  $\mathbf{v}(\mathbf{x}, t)$  as follows:

Let  $\mathbf{x} = \mathbf{x}(s)$  be the parametric equation for the streamline at time  $t$ , which passes through a given point  $\mathbf{x}_0$ . Then an  $s$  can always be chosen such that

$$\frac{d\mathbf{x}}{ds} = \mathbf{v}(\mathbf{x}, t) \quad (6.10.1a)$$

$$\mathbf{x}(0) = \mathbf{x}_0 \quad (6.10.1b)$$

#### Example 6.10.1

Given the velocity field in dimensionless form<sup>†</sup>

$$v_1 = \frac{x_1}{1+t}, \quad v_2 = x_2, \quad v_3 = 0 \quad (i)$$

find the streamline which passes through the point  $(\alpha_1, \alpha_2, \alpha_3)$  at time  $t$

*Solution.* From

$$\frac{dx_1}{ds} = \frac{x_1}{1+t},$$

we have

$$\int_{\alpha_1}^{x_1} \frac{dx_1}{x_1} = \int_0^s \frac{1}{1+t} ds \quad (ii)$$

Thus,

$$\ln x_1 - \ln \alpha_1 = \frac{s}{1+t}$$

i.e.,

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<sup>†</sup> The example is chosen to demonstrate the differences between streamlines, pathlines and streaklines. The velocity field obviously does not correspond to an incompressible fluid.

$$x_1 = \alpha_1 \exp \left[ \frac{s}{1+t} \right] \quad (\text{iii})$$

Similarly, from  $dx_2/ds = x_2$ , we have

$$\int_{\alpha_2}^{x_2} \frac{dx_2}{x_2} = \int_0^s ds \quad (\text{iv})$$

Thus,  $x_2 = \alpha_2 e^s$ . Obviously,  $x_3 = \alpha_3$ .

### (ii) Pathline

A **pathline** is the path traversed by a fluid particle. To photograph a pathline, it is necessary to use long time exposure of a reflecting particle. Mathematically, the pathline of a particle which was at  $\mathbf{X}$  at time  $t_0$  can be obtained from the velocity field  $\mathbf{v}(\mathbf{x}, t)$  as follows:

Let  $\mathbf{x} = \mathbf{x}(t)$  be the pathline, then

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t) \quad (6.10.2a)$$

$$\mathbf{x}(t_0) = \mathbf{X} \quad (6.10.2b)$$

### Example 6.10.2

For the velocity field of the previous example, find the pathline for a particle which was at  $(X_1, X_2, X_3)$  at time  $t_0$

*Solution* From

$$\frac{dx_1}{dt} = \frac{x_1}{1+t} \quad (\text{i})$$

we have

$$\int_{X_1}^{x_1} \frac{dx_1}{x_1} = \int_{t_0}^t \frac{dt}{1+t} \quad (\text{ii})$$

Thus,

$$\ln x_1 - \ln X_1 = \ln(1+t) - \ln(1+t_0), \quad (\text{iii})$$

i.e.,

$$x_1 = X_1 \frac{1+t}{1+t_0} \quad (\text{iv})$$

Similarly from  $dx_2/dt = x_2$ , we have

$$\int_{X_2}^{x_2} \frac{dx_2}{x_2} = \int_{t_0}^t dt \tag{v}$$

thus

$$x_2 = X_2 e^{t-t_0} \tag{vi}$$

and obviously,  $x_3 = X_3$

(iii) *Streakline*

A **streakline** through a fixed point  $\mathbf{x}_0$  is the line at time  $t$  formed by all the particles which passed through  $\mathbf{x}_0$  at  $\tau < t$ .

Let  $\mathbf{X} = \mathbf{X}(\mathbf{x}, t)$  denote the inverse of  $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ , then the particle which was at  $\mathbf{x}_0$  at time  $\tau$ , has the material coordinates given by  $\mathbf{X} = \mathbf{X}(\mathbf{x}_0, \tau)$ ; this same particle is then at  $\mathbf{x} = \mathbf{x}(\mathbf{X}(\mathbf{x}_0, \tau), t)$  at time  $t$ . Thus, the streakline at time  $t$  is given by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}(\mathbf{x}_0, \tau), t) \text{ for fixed } t \text{ and variable } \tau \tag{6.10.3}$$

Example 6.10.3

Given the dimensionless velocity field

$$v_1 = \frac{x_1}{1+t}, \quad v_2 = x_2, \quad v_3 = 0 \tag{i}$$

find the streakline formed by the particles which passed through the spatial position  $(\alpha_1, \alpha_2, \alpha_3)$ .

*Solution.* The pathline equations for this velocity field was obtained in Example 6.10.2 to be

$$x_1 = X_1 \frac{1+t}{1+t_0}, \quad x_2 = X_2 e^{t-t_0}, \quad x_3 = X_3 \tag{ii}$$

From which we obtain the inverse equations

$$X_1 = x_1 \frac{1+t_0}{1+t}, \quad X_2 = x_2 e^{-t+t_0}, \quad X_3 = x_3 \tag{iii}$$

Thus, the particle which passes through  $\alpha_1, \alpha_2, \alpha_3$  at time  $\tau$  is given by

$$X_1 = \alpha_1 \frac{1+t_0}{1+\tau}, \quad X_2 = \alpha_2 e^{-\tau+t_0}, \quad X_3 = \alpha_3 \tag{iv}$$

Substituting Eq. (iv) into Eq. (ii), we obtain the parametric equations for the streakline to be

$$x_1 = \alpha_1 \frac{1+t}{1+\tau}, \quad x_2 = \alpha_2 e^{t-\tau}, \quad x_3 = \alpha_3$$

Example 6.10.4

Given the two dimensional problem

$$v_1 = kx_2, \quad v_2 = 0 \tag{i}$$

Obtain (a) the streamline passing through the point  $(\alpha_1, \alpha_2)$

(b) the pathline for the particle  $(X_1, X_2)$  and

(c) the streakline for the particles which passed through the point  $(\alpha_1, \alpha_2)$

*Solution.* (a) From Eq. (i), we have

$$\frac{dx_1}{ds} = kx_2, \quad \frac{dx_2}{ds} = 0 \tag{ii}$$

thus,

$$x_2 = \alpha_2, \quad x_1 = \alpha_1 + k\alpha_2 s, \quad 0 \leq s < \infty \tag{iii}$$

This is obviously a straight line parallel to the  $x_1$  axis.

(b) from (i), we have

$$\frac{dx_1}{dt} = kx_2, \quad \frac{dx_2}{dt} = 0 \tag{iv}$$

thus,

$$x_2 = X_2, \quad x_1 = X_1 + kX_2 t, \quad 0 \leq t < \infty \tag{v}$$

Again, this is a straight line parallel to the  $x_1$  axis.

(c) From the results of (b), we have

$$X_1 = x_1 - kx_2 t, \quad X_2 = x_2 \tag{vi}$$

therefore,

$$X_1 = \alpha_1 - k\alpha_2 \tau, \quad X_2 = \alpha_2 \tag{vii}$$

Substituting Eq. (vii) into Eq. (v), we obtain

$$x_1 = \alpha_1 + k\alpha_2 (t-\tau), \quad x_2 = \alpha_2, \quad -\infty \leq \tau < t \tag{viii}$$

Again, this is a straight line parallel to the  $x_1$  axis.

(iv) *Steady and Unsteady Flow*

A flow is called **steady** if at every fixed location nothing changes with time. Otherwise, the flow is called **unsteady**. It is important to note, however that in a steady flow, the velocity, acceleration, temperature etc. of a given fluid particle in general changes with time. In other words, let  $\Psi$  be any dependent variable, then in a steady flow,  $(\partial\Psi/\partial t)_{x-\text{fixed}} = 0$ , but  $D\Psi/Dt$  is in general not zero. For example, the steady flow given by the velocity field

$$v_1 = x_1, v_2 = -x_2, v_3 = 0 \quad (\text{i})$$

has an acceleration field given by

$$a_1 = \frac{\partial v_1}{\partial t} + v_1 \frac{\partial v_1}{\partial x_1} + v_2 \frac{\partial v_1}{\partial x_2} + v_3 \frac{\partial v_1}{\partial x_3} = 0 + x_1(1) + 0 + 0 = x_1 \quad (\text{ii})$$

$$a_2 = \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3} = 0 + 0 + (-x_2)(-1) + 0 = x_2 \quad (\text{iii})$$

$$a_3 = 0 \quad (\text{iv})$$

We note that for steady flow the pathlines coincide with the streamlines and streaklines.

(v) *Laminar and Turbulent Flow*

A **laminar** flow is a very orderly flow in which the fluid particles move in smooth layers, or laminae, sliding over particles in adjacent laminae without mixing with them. Such flow are generally realized at slow speed. For the case of water flowing through a tube of circular cross-section, it was found by Reynolds who observed the thin filaments of dye in the tube, that when the dimensionless parameter  $N_R$  (now known as **Reynolds number**) defined by

$$N_R = \frac{v_m \rho d}{\mu} \quad (6.10.3)$$

[where  $v_m$  is the average velocity in the pipe,  $d$  the diameter of the pipe, and  $\rho$  and  $\mu$  the density and viscosity of the fluid], is less than a certain value (approximately 2100), the thin filament of dye was maintained intact throughout the tube, forming straight lines parallel to the axis of the tube. Any accidental disturbances were rapidly obliterated. As the Reynolds number is increased the flow becomes increasingly sensitive to small perturbations until a stage is reached wherein the dye filament broke and diffused through the flowing water. This phenomenon of irregular intermingling of fluid particle in the flow is termed **turbulent**. In the case of pipe flow, the upper limit of the Reynolds number, beyond which the flow is turbulent, is indeterminate. Depending on the experimental setup and the initial quietness of fluid, this upper limit can be as high as 100,000.

In the following sections, we restrict ourselves to the study of laminar flows only. It is therefore to be understood that the solutions to be presented are valid only within certain limits of some parameter (such as Reynolds number) governing the stability of the flow.

In the following sections, we shall present some examples of laminar flows of an incompressible Newtonian fluid.

### 6.11 Plane Couette Flow

The steady unidirectional flow, under zero pressure gradient in the flow direction, of an incompressible viscous fluid between two horizontal plates of infinite extent, one fixed and the other moving in its own plane with a constant velocity  $v_0$  is known as the **plane Couette flow** (Fig. 6.6).

Let  $x_1$  be the direction of the flow. Then  $v_2 = v_3 = 0$ . It follows from the continuity equation that  $v_1$  can not depend on  $x_1$ . Let  $x_1x_2$  plane be the plane of flow, then the velocity field for the plane Couette flow is of the form

$$v_1 = v(x_2), \quad v_2 = 0, \quad v_3 = 0 \quad (i)$$

From the Navier-Stokes equation and the boundary conditions  $v(0) = 0$  and  $v(d) = v_0$ , it can be shown (we leave it as an exercise) that

$$v(x_2) = \frac{v_0 x_2}{d} \quad (6.11.1)$$

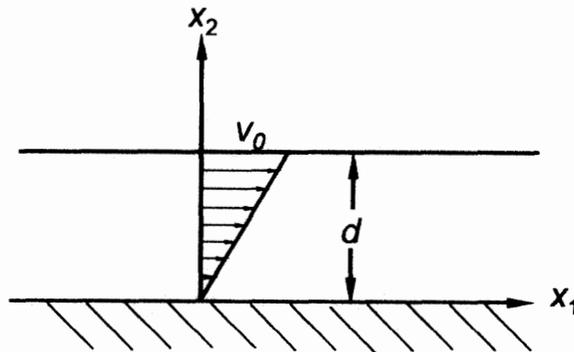


Fig. 6.6

### 6.12 Plane-Poiseuille Flow

The plane Poiseuille flow is the two-dimensional steady unidirectional flow between two fixed plates of infinite extent. Let  $x_1$  be the direction of flow,  $x_2$  be perpendicular to the boundary plates and the flow be unbounded in the  $x_3$  direction. Then the velocity field is of the following form:

$$v_1 = v(x_2), \quad v_2 = 0 \quad \text{and} \quad v_3 = 0$$

Let us first consider the case where gravity is neglected. We shall show later that the presence of gravity does not at all affect the flow field, it only modifies the pressure field.

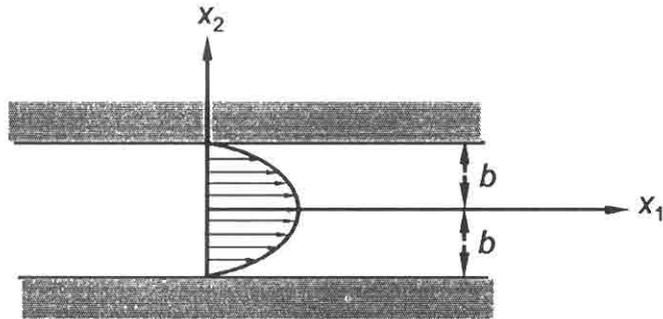


Fig. 6.7

In the absence of body forces, the Navier-Stokes equations, Eqs. (6.7.2) yield:

$$\frac{\partial p}{\partial x_1} = \mu \frac{d^2 v}{dx_2^2} \tag{6.12.1a}$$

$$\frac{\partial p}{\partial x_2} = 0 \tag{6.12.1b}$$

$$\frac{\partial p}{\partial x_3} = 0 \tag{6.12.1c}$$

Equations (6.12.1b) and (6.12.1c) state that  $p$  does not depend on  $x_2$  and  $x_3$ . If we differentiate Eq. (6.12.1a) with respect to  $x_1$ , and noting that the right hand side is a function of  $x_2$  only, we obtain

$$\frac{d^2 p}{dx_1^2} = 0 \tag{i}$$

Thus,

$$\frac{dp}{dx_1} = \text{a constant} \quad (\text{ii})$$

i.e., in plane Poiseuille flow, the pressure gradient is a constant along the flow direction. This pressure gradient is the driving force for the flow. Let

$$\frac{dp}{dx_1} = -\alpha \quad (6.12.2)$$

so that a positive  $\alpha$  corresponds to the case where the pressure decreases along the flow direction, then Eq. (6.12.1a) becomes

$$-\alpha = \mu \frac{d^2v}{dx_2^2} \quad (\text{iii})$$

Integrating, one gets

$$\mu \frac{dv}{dx_2} = -\alpha x_2 + C$$

and

$$\mu v = -\frac{\alpha x_2^2}{2} + Cx_2 + D$$

Referring to Fig. 6.7, the boundary conditions are:

$$v(-b) = v(+b) = 0 \quad (\text{iv})$$

thus, the solution is:

$$v(x_2) = \frac{\alpha}{2\mu}(b^2 - x_2^2) \quad (6.12.3)$$

Thus, the velocity profile is a parabola, with a maximum velocity at the mid-channel given by

$$v_{\max} = \frac{\alpha}{2\mu}b^2 \quad (6.12.4)$$

The flow volume per unit time per unit width passing any cross-section can be obtained by integration:

$$Q = \int_{-b}^b v dx_2 = \frac{\alpha}{\mu} \left( \frac{2b^3}{3} \right) \quad (6.12.5)$$

The average velocity is

$$\bar{v} = \frac{Q}{2b} = \frac{\alpha}{\mu} \frac{b^2}{3} \quad (6.12.6)$$

### 374 Plane-Poiseuille Flow

We shall now prove that in the presence of gravity and independent of the inclination of the channel, the Poiseuille flow always has the velocity profile given by Eq. (6.12.3).

Let  $\mathbf{k}$  be a unit vector pointing upward in the vertical direction, then the body force is:

$$\mathbf{B} = -g\mathbf{k} \quad (6.12.7)$$

and the components of the body force in the  $x_1, x_2$  and  $x_3$  directions are:

$$B_1 = -g\mathbf{e}_1 \cdot \mathbf{k} \quad B_2 = -g\mathbf{e}_2 \cdot \mathbf{k} \quad B_3 = -g\mathbf{e}_3 \cdot \mathbf{k} \quad (v)$$

Let  $\mathbf{r}$  be the position vector of a fluid particle and let  $y$  be its vertical coordinate. Then

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \quad (vi)$$

and

$$y = \mathbf{r} \cdot \mathbf{k} = x_1(\mathbf{e}_1 \cdot \mathbf{k}) + x_2(\mathbf{e}_2 \cdot \mathbf{k}) + x_3(\mathbf{e}_3 \cdot \mathbf{k}) \quad (vii)$$

Now, using Eq. (vii) we can write the body force components Eq. (v) as follows:

$$B_1 = -g \frac{\partial y}{\partial x_1} \quad B_2 = -g \frac{\partial y}{\partial x_2} \quad B_3 = -g \frac{\partial y}{\partial x_3} \quad (viii)$$

Thus, the Navier-Stokes equations can be written

$$0 = -\frac{\partial(p + \rho gy)}{\partial x_1} + \mu \frac{\partial^2 v}{\partial x_2^2} \quad (6.12.8a)$$

$$0 = -\frac{\partial(p + \rho gy)}{\partial x_2} \quad (6.12.8b)$$

$$0 = -\frac{\partial(p + \rho gy)}{\partial x_3} \quad (6.12.8c)$$

These equations are the same as Eqs. (6.12.1) except that the pressure  $p$  is replaced by  $p + \rho gy$ . From these equations, one clearly will obtain the same parabolic velocity profile, except that the driving force in this case is the gradient of  $p + \rho gy$  in the flow direction, instead of simply the gradient of  $p$ . We note that  $[p / (\rho g) + y]$  has been defined in Example 6.7.2 as the **piezometric head**. We can also say that the driving force is the gradient of the piezometric head and the piezometric head is a constant along any direction perpendicular to the flow.

### 6.13 Hagen-Poiseuille Flow

The so-called **Hagen-Poiseuille flow** is a steady unidirectional axisymmetric flow in a circular cylinder. Thus, we look for the velocity field in cylindrical coordinates in the following form

$$v_r = 0, \quad v_\theta = 0, \quad v_z = v(r) \quad (i)$$

The velocity field given by Eq. (i) obviously satisfies the equation of continuity:

$$\frac{1}{r} \frac{\partial}{\partial r}(rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \tag{ii}$$

for any  $v(r)$ .

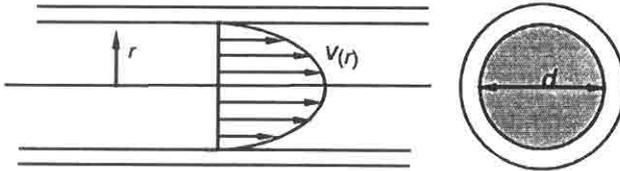


Fig. 6.8

In the absence of body forces, the Navier-Stokes equations, in cylindrical coordinates for the velocity field of Eq. (i) are :

$$0 = -\frac{\partial p}{\partial r} \tag{6.13.1a}$$

$$0 = -\frac{\partial p}{\partial \theta} \tag{6.13.1b}$$

$$0 = -\frac{\partial p}{\partial z} + \mu \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dv}{dr} \right) \right] \tag{6.13.1c}$$

From Eqs. (6.13.1a) and (6.13.1b), we see that  $p$  depends only on  $z$  and from Eq. (6.13.1c), we have

$$\frac{d^2 p}{dz^2} = 0 \tag{i}$$

thus,  $dp/dz$  is a constant. Let

$$\alpha \equiv -\frac{dp}{dz} \tag{6.13.2}$$

then

$$-\frac{\alpha}{\mu} = \frac{1}{r} \frac{d}{dr} \left( r \frac{dv}{dr} \right) \tag{ii}$$

Thus,

$$\frac{dv}{dr} = -\frac{\alpha r}{2\mu} + \frac{b}{r} \quad (\text{iii})$$

and

$$v = -\frac{\alpha r^2}{4\mu} + b \ln r + c \quad (\text{iv})$$

Since  $v$  must be bounded in the flow region, the integration constant  $b$  must be zero. Now, the nonslip condition on the cylindrical wall demands that

$$v = 0 \text{ at } r = \frac{d}{2} \quad (\text{v})$$

where  $d$  is the diameter of the pipe, thus

$$c = \frac{\alpha}{\mu} \left( \frac{d^2}{16} \right) \quad (\text{vi})$$

and

$$v = \frac{\alpha}{4\mu} \left( \frac{d^2}{4} - r^2 \right) \quad (6.13.3)$$

The above equation states that the velocity over the cross-section is distributed in the form of a paraboloid of revolution.

The maximum velocity is (at  $r = 0$ )

$$v_{\max} = \frac{\alpha d^2}{16\mu} \quad (6.13.4)$$

The mean velocity  $\bar{v}$  is

$$\bar{v} = \frac{1}{(\pi d^2/4)_A} \int v dA = \frac{\alpha d^2}{32\mu} = \frac{v_{\max}}{2} \quad (6.13.5)$$

and the volume rate of flow  $Q$  is

$$Q = \left( \frac{\pi d^2}{4} \right) \cdot \bar{v} = \frac{\alpha \pi d^4}{128\mu} \quad (6.13.6a)$$

where

$$\alpha = -dp/dz \quad (6.13.6b)$$

As in the case of plane Poiseuille flow, if the effect of gravity is included, the velocity profile in the pipe remains the same as that given by Eq. (6.13.3), however, the driving force now is

the gradient of  $(p + \rho gy)$  where  $y$  is the vertical height measured from some reference datum, and the piezometric head  $(p/\rho g + y)$  is a constant along any direction perpendicular to the flow. [see Example 6.7.2].

### 6.14 Plane Couette Flow of Two Layers of Incompressible Fluids

Let the viscosity and the density of the top layer be  $\mu_1$  and  $\rho_1$  and those of the bottom layer be  $\mu_2$  and  $\rho_2$ . Let  $x_1$  be the direction of flow and let  $x_2 = 0$  be the interface. We look for steady unidirectional flows of the two layers between the infinite plates  $x_2 = +b_1$  and  $x_2 = -b_2$ . The plate  $x_2 = -b_2$  is fixed and the plate  $x_2 = +b_1$  is moving on its own plane with velocity  $v_0$ . The pressure gradient in the flow direction is assumed to be zero. (Fig. 6.9).

Let the velocity distribution in the top layer be

$$v_1^{(1)} = v^{(1)}(x_2), \quad v_2^{(1)} = v_3^{(1)} = 0 \quad (\text{i})$$

and that in the bottom layer be

$$v_1^{(2)} = v^{(2)}(x_2), \quad v_2^{(2)} = v_3^{(2)} = 0 \quad (\text{ii})$$

the equation of continuity is clearly satisfied for each layer. The Navier-Stokes equations give:

For layer 1,

$$0 = \mu_1 \frac{d^2 v^{(1)}}{dx_2^2} \quad (6.14.1a)$$

$$0 = -\frac{\partial p^{(1)}}{\partial x_2} - \rho_1 g \quad (6.14.1b)$$

$$0 = -\frac{\partial p^{(1)}}{\partial x_3} \quad (6.14.1c)$$

For layer 2,

$$0 = \mu_2 \frac{d^2 v^{(2)}}{dx_2^2} \quad (6.14.2a)$$

$$0 = -\frac{\partial p^{(2)}}{\partial x_2} - \rho_2 g \quad (6.14.2b)$$

$$0 = -\frac{\partial p^{(2)}}{\partial x_3} \quad (6.14.2c)$$

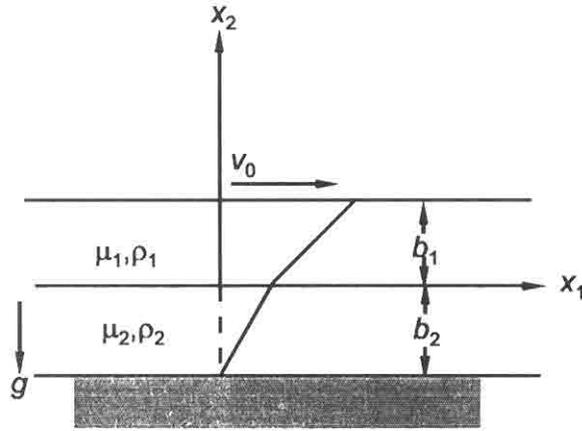


Fig. 6.9

From Eqs. (6.14.1),

$$v^{(1)} = A_1 x_2 + B_1 \text{ and } p^{(1)} = -\rho_1 g x_2 + C_1 \tag{iii}$$

From Eqs. (6.14.2),

$$v^{(2)} = A_2 x_2 + B_2 \text{ and } p^{(2)} = -\rho_2 g x_2 + C_2 \tag{iv}$$

Since the bottom plate is fixed

$$v^{(2)} = 0 \text{ at } x_2 = -b_2 \tag{v}$$

and we have

$$B_2 = A_2 b_2 \tag{vi}$$

Since the top plate is moving with  $v_0$  to the right, therefore  $v^{(1)} = v_0$  at  $x_2 = +b_1$  and we have

$$B_1 = v_0 - A_1 b_1 \tag{vii}$$

At the interface  $x_2 = 0$ , we must have  $v^{(1)} = v^{(2)}$  so that there is no slipping at the fluid interface. Therefore,

$$B_1 = B_2 \tag{viii}$$

Furthermore, from Newton's third law, we have, on  $x_2 = 0$ , the stress vectors on the two layers are related by

$$\mathbf{t}_{-e_2}^{(1)} = -\mathbf{t}_{+e_2}^{(2)} \tag{ix}$$

In terms of stress tensors, we have  $\mathbf{T}^{(1)}\mathbf{e}_2 = \mathbf{T}^{(2)}\mathbf{e}_2$ . That is

$$T_{12}^{(1)} = T_{12}^{(2)}, T_{22}^{(1)} = T_{22}^{(2)}, T_{32}^{(1)} = T_{32}^{(2)} \tag{x}$$

In other words, these stress components must be continuous across the fluid interface. Since

$$T_{12}^{(1)} = 2\mu_1 D_{12}^{(1)} = \mu_1 \frac{dv^{(1)}}{dx_2} = \mu_1 A_1 \tag{xi}$$

$$T_{12}^{(2)} = 2\mu_2 D_{12}^{(2)} = \mu_2 \frac{dv^{(2)}}{dx_2} = \mu_2 A_2 \tag{xii}$$

the condition  $T_{12}^{(1)} = T_{12}^{(2)}$  gives

$$\mu_1 A_1 = \mu_2 A_2 \tag{xiii}$$

Note that this condition means that the slope of the velocity profile is not continuous at  $x_2 = 0$ . Also

$$T_{22}^{(1)} = -p^{(1)} + 2\mu_1 D_{22}^{(1)} = -p^{(1)} \tag{xiv}$$

and

$$T_{22}^{(2)} = -p^{(2)} \tag{xv}$$

so that  $T_{22}^{(1)} = T_{22}^{(2)}$  at  $x_2 = 0$  gives  $C_1 = C_2 = p_o$ , the pressure at the interface. Since  $T_{32}^{(1)} = 0$  and  $T_{32}^{(2)} = 0$ , the condition  $T_{32}^{(1)} = T_{32}^{(2)}$  is clearly satisfied. From Eqs. (vi,vii,viii,xiii), we obtain

$$A_1 = \frac{\mu_2 v_o}{(\mu_2 b_1 + \mu_1 b_2)} \tag{xvi}$$

$$A_2 = \frac{\mu_1}{\mu_2} A_1 = \frac{\mu_1 v_o}{(\mu_2 b_1 + \mu_1 b_2)} \tag{xvii}$$

and

$$B_2 = B_1 = \frac{\mu_1 b_2 v_o}{(\mu_2 b_1 + \mu_1 b_2)} \tag{xviii}$$

Thus, the velocity distributions are

$$v_1^{(1)} = \frac{(\mu_2 x_2 + \mu_1 b_2)v_o}{(\mu_2 b_1 + \mu_1 b_2)}, \quad v_2^{(1)} = v_3^{(1)} = 0 \tag{6.14.3a}$$

and

$$v_1^{(2)} = \frac{(\mu_1 x_2 + \mu_1 b_2)v_o}{(\mu_2 b_1 + \mu_1 b_2)}, \quad v_2^{(2)} = v_3^{(2)} = 0 \tag{6.14.3b}$$

Note that in the case of  $b_2 = 0$ ,  $v_1^{(1)} = (v_o/b_1)x_2$ , which is the case of plane Couette flow of a single fluid.

**6.15 Couette Flow**

The laminar steady two-dimensional flow of an incompressible Newtonian fluid between two coaxial infinitely long cylinders caused by the rotation of either one or both cylinders with constant angular velocities is known as **Couette flow**.

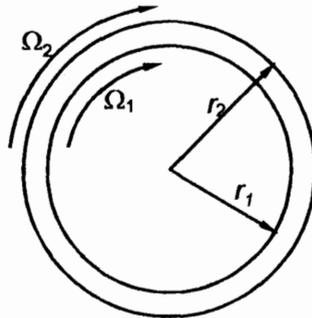
For this flow, we look for the velocity field in the following form in cylindrical coordinates

$$v_r = 0, \quad v_\theta = v(r), \quad v_z = 0 \tag{i}$$

This velocity field obviously satisfies the equation of continuity [Eq. (6.8.2)] for any  $v(r)$ .

In the absence of body forces and taking into account the rotational symmetry of the flow (i.e., nothing depends on  $\theta$ ), we have, from the second Navier Stokes-equation of motion, Eq. (6.8.1b), for the two-dimensional flow,

$$\frac{d^2 v}{dr^2} + \frac{1}{r} \frac{dv}{dr} - \frac{v}{r^2} = 0 \tag{6.15.1}$$



**Fig. 6.10**

It is easily verified that  $v = r$  and  $v = 1/r$  satisfy the above equation. Thus, the general solution is

$$v = Ar + \frac{B}{r} \quad (6.15.2)$$

where A and B are arbitrary constants

Let  $r_1$  and  $r_2$  denote the radii of the inner and outer cylinders, respectively,  $\Omega_1$  and  $\Omega_2$  their respective angular velocities. Then

$$r_1\Omega_1 = Ar_1 + \frac{B}{r_1} \quad (ii)$$

and

$$r_2\Omega_2 = Ar_2 + \frac{B}{r_2} \quad (iii)$$

from which the constants A and B can be obtained to be

$$A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2}, \quad B = \frac{r_1^2 r_2^2 (\Omega_1 - \Omega_2)}{r_2^2 - r_1^2} \quad (iv)$$

so that

$$v_\theta = v = \frac{1}{r_2^2 - r_1^2} \left[ r (\Omega_2 r_2^2 - \Omega_1 r_1^2) - \frac{r_1^2 r_2^2}{r} (\Omega_2 - \Omega_1) \right] \quad (6.15.3)$$

and

$$v_r = v_z = 0$$

The shearing stress at the walls is equal to

$$|\tau_{r\theta}| = |2\mu D_r \theta| = \mu r \left| \frac{d}{dr} \left( \frac{v_\theta}{r} \right) \right|_{r=r_1, r_2} = \left| \frac{2B}{r^2} \right|_{r=r_1, r_2} \quad (6.15.4)$$

It can be obtained (see Prob. 6.27) that the torque per unit length which must be applied to the cylinders ( equal and opposite for the two cylinders ) to maintain the flow is given by

$$M = \frac{4\pi\mu r_1^2 r_2^2 |\Omega_1 - \Omega_2|}{r_2^2 - r_1^2} \quad (6.15.5)$$

## 6.16 Flow Near an Oscillating Plate

Let us consider the following unsteady parallel flow near an oscillating plate:

$$v_1 = v(x_2, t), \quad v_2 = 0, \quad v_3 = 0 \quad (i)$$

Omitting body forces and assuming a constant pressure field, the only nontrivial Navier-Stokes equation is

$$\rho \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial x_2^2} \quad (6.16.1)$$

It can be easily verified that

$$v = ae^{-\beta x_2} \cos(\omega t - \beta x_2 + \varepsilon) \quad (6.16.2a)$$

satisfies the above equation if

$$\beta = \sqrt{\rho\omega / 2\mu} \quad (6.16.2b)$$

From Eq. (6.16.2a), the fluid velocity at  $x_2 = 0$  is

$$v = a \cos(\omega t + \varepsilon) \quad (6.16.3)$$

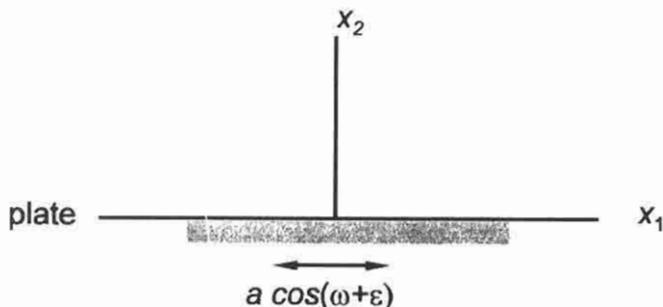


Fig. 6.11

Thus, the solution Eq. (6.16.2) represents the velocity field of an infinite extent of liquid lying in the region  $x_2 \geq 0$  and bounded by a plate at  $x_2 = 0$  which executes simple harmonic oscillations of amplitude  $a$  and circular frequency  $\omega$ . It represents a transverse wave of wavelength  $\frac{2\pi}{\beta}$ , propagating inward from the boundary with a phase velocity  $\frac{\omega}{\beta}$ , but with rapidly diminishing amplitude (the falling off within a wavelength being in the ratio  $e^{-2\pi} = 1/535$ ). Thus, we see that the influence of viscosity extends only to a short distance from the plate performing rapid oscillation of small amplitude  $a$ .

### 6.17 Dissipation Functions for Newtonian Fluids

The rate of work done  $P$  by the stress vectors and the body forces on a material particle of a continuum was derived in Chapter 4, Section 4.12 to be given by

$$P = \frac{D}{Dt}(K.E.) + P_s dV, \quad P_s = T_{ij} \frac{\partial v_i}{\partial x_j} \quad (6.17.1)$$

where  $dV$  is the volume of the material particle. In Eq. (6.17.1), the first term in the right side is the rate of change of the kinetic energy ( $K.E.$ ) and the second term  $P_s dV$  is the rate of work done to change the volume and shape of the “particle” of volume  $dV$ . Per unit volume, this rate is denoted by  $P_s$  and is known as the stress working or stress power.

In this section, we shall compute the stress power for a Newtonian fluid.

#### (A) Incompressible Newtonian Fluid.

We have,

$$T_{ij} = -p\delta_{ij} + T'_{ij}, \quad (i)$$

thus

$$T_{ij} \frac{\partial v_i}{\partial x_j} = -p \frac{\partial v_i}{\partial x_i} + T'_{ij} \frac{\partial v_i}{\partial x_j} \quad (6.17.2)$$

Since the fluid is incompressible,  $\partial v_i / \partial x_i = 0$ , therefore,

$$T_{ij} \frac{\partial v_i}{\partial x_j} = T'_{ij} \frac{\partial v_i}{\partial x_j} = 2\mu D_{ij} \frac{\partial v_i}{\partial x_j} = 2\mu D_{ij}(D_{ij} + W_{ij}) = 2\mu D_{ij}D_{ij} \quad (ii)$$

i.e.,

$$P_s = 2\mu(D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{13}^2 + 2D_{23}^2) \quad (6.17.3)$$

This is the work per unit volume per unit time done to change the shape and this part of the work accumulates with time regardless of how  $D_{ij}$  vary with time ( $P_s$  is always positive and is zero only for rigid body motions). Thus, the function

$$\Phi_{inc} = 2\mu(D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{13}^2 + 2D_{23}^2) = 2\mu D_{ij}D_{ij} \quad (6.17.4)$$

is known as the **dissipation function** for an incompressible Newtonian fluid. It represents the rate at which work is converted into heat.

#### (B) Newtonian Compressible Fluid

For this case, we have, with  $\Delta$  denoting  $\frac{\partial v_i}{\partial x_i}$

### 384 Energy Equation For a Newtonian Fluid

$$T_{ij} \frac{\partial v_i}{\partial x_j} = -p\Delta + \lambda\Delta^2 + \Phi_{inc} \equiv -p\Delta + \Phi \quad (6.17.5)$$

where

$$\Phi = \lambda(D_{11} + D_{22} + D_{33})^2 + \Phi_{inc} \quad (6.17.6a)$$

is the dissipation function for a compressible fluid. We leave it as an exercise [see Prob. 6.39] to show that the dissipation function  $\Phi$  can be written

$$\begin{aligned} \Phi = & (\lambda + \frac{2}{3}\mu)(D_{11} + D_{22} + D_{33})^2 + \frac{2}{3}\mu[(D_{11} - D_{22})^2 + (D_{11} - D_{33})^2 + (D_{22} - D_{33})^2] \\ & + 4\mu(D_{12}^2 + D_{13}^2 + D_{23}^2) \end{aligned} \quad (6.17.6b)$$

#### Example 6.17.1

For the simple shearing flow with

$$v_1 = kx_2, \quad v_2 = 0, \quad v_3 = 0 \quad (i)$$

Find the rate at which work is converted into heat if the liquid inside of the plates is water with  $\mu = 2 \times 10^{-5}$  lb.s./ft<sup>2</sup> (0.958 mPa.s), and  $k = 1 \text{ s}^{-1}$ .

*Solution* Since the only nonzero component of the rate of deformation tensor is

$$D_{12} = \frac{k}{2} \quad (ii)$$

Thus, from Eq. (6.17.4),

$$\Phi_{inc} = 4\mu D_{12}^2 = \mu k^2 = 2 \times 10^{-5} \frac{\text{ft} \cdot \text{lb}}{(\text{ft})^3 \text{s}} \quad [\text{or}, 0.958 \times 10^{-3} \frac{\text{N} \cdot \text{m}}{\text{m}^3 \text{s}}]$$

Thus, in one second, per cubic feet of water, the heat generated by viscosities is  $2.5 \times 10^{-8}$  B.T.U. [or,  $0.958 \times 10^{-3}$  joule per cubic meter per second].

### 6.18 Energy Equation For a Newtonian Fluid

In Section 4.14 of chapter 4, we derived the energy equation for a continuum to be

$$\rho \frac{Du}{Dt} = T_{ij} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + q_s \quad (6.18.1)$$

where  $u$  is the internal energy per unit mass,  $\rho$  is density,  $q_i$  is the component of heat flux vector,  $q_s$  is the heat supply due to external sources.

If the only heat flow taking place is that due to conduction governed by Fourier's law  $\mathbf{q} = -\kappa \nabla \Theta$ , where  $\Theta$  is the temperature, then Eq. (6.18.1) becomes, assuming a constant coefficient of thermoconductivity  $\kappa$

$$\rho \frac{Du}{Dt} = T_{ij} \frac{\partial v_i}{\partial x_j} + \kappa \frac{\partial^2 \Theta}{\partial x_j \partial x_j} \quad (6.18.2)$$

For an incompressible Newtonian fluid, if it is assumed that the internal energy per unit mass is given by  $c\Theta$ , where  $c$  is the specific heat per unit mass, then Eq. (6.18.2) becomes

$$\rho c \frac{D\Theta}{Dt} = \Phi_{inc} + \kappa \frac{\partial^2 \Theta}{\partial x_j \partial x_j} \quad (6.18.3)$$

where from Eq. (6.17.4),  $\Phi_{inc} = 2\mu(D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{13}^2 + 2D_{23}^2)$ , representing the heat generated through viscous forces.

There are many situations in which the heat generated through viscous action is very small compared with that arising from the heat conduction from the boundaries, in which case, Eq. (6.18.3) simplifies to

$$\frac{D\Theta}{Dt} = \alpha \frac{\partial^2 \Theta}{\partial x_j \partial x_j} \quad (6.18.4)$$

where  $\alpha = \kappa/\rho c =$  thermal diffusivity.

#### Example 6.18.1

A fluid is at rest between two plates of infinite dimension. If the lower plate is kept at constant temperature  $\Theta_l$  and the upper plate at  $\Theta_u$ , find the steady-state temperature distribution. Neglect the heat generated through viscous action.

*Solution.* The steady-state distribution is governed by the Laplace equation

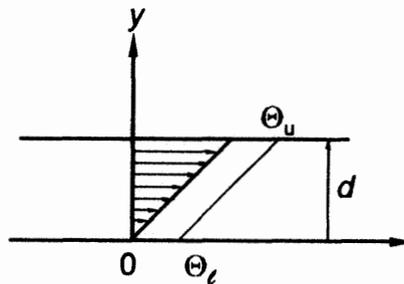


Fig. 6.12

$$\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} + \frac{\partial^2 \Theta}{\partial z^2} = 0 \quad (i)$$

which in this problem reduces to

$$\frac{d^2 \Theta}{dy^2} = 0 \quad (ii)$$

Thus,

$$\frac{d \Theta}{dy} = C_1$$

and

$$\Theta = C_1 y + C_2 \quad (iii)$$

Using the boundary condition  $\Theta = \Theta_l$  at  $y = 0$  and  $\Theta = \Theta_u$  at  $y = d$ , the constants of integration are determined to be

$$C_1 = \frac{\Theta_u - \Theta_l}{d}$$

$$C_2 = \Theta_l$$

It is noted here that when the values of  $\Theta$  are prescribed on the plates, the values of  $\frac{d\Theta}{dy}$  on the plates are completely determined. In fact,  $\frac{d\Theta}{dy} = (\Theta_u - \Theta_l)/d$ . This serves to illustrate that, in steady-state heat conduction problem (governed by the Laplace equation) it is in general not possible to prescribe both the values of  $\Theta$  and the normal derivatives of  $\Theta$  at the same points of the complete boundary unless they happen to be consistent with each other.

### Example 6.18.2

The plane Couette flow is given by the following velocity distribution:

$$v_1 = ky, \quad v_2 = 0, \quad v_3 = 0 \quad (i)$$

If the temperature at the lower plate is kept at  $\Theta_l$  and that at the upper plate at  $\Theta_u$ , find the steady-state temperature distribution.

*Solution.* We seek a temperature distribution that depends only on  $y$ . From Eq. (6.18.3), we have, since  $D_{12} = k/2$

$$0 = \mu k^2 + \kappa \frac{d^2 \Theta}{dy^2} \quad (ii)$$

Thus,

$$\frac{d^2\Theta}{dy^2} = -\frac{\mu k^2}{\kappa} \quad (\text{iii})$$

which gives

$$\Theta = -\frac{\mu k^2 y^2}{2\kappa} + C_1 y + C_2$$

where  $C_1$  and  $C_2$  are constants of integration. Now at  $y = 0$ ,  $\Theta = \Theta_l$  and at  $y = d$ ,  $\Theta = \Theta_u$ , therefore,

$$C_1 = \frac{\Theta_u}{d} + \frac{\mu k^2 d}{2\kappa} - \frac{\Theta_l}{d} \quad \text{and} \quad C_2 = \Theta_l$$

The temperature distribution is therefore given by

$$\Theta = -\frac{\mu k^2}{2\kappa} y^2 + \left( \frac{\Theta_u - \Theta_l}{d} + \frac{\mu k^2 d}{2\kappa} \right) y + \Theta_l \quad (\text{iv})$$

## 6.19 Vorticity Vector

We recall from Chapter 3, Section 3.13 and 14 that the antisymmetric part of the velocity gradient ( $\nabla \mathbf{v}$ ) is defined as the spin tensor  $\mathbf{W}$ . Being antisymmetric, the tensor  $\mathbf{W}$  is equivalent to a vector  $\boldsymbol{\omega}$  in the sense that  $\mathbf{W}\mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}$  (see Sect. 2B16). In fact,

$$\boldsymbol{\omega} = -(W_{23}\mathbf{e}_1 + W_{31}\mathbf{e}_2 + W_{12}\mathbf{e}_3).$$

Since (see Eq. (3.14.4),

$$\frac{D}{Dt}(d\mathbf{x}) = (\nabla \mathbf{v})d\mathbf{x} = \mathbf{D}d\mathbf{x} + \mathbf{W}d\mathbf{x} = \mathbf{D}d\mathbf{x} + \boldsymbol{\omega} \times d\mathbf{x} \quad (6.19.1)$$

the vector  $\boldsymbol{\omega}$  is the angular velocity vector of that part of the motion, representing the rigid body rotation in the infinitesimal neighborhood of a material point. Further,  $\boldsymbol{\omega}$  is the angular velocity vector of the principal axes of  $\mathbf{D}$ , which we show below:

Let  $d\mathbf{x}$  be a material element in the direction of the unit vector  $\mathbf{n}$  at time  $t$ , i.e.,

$$\mathbf{n} = \frac{d\mathbf{x}}{ds} \quad (6.19.2)$$

where  $ds$  is the length of  $d\mathbf{x}$ . Now

$$\frac{D}{Dt}\mathbf{n} = \frac{D}{Dt}\left(\frac{d\mathbf{x}}{ds}\right) = \frac{1}{ds}\left(\frac{D}{Dt}d\mathbf{x}\right) - \frac{1}{ds^2}\left[\frac{D}{Dt}(ds)\right]d\mathbf{x} \quad (\text{i})$$

But, from Eq. (3.13.6) of Chapter 3, we have

$$\frac{1}{ds} \frac{D}{Dt}(ds) = \mathbf{n} \cdot \mathbf{D}\mathbf{n} \quad (\text{ii})$$

Using Eq. (6.19.1) and (ii), Eq. (i) becomes

$$\frac{D}{Dt}\mathbf{n} = (\mathbf{D} + \mathbf{W})\mathbf{n} - (\mathbf{n} \cdot \mathbf{D}\mathbf{n})\mathbf{n} \quad (6.19.3)$$

Now, if  $\mathbf{n}$  is an eigenvector of  $\mathbf{D}$ , then

$$\mathbf{D}\mathbf{n} = \lambda\mathbf{n} \quad (6.19.4)$$

and

$$\mathbf{n} \cdot \mathbf{D}\mathbf{n} = \lambda \quad (6.19.5)$$

and Eq. (6.19.3) becomes

$$\frac{D}{Dt}\mathbf{n} = \mathbf{W}\mathbf{n} = \boldsymbol{\omega} \times \mathbf{n} \quad (6.19.6)$$

which is the desired result.

Eq. (6.19.6) and Eq. (6.19.1) state that the material elements which are in the principal directions of  $\mathbf{D}$  rotate with angular velocity  $\boldsymbol{\omega}$  while at the same time changing their lengths.

In rectangular Cartesian coordinates,

$$\boldsymbol{\omega} = \frac{1}{2} \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) \mathbf{e}_1 + \frac{1}{2} \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \mathbf{e}_2 + \frac{1}{2} \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \mathbf{e}_3 \quad (6.19.7)$$

Conventionally, the factor of 1/2 is dropped and one defines the so-called **vorticity vector**  $\boldsymbol{\zeta}$  as

$$\boldsymbol{\zeta} = 2\boldsymbol{\omega} = \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) \mathbf{e}_1 + \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \mathbf{e}_2 + \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \mathbf{e}_3 \quad (6.19.8)$$

The tensor  $2\mathbf{W}$  is known as the **vorticity tensor**.

It can be easily seen that in indicial notation, the Cartesian components of  $\boldsymbol{\zeta}$  are

$$\zeta_i = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}, \quad \text{or equivalently} \quad \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} = -\varepsilon_{kij} \zeta_k \quad (6.19.9)$$

and in invariant notation,

$$\boldsymbol{\zeta} = \text{curl } \mathbf{v} \quad (6.19.10)$$

In cylindrical coordinates  $(r, \theta, z)$

$$\boldsymbol{\zeta} = \left( \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right) \mathbf{e}_r + \left( \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \mathbf{e}_\theta + \left( \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) \mathbf{e}_z \quad (6.19.11)$$

In spherical coordinates  $(r, \theta, \varphi)$

$$\boldsymbol{\zeta} = \left( \frac{1}{r} \frac{\partial v_\varphi}{\partial \theta} + \frac{v_\varphi \cot \theta}{r} - \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \varphi} \right) \mathbf{e}_r + \left( \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r} - \frac{\partial v_\varphi}{\partial r} \right) \mathbf{e}_\theta + \left( \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) \mathbf{e}_\varphi \quad (6.19.12)$$

Example 6.19.1

Find the vorticity vector for the simple shearing flow:

$$v_1 = kx_2, \quad v_2 = v_3 = 0 \quad (i)$$

*Solution.* We have

$$\zeta_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = 0, \quad \zeta_2 = \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} = 0$$

and

$$\zeta_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = -k$$

That is,

$$\boldsymbol{\zeta} = -k\mathbf{e}_3 \quad (ii)$$

We see that the angular velocity vector ( $= \boldsymbol{\zeta} / 2$ ) is normal to the  $x_1 x_2$  plane and the minus sign simply means that the spinning is clockwise looking from the positive side of  $x_3$ .

Example 6.19.2

Find the distribution of the vorticity vector in the Couette flow discussed in Section 6.15.

*Solution.* With  $v_r = v_z = 0$  and  $v_\theta = Ar + (B/r)$ . It is obvious that the only nonzero vorticity component is in the  $z$  direction.

From Eq. (6.19.11),

$$\zeta_z = \frac{dv_\theta}{dr} + \frac{v_\theta}{r} = \frac{1}{r} \frac{d}{dr}(rv_\theta) \quad (iv)$$

Now,

$$\frac{d}{dr}(rv_\theta) = \frac{d}{dr}(Ar^2 + B) = 2Ar \quad (v)$$

Thus,

$$\zeta_z = 2A = 2 \frac{(\Omega_2 r_2^2 - \Omega_1 r_1^2)}{r_2^2 - r_1^2} \quad (\text{vi})$$

## 6.20 Irrotational Flow

If the vorticity vector ( or equivalently, vorticity tensor) corresponding to a velocity field, is zero in some region and for some time interval, the flow is called irrotational in that region and in that time interval.

Let  $\varphi(x_1, x_2, x_3, t)$  be a scalar function and let the velocity components be derived from  $\varphi$  by the following equation:

$$v_1 = -\frac{\partial \varphi}{\partial x_1} \quad v_2 = -\frac{\partial \varphi}{\partial x_2} \quad v_3 = -\frac{\partial \varphi}{\partial x_3} \quad (6.20.1)$$

i.e.,

$$v_i = -\frac{\partial \varphi}{\partial x_i} \quad (6.20.2)$$

Then the vorticity component

$$\zeta_1 = \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} = -\frac{\partial^2 \varphi}{\partial x_3 \partial x_2} + \frac{\partial^2 \varphi}{\partial x_2 \partial x_3} = 0$$

and similarly

$$\zeta_2 = \zeta_3 = 0$$

That is, a scalar function  $\varphi(x_1, x_2, x_3, t)$  defines an irrotational flow field through the Eq. (6.20.2). Obviously, not all arbitrary functions  $\varphi$  will give rise to velocity fields that are physically possible. For one thing, the equation of continuity, expressing the principle of conservation of mass, must be satisfied. For an incompressible fluid, the equation of continuity reads

$$\frac{\partial v_i}{\partial x_i} = 0 \quad (6.20.3)$$

Thus, combining Eq. (6.20.2) with Eq. (6.20.3), we obtain the Laplacian equation for  $\varphi$ ,

$$\frac{\partial^2 \varphi}{\partial x_j \partial x_j} = 0 \quad (6.20.4a)$$

i.e.,

$$\nabla^2 \varphi \equiv \frac{\partial^2 \varphi}{\partial x_1^2} + \frac{\partial^2 \varphi}{\partial x_2^2} + \frac{\partial^2 \varphi}{\partial x_3^2} = 0 \quad (6.20.4b)$$

In the next two sections, we shall discuss the conditions under which irrotational flows are dynamically possible for an inviscid and viscous fluid.

### 6.21 Irrotational Flow of an Inviscid Incompressible Fluid of Homogeneous Density

An inviscid fluid is defined by

$$T_{ij} = -p\delta_{ij} \quad (6.21.1)$$

obtained by setting the viscosity  $\mu = 0$  in the constitutive equation for Newtonian viscous fluid.

The equations of motion for an inviscid fluid are

$$\rho \left( \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \rho B_i \quad (6.21.2a)$$

or

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla p + \rho \mathbf{B} \quad (6.21.2b)$$

Equations (6.21.2) are known as the **Euler's equation of motion**. We now show that irrotational flows are always dynamically possible for an inviscid, incompressible fluid with homogeneous density provided that the body forces acting are derivable from a potential  $\Omega$  by the formulas:

$$B_i = -\frac{\partial \Omega}{\partial x_i} \quad (6.21.3)$$

For example, in the case of gravity force, with  $x_3$  axis pointing vertically upward,

$$\Omega = gx_3 \quad (6.21.4)$$

so that

$$B_1 = 0, B_2 = 0, B_3 = -g \quad (6.21.5)$$

Using Eq. (6.21.3), and noting that  $\rho = \text{constant}$  for a homogeneous fluid. Eq. (6.21.2) can be written as

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial}{\partial x_i} \left( \frac{p}{\rho} + \Omega \right) \quad (6.21.6)$$

For an irrotational flow

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i} \quad (i)$$

so that

$$v_j \frac{\partial v_i}{\partial x_j} = v_j \frac{\partial v_j}{\partial x_i} = \frac{1}{2} \frac{\partial}{\partial x_i} v_j v_j = \frac{1}{2} \frac{\partial}{\partial x_i} v^2 \tag{ii}$$

where  $v^2 = v_1^2 + v_2^2 + v_3^2$  is the square of the speed. Therefore Eq. (6.21.6) becomes

$$\frac{\partial}{\partial x_i} \left( -\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} + \Omega \right) = 0 \tag{6.21.7}$$

Thus

$$-\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + \frac{p}{\rho} + \Omega = f(t) \tag{6.21.8}$$

where  $f(t)$  is an arbitrary function of  $t$ .

If the flow is also steady then we have

$$\frac{v^2}{2} + \frac{p}{\rho} + \Omega = \text{constant.} \tag{6.21.9}$$

Equation (6.21.8) and the special case (6.21.9) are known as the **Bernoulli's equations**. In addition to being a very useful formula in problems where the effect of viscosity can be neglected, the above derivation of the formula shows that irrotational flows are always dynamically possible under the conditions stated earlier. For whatever function  $\varphi$ , so long as  $v_i = -\frac{\partial \varphi}{\partial x_i}$  and  $\nabla^2 \varphi = 0$ , the dynamic equations of motion can always be integrated to give Bernoulli's equation from which the pressure distribution is obtained, corresponding to which the equations of motion are satisfied.

Example 6.21.1

Given  $\varphi = x^3 - 3xy^2$ .

- (a) Show that  $\varphi$  satisfies the Laplace equation.
- (b) Find the irrotational velocity field.
- (c) Find the pressure distribution for an incompressible homogeneous fluid, if at  $(0,0,0)$   $p = p_o$  and  $\Omega = gz$ .
- (d) If the plane  $y = 0$  is a solid boundary, find the tangential component of velocity on the plane.

*Solution.* (a) We have

$$\frac{\partial^2 \varphi}{\partial x^2} = 6x, \quad \frac{\partial^2 \varphi}{\partial y^2} = -6x, \quad \frac{\partial^2 \varphi}{\partial z^2} = 0 \tag{i}$$

therefore,

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 6x - 6x = 0 \tag{ii}$$

(b) From  $v_i = -\frac{\partial \varphi}{\partial x_i}$ , we have

$$v_1 = -\frac{\partial \varphi}{\partial x} = -3x^2 + 3y^2, \quad v_2 = -\frac{\partial \varphi}{\partial y} = 6xy, \quad v_3 = 0 \tag{iii}$$

(c) We have, at  $(0,0,0)$ ,  $v_1 = 0$ ,  $v_2 = 0$ ,  $v_3 = 0$ ,  $p = p_o$ , and  $\Omega = 0$

therefore, from the Bernoulli's equation, [Eq. (6.21.9)]

$$\frac{1}{2}v^2 + \frac{p}{\rho} + \Omega = C \tag{iv}$$

we have

$$C = \frac{p_o}{\rho} \tag{v}$$

and

$$p = p_o - \frac{\rho}{2}(v_1^2 + v_2^2) - \rho gz \tag{vi}$$

or

$$p = p_o - \frac{\rho}{2}[9(y^2 - x^2)^2 + 36x^2y^2] - \rho gz \tag{vii}$$

(d) On the plane  $y = 0$ ,  $v_1 = -3x^2$  and  $v_2 = 0$ . Now,  $v_2 = 0$  means that the normal components of velocity are zero on the plane, which is what it should be if  $y = 0$  is a solid fixed boundary. Since  $v_1 = -3x^2$ , the tangential components of velocity are not zero on the plane, that is, the fluid slips on the boundary. In inviscid theory, consistent with the assumption of zero viscosity, the slipping of fluid on a solid boundary is allowed. More discussion on this point will be given in the next section.

### Example 6.21.2

A liquid is being drained through a small opening as shown. Neglect viscosity and assume that the falling of the free surface is so slow that the flow can be treated as a steady one. Find the exit speed of the liquid jet as a function of  $h$ .

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*Solution.* For a point on the free surface such as the point  $A$ ,  $p = p_o$ ,  $v \approx 0$  and  $z = h$ . Therefore, from Eq. (6.21.9)

$$\frac{1}{2}v^2 + \frac{p}{\rho} + gz = \frac{p_o}{\rho} + gh \quad (i)$$

At a point on the exit jet, such as the point  $B$ ,  $z = 0$  and  $p = p_o$ . Thus,

$$\frac{1}{2}v^2 + \frac{p_o}{\rho} = \frac{p_o}{\rho} + gh \quad (ii)$$

from which

$$v = \sqrt{2gh} \quad (iii)$$

This is the well known *Toricelli's* formula.

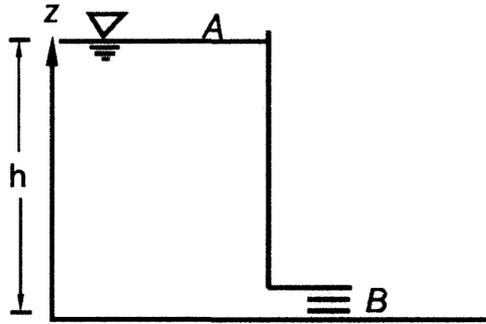


Fig. 6.13

### 6.22 Irrotational Flows as Solutions of Navier-Stokes Equation

For an incompressible Newtonian fluid, the equations of motion are the Navier-Stokes equations:

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \frac{\mu}{\rho} \frac{\partial^2 v_i}{\partial x_j \partial x_j} + B_i \quad (6.22.1)$$

For irrotational flows

$$v_i = -\frac{\partial \phi}{\partial x_i} \quad (6.22.2)$$

so that

$$\frac{\partial^2 v_i}{\partial x_j \partial x_j} = -\frac{\partial^2}{\partial x_j \partial x_j} \left( \frac{\partial \varphi}{\partial x_i} \right) = -\frac{\partial}{\partial x_i} \frac{\partial^2 \varphi}{\partial x_j \partial x_j}$$

But, from Eq. (6.20.4)  $\frac{\partial^2 \varphi}{\partial x_j \partial x_j} = 0$ . Therefore, the terms involving viscosity in the Navier-

Stokes equation drop out in the case of irrotational flows so that the equations take the same form as the Euler's equation for an inviscid fluid. Thus, if the *viscous* fluid has homogeneous density and if the body forces are conservative ( i.e.,  $B_i = -\frac{\partial \Omega}{\partial x_i}$ ), the results of the last sections

show that irrotational flows are dynamically possible also for a viscous fluid. However, in any physical problems, there are always solid boundaries. A viscous fluid adheres to the boundary so that both the tangential and the normal components of the fluid velocity at the boundary should be those of the boundary. This means that both velocity components at the boundary are to be prescribed. For example, if  $y = 0$  is a solid boundary at rest, then on the boundary, the tangential components,  $v_x = v_z = 0$ , and the normal components  $v_y = 0$ . For irrotational flow, the conditions to be prescribed for  $\varphi$  on the boundary are  $\varphi = \text{constant}$  at  $y = 0$  ( so that

$v_x = v_z = 0$ ) and  $\frac{\partial \varphi}{\partial y} = 0$  at  $y = 0$ . But it is known (e.g., see Example 6.18.1, or from the potential theory) that in general there does not exist solution of the Laplace equation satisfying

both the conditions  $\varphi = \text{constant}$  and  $\nabla \varphi \cdot \mathbf{n} = \frac{\partial \varphi}{\partial n} = 0$  on the complete boundaries. There-

fore, unless the motion of solid boundaries happens to be consistent with the requirements of irrotationality, vorticity will be generated on the boundary and diffuse into the flow field according to vorticity equations to be derived in the next section. However, in certain problems under suitable conditions, the vorticity generated by the solid boundaries is confined to a thin layer of fluid in the vicinity of the boundary so that outside of the layer the flow is irrotational if it originated from a state of irrotationality. We shall have more to say about this in the next two sections.

### Example 6.22.1

For the Couette flow between two coaxial infinitely long cylinders, how should the ratio of the angular velocities of the two cylinders be, so that the *viscous* fluid will be having irrotational flow?

*Solution.* From Example 19.2 of Section 6.19, the only nonzero vorticity component in the Couette flow is

$$\xi_z = 2 \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2} \tag{i}$$

where  $\Omega_i$  denotes the angular velocities. If  $\Omega_2 r_2^2 - \Omega_1 r_1^2 = 0$ , the flow is irrotational. Thus,

$$\frac{\Omega_2}{\Omega_1} = \frac{r_1^2}{r_2^2} \tag{ii}$$

It should be noted that even though the viscous terms drop out from the Navier-Stokes equations in the case of irrotational flows, it does not mean that there is no viscous dissipation in an irrotational flow of a viscous fluid. In fact, so long as there is one nonzero rate of deformation component, there is viscous dissipation [given by Eq. (6.17.4)] and the rate of work done to maintain the irrotational flow exactly compensates the viscous dissipations.

### 6.23 Vorticity Transport Equation for Incompressible Viscous Fluid with a Constant Density

In this section, we derive the equation governing the vorticity vector for an incompressible homogeneous viscous fluid. First, we assume that the body force is derivable from a potential  $\Omega$ , i.e.,  $B_i = -\frac{\partial \Omega}{\partial x_i}$ . Now, with  $\rho = \text{constant}$  and  $B_i = -\frac{\partial \Omega}{\partial x_i}$ , the Navier-Stokes equation can be written

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial}{\partial x_i} \left( \frac{p}{\rho} + \Omega \right) + \nu \frac{\partial^2 v_i}{\partial x_j \partial x_j} \tag{6.23.1}$$

where  $\nu = \mu/\rho$  is called the **kinematic viscosity**. If we operate on Eq. (6.23.1) by the differential operator  $\epsilon_{mni} \frac{\partial}{\partial x_n}$  [i.e, taking the curl of both sides of Eq. (6.23.1)]. We have, since

$$\epsilon_{mni} \frac{\partial}{\partial x_n} \left( \frac{\partial v_i}{\partial t} \right) = \frac{\partial}{\partial t} \left( \epsilon_{mni} \frac{\partial v_i}{\partial x_n} \right) = \frac{\partial \zeta_m}{\partial t} \tag{i}$$

$$\begin{aligned} \epsilon_{mni} \frac{\partial}{\partial x_n} \left( v_j \frac{\partial v_i}{\partial x_j} \right) &= \epsilon_{mni} \frac{\partial v_j}{\partial x_n} \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial}{\partial x_j} \left( \epsilon_{mni} \frac{\partial v_i}{\partial x_n} \right) \\ &= \epsilon_{mni} \frac{\partial v_j}{\partial x_n} \frac{\partial v_i}{\partial x_j} + v_j \frac{\partial \zeta_m}{\partial x_j} \end{aligned} \tag{ii.}$$

$$\epsilon_{mni} \frac{\partial^2}{\partial x_n \partial x_i} \left( \frac{p}{\rho} + \Omega \right) = 0 \text{ [i.e., } \text{curl} \nabla \left( \frac{p}{\rho} + \Omega \right) = 0] \tag{6.23.2}$$

and

$$\epsilon_{mni} \frac{\partial}{\partial x_n} \left( \frac{\partial^2 v_i}{\partial x_j \partial x_j} \right) = \frac{\partial^2}{\partial x_j \partial x_j} \left( \epsilon_{mni} \frac{\partial v_i}{\partial x_n} \right) = \frac{\partial^2 \zeta_m}{\partial x_j \partial x_j} \tag{iii}$$

The Navier-Stokers equation therefore, takes the form

$$\frac{\partial \xi_m}{\partial t} + v_j \frac{\partial \xi_m}{\partial x_j} + \varepsilon_{mni} \frac{\partial v_j}{\partial x_n} \frac{\partial v_i}{\partial x_j} = \nu \frac{\partial^2 \xi_m}{\partial x_j \partial x_j} \quad (6.23.3)$$

We now show that the third term on the left-hand side is equal to  $-\frac{\partial v_m}{\partial x_n} \xi_n$ .

From Eq. (6.19.9), we have

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial v_j}{\partial x_i} + \varepsilon_{pji} \zeta_p \quad (iv)$$

Thus,

$$\varepsilon_{mni} \frac{\partial v_j}{\partial x_n} \frac{\partial v_i}{\partial x_j} = \varepsilon_{mni} \frac{\partial v_j}{\partial x_n} \frac{\partial v_j}{\partial x_i} + \varepsilon_{mni} \varepsilon_{pji} \frac{\partial v_j}{\partial x_n} \zeta_p \quad (v)$$

But it can be easily verified that for any  $A_{ij}$ ,  $\varepsilon_{mni} A_{jn} A_{ji} = 0$ , thus

$$\varepsilon_{mni} \frac{\partial v_j}{\partial x_n} \frac{\partial v_j}{\partial x_i} = 0 \quad (vi)$$

and since  $\varepsilon_{mni} \varepsilon_{pji} = (\delta_{mp} \delta_{nj} - \delta_{mj} \delta_{np})$  [see Prob. 2A7]

$$\begin{aligned} \varepsilon_{mni} \varepsilon_{pji} \frac{\partial v_j}{\partial x_n} \zeta_p &= (\delta_{mp} \delta_{nj} - \delta_{mj} \delta_{np}) \frac{\partial v_j}{\partial x_n} \zeta_p \\ &= \frac{\partial v_n}{\partial x_n} \zeta_m - \frac{\partial v_m}{\partial x_n} \zeta_n = -\frac{\partial v_m}{\partial x_n} \zeta_n \end{aligned} \quad (vii)$$

where we have used the equation of continuity  $\frac{\partial v_n}{\partial x_n} = 0$ . Therefore, we have

$$\frac{\partial \xi_m}{\partial t} + v_j \frac{\partial \xi_m}{\partial x_j} = \frac{\partial v_m}{\partial x_n} \zeta_n + \nu \frac{\partial^2 \xi_m}{\partial x_j \partial x_j} \quad (6.23.4a)$$

or,

$$\frac{D \xi_m}{Dt} = \frac{\partial v_m}{\partial x_n} \zeta_n + \nu \frac{\partial^2 \xi_m}{\partial x_j \partial x_j} \quad (6.23.4b)$$

which can be written in the following invariant form:

$$\frac{D \xi}{Dt} = (\nabla \mathbf{v}) \xi + \nu \nabla^2 \xi \quad (6.23.5)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x_j \partial x_j}$$

Example 6.23.1

Reduce, from Eq. (6.23.5) the vorticity transport equation for the case of two-dimensional flow.

*Solutions.* Let the velocity field be:

$$v_1 = v_1(x_1, x_2, t), \quad v_2 = v_2(x_1, x_2, t), \quad v_3 = 0 \tag{i}$$

Then

$$\boldsymbol{\zeta} = \left( \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \right) \mathbf{e}_1 + \left( \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \right) \mathbf{e}_2 + \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \mathbf{e}_3 \tag{ii}$$

becomes

$$\boldsymbol{\zeta} = \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right) \mathbf{e}_3 = \zeta_3 \mathbf{e}_3 \tag{iii}$$

That is, the angular velocity vector ( $\boldsymbol{\zeta} / 2$ ) is perpendicular to the plane of flow as expected.

Now,

$$[(\nabla \cdot \boldsymbol{\zeta})] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & 0 \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \zeta_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{iv}$$

Thus, Eq. (6.23.5) reduces to the scalar equation

$$\frac{D\zeta_3}{Dt} = \nu \nabla^2 \zeta_3 \tag{6.23.6}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} \quad \text{and} \quad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$


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## Example 6.23.2

The velocity field for the plane Poiseuille flow is given by

$$v_1 = C \left( \frac{h^2}{4} - x_2^2 \right), \quad v_2 = 0, \quad v_3 = 0$$

- (a) Find the vorticity components.  
 (b) Verify that Eq. (6.23.6) is satisfied.

*Solution.* The only nonzero vorticity component is

$$\zeta_3 = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 2Cx_2 \quad (i)$$

- (b) We have, letting  $\zeta_3 = \zeta$

$$\frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} + v_1 \frac{\partial \zeta}{\partial x_1} + v_2 \frac{\partial \zeta}{\partial x_2} = 0 + (v_1)(0) + 0 = 0 \quad (ii)$$

and

$$\nabla^2 \zeta = \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) (2Cx_2) = 0 \quad (iv)$$

so that Eq. (6.23.6) is satisfied.

## 6.24 Concept of a Boundary Layer

In this section we shall describe, qualitatively, the concept of viscous boundary layer by means of an analogy. In Example 6.23.1, we derived the vorticity equation for two-dimensional flow of an incompressible viscous fluid to be the following:

$$\frac{D\zeta}{Dt} = \nu \nabla^2 \zeta \quad (6.24.1)$$

where  $\zeta$  is the only nonzero vorticity component for the two-dimensional flow and  $\nu$  is kinematic viscosity ( $\nu = \mu/\rho$ ).

In Section 6.18 we saw that, if the heat generated through viscous dissipation is neglected, the equation governing the temperature distribution in the flow field due to heat conduction through the boundaries of a hot body is given by [Eq. (6.18.4)]

$$\frac{D\Theta}{Dt} = \alpha \nabla^2 \Theta \quad (6.24.2)$$

where  $\Theta$  is temperature and  $\alpha$ , the thermal diffusivity, is related to conductivity  $\kappa$ , density  $\rho$  and specific heat per unit mass  $c$  by the formulas  $\alpha = \kappa/\rho c$ .

Suppose now we have the problem of a uniform stream flowing past a hot body whose temperature in general varies along the boundary. Let the temperature at large distance from the body be  $\Theta_\infty$ , then defining  $\Theta' = \Theta - \Theta_\infty$ , we have

$$\frac{D\Theta'}{Dt} = \alpha \nabla^2 \Theta' \tag{6.24.3}$$

with  $\Theta' = 0$  at  $x^2 + y^2 \rightarrow \infty$ . On the other hand, the distribution of vorticity around the body is governed by

$$\frac{D\zeta}{Dt} = \nu \nabla^2 \zeta \tag{6.24.4}$$

with  $\zeta = 0$  at  $x^2 + y^2 \rightarrow \infty$ , where the variation of  $\zeta$ , being due to vorticity generated on the solid boundary and diffusing into the field, is much the same as the variation of temperature, being due to heat diffusing from the hot body into the field.

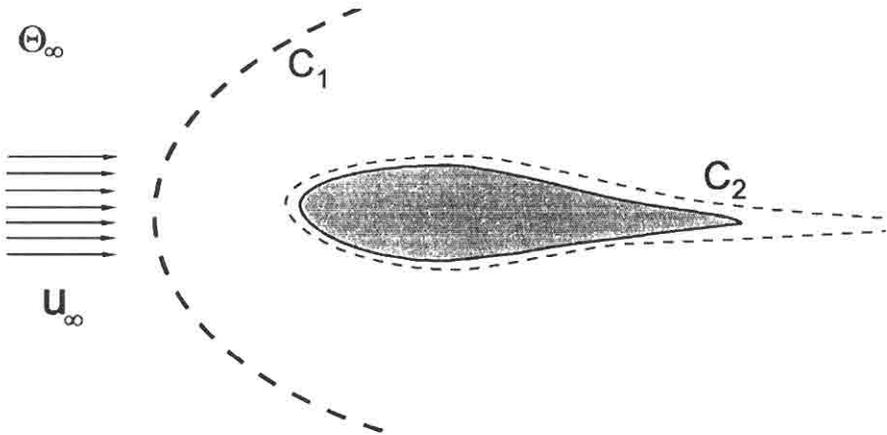


Fig. 6.14

Now, it is intuitively clear that in the case of the temperature distribution, the influence of the hot temperature of the body in the field depends on the speed of the stream. At very low speed, conduction dominates over the convection of heat so that its influence will extend deep into the fluid in all directions as shown by the curve  $C_1$  in Fig. 6.14, whereas at high speed, the heat is convected away by the fluid so rapidly that the region affected by the hot body will be confined to a thin layer in the immediate neighborhood of the body and a tail of heated fluid behind it, as is shown by the curve  $C_2$  in Fig. 6.14.

Analogously, the influence of viscosity, which is responsible for the generation of vorticity on the boundary, depends on the speed  $U_\infty$  far upstream. At low speed, the influence will be deep into the field in all directions so that essentially the whole flow field is having vorticity. On the other hand, at high speed, the effect of viscosity is confined in a thin layer ( known as a **boundary layer**) near the body and behind it. Outside of the layer, the flow is essentially irrotational. This concept enables one to solve a fluid flow problem by dividing the flow region into an irrotational external flow region and a viscous boundary layer. Such a method simplifies considerably the complexity of the mathematical problem involving the full Navier-Stokes equations. We shall not go into the methods of solution and of the matching of the regions as they belong to the boundary layer theory.

### 6.25 Compressible Newtonian Fluid

For a compressible fluid, to be consistent with the state of stress corresponding to the state of rest and also to be consistent with the definition that  $p$  is not to depend explicitly on any kinematic quantities when in motion, we shall regard  $p$  as having the same value as the thermodynamic equilibrium pressure. Therefore, for a particular density  $\rho$  and temperature  $\Theta$ , the pressure is determined by the equilibrium equation of state

$$p = p(\rho, \Theta) \quad (6.25.1)$$

For example, for an ideal gas  $p = R\rho\Theta$ . Thus

$$T_{ij} = -p(\rho, \Theta)\delta_{ij} + \lambda\Delta\delta_{ij} + 2\mu D_{ij} \quad (6.25.2)$$

Since

$$\frac{T_{ii}}{3} = -p + \left(\lambda + \frac{2}{3}\mu\right)\Delta \quad (6.25.3)$$

it is clear that the “ pressure”  $p$  in this case does not have the meaning of mean normal compressive stress. It does have the meaning if

$$k = \lambda + \frac{2}{3}\mu = 0 \quad (6.25.4)$$

which is known to be true for monatomic gases.

Written in terms of  $\mu$  and  $k = \lambda + \frac{2}{3}\mu$ , the constitutive equation reads

$$T_{ij} = -p\delta_{ij} - \frac{2}{3}\mu\Delta\delta_{ij} + 2\mu D_{ij} + k\Delta\delta_{ij} \quad (6.25.5)$$

With  $T_{ij}$  given by the above equation, the equations of motion become ( assuming constant  $\mu$  and  $k$  )

$$\rho \frac{Dv_i}{Dt} = \rho B_i - \frac{\partial p}{\partial x_i} + \frac{\mu}{3} \frac{\partial}{\partial x_i} \left( \frac{\partial v_j}{\partial x_j} \right) + \mu \frac{\partial^2 v_i}{\partial x_j \partial x_j} + k \frac{\partial}{\partial x_i} \left( \frac{\partial v_j}{\partial x_j} \right) \quad (6.25.6)$$

## 402 Energy Equation in Terms of Enthalpy

Equations (6.25.1) and (6.25.6) are four equations for six unknowns  $v_1, v_2, v_3, p, \rho, \Theta$ ; the fifth equation is given by the equation of continuity

$$\frac{D\rho}{Dt} + \rho \frac{\partial v_j}{\partial x_j} = 0 \quad (6.25.7)$$

and the sixth equation is supplied by the energy equation

$$\rho \frac{Du}{Dt} = T_{ij} \frac{\partial v_i}{\partial x_j} + \kappa \frac{\partial^2 \Theta}{\partial x_j \partial x_j} \quad (6.25.8)$$

where  $T_{ij}$  is given by Eq. (6.25.5) and the dependence of the internal energy  $u$  on  $\rho$  and  $\Theta$  is assumed to be the same as when the fluid is in the equilibrium state, for example, for ideal gas

$$u = c_v \Theta \quad (6.25.9)$$

where  $c_v$  is the specific heat at constant volume.

In general, we have

$$u = u(\rho, \Theta) \quad (6.25.10)$$

Equations (6.25.1), (6.25.6), (6.25.7), (6.25.8), and (6.25.10) form a system of seven scalar equations for the seven unknowns  $v_1, v_2, v_3, p, \rho, \Theta$ , and  $u$ .

## 6.26 Energy Equation in Terms of Enthalpy

**Enthalpy** per unit mass is defined as

$$h = u + \frac{p}{\rho} \quad (6.26.1)$$

where  $u$  is the internal energy per unit mass,  $p$  the pressure,  $\rho$  the density.

Let  $h_o = h + v^2/2$ , ( $h_o$  is known as the **stagnation enthalpy**). We shall show that in terms of  $h_o$ , the energy equation becomes (neglecting body forces)

$$\rho \frac{Dh_o}{Dt} = \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_j} (T_{ij}' v_i - q_j) \quad (6.26.2)$$

where  $T_{ij}'$  is the viscous stress tensor,  $q_j$  the heat flux vector. First, by definition,

$$\frac{Dh_o}{Dt} = \frac{D}{Dt} \left( u + \frac{p}{\rho} + \frac{v_j v_j}{2} \right) = \frac{Du}{Dt} + \frac{D}{Dt} \left( \frac{p}{\rho} \right) + v_j \frac{Dv_j}{Dt} \quad (6.26.3).$$

From the energy equation [Eq. (6.18.1)], with  $q_s = 0$ , we have

$$\rho \frac{Du}{Dt} = T_{ij} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} = (-p\delta_{ij} + T'_{ij}) \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} = -p \frac{\partial v_i}{\partial x_i} + T'_{ij} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} \quad (6.26.4)$$

Also, we have

$$\frac{D}{Dt} \left( \frac{p}{\rho} \right) = \frac{1}{\rho} \frac{Dp}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt} \quad (ii)$$

and the equation of motion ( in the absence of body forces)

$$\rho \frac{Dv_i}{Dt} = \frac{\partial T_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\partial T'_{ij}}{\partial x_j} \quad (6.26.5)$$

Thus,

$$\rho \frac{Dh_o}{Dt} = -p \frac{\partial v_i}{\partial x_i} + T'_{ij} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_i}{\partial x_i} + \frac{Dp}{Dt} - \frac{p}{\rho} \frac{D\rho}{Dt} - v_i \frac{\partial p}{\partial x_i} + \frac{\partial T'_{ij}}{\partial x_j} v_i \quad (6.26.6)$$

Noting that

$$\frac{Dp}{Dt} - v_i \frac{\partial p}{\partial x_i} = \frac{\partial p}{\partial t} \quad (iii)$$

and

$$-p \frac{\partial v_i}{\partial x_i} - \frac{p}{\rho} \frac{D\rho}{Dt} = -\frac{p}{\rho} \left( \frac{D\rho}{Dt} + \rho \frac{\partial v_i}{\partial x_i} \right) = 0 \quad (iv)$$

we have,

$$\rho \frac{Dh_o}{Dt} = T'_{ij} \frac{\partial v_i}{\partial x_j} + \frac{\partial T'_{ij}}{\partial x_j} v_i - \frac{\partial q_i}{\partial x_i} + \frac{\partial p}{\partial t} \quad (6.26.7)$$

or,

$$\rho \frac{Dh_o}{Dt} = \frac{\partial}{\partial x_j} (T'_{ij} v_i) - \frac{\partial q_i}{\partial x_i} + \frac{\partial p}{\partial t}$$

which is Equation (6.26.2).

### Example 6.26.1

Show that for steady flow of an inviscid non-heat conducting fluid, if the flow originates from a homogeneous state, then

- (a)  $h + (v^2/2) = \text{constant}$ , and
- (b) if the fluid is an ideal gas then

$$\frac{\gamma}{\gamma-1} \frac{p}{\rho} + \frac{v^2}{2} = \text{constant.} \quad (6.26.8)$$

where  $\gamma = c_p/c_v$ , the ratio of specific heat under constant pressure and constant volume.

*Solution.* (a) Since the flow is steady, therefore,  $\partial p/\partial t = 0$ . Since the fluid is inviscid and non-heat conducting, therefore  $T_{ij}' = 0$  and  $q_i = 0$ . Thus, the energy equation (6.26.2) reduces to

$$\frac{Dh_o}{Dt} = 0 \quad (i)$$

In other words,  $h_o$  is a constant for each particle. But since the flow originates from a homogeneous state, therefore

$$h_o = h + \frac{v^2}{2} = \frac{p}{\rho} + u + \frac{v^2}{2} = \text{constant} \quad (6.26.9)$$

in the whole flow field.

(b) For an ideal gas  $p = \rho R\Theta$ ,  $u = c_v\Theta$ , and  $R = c_p - c_v$ , therefore

$$u = \frac{p}{\rho} \left( \frac{1}{\gamma-1} \right) \quad (6.26.10)$$

where

$$\gamma = \frac{c_p}{c_v} \quad (6.26.11)$$

and

$$h_o = \frac{p}{\rho} \left( \frac{\gamma}{\gamma-1} \right) + \frac{v^2}{2} = \text{constant} \quad (6.26.12)$$

## 6.27 Acoustic Wave

The propagation of sound can be approximated by considering the propagation of infinitesimal disturbances in a compressible inviscid fluid. For an inviscid fluid, neglecting body forces, the equations of motion are

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} \quad (6.27.1)$$

Let us suppose that the fluid is initially at rest with

$$v_i = 0, \rho = \rho_o, p = p_o \quad (6.27.2)$$

Now suppose that the fluid is perturbed from rest such that

$$v_i = v'_i(\mathbf{x}, t), \quad \rho = \rho_o + \rho'(\mathbf{x}, t), \quad p = p_o + p'(\mathbf{x}, t) \quad (6.27.3)$$

Substituting Eq. (6.27.3) into Eq. (6.27.1),

$$\frac{\partial v'_i}{\partial t} + v'_j \frac{\partial v'_i}{\partial x_j} = - \frac{1}{\rho_o(1 + \rho'/\rho_o)} \frac{\partial p'}{\partial x_i} \quad (i) \quad (6.27.4)$$

Since we assumed infinitesimal disturbances, the terms  $v'_j(\partial v'_i/\partial x_j)$  and  $\rho'/\rho_o$  are negligible and the equations of motion now take the linearized form

$$\frac{\partial v'_i}{\partial t} = - \frac{1}{\rho_o} \frac{\partial p'}{\partial x_i} \quad (6.27.4)$$

In a similar manner, we consider the mass conservation equation

$$\frac{\partial \rho'}{\partial t} + v'_i \frac{\partial \rho'}{\partial x_i} + \rho_o(1 + \rho'/\rho_o) \frac{\partial v'_i}{\partial x_i} = 0 \quad (ii) \quad (6.27.5)$$

and obtain the linearized equation

$$\frac{\partial v'_i}{\partial x_i} = - \frac{1}{\rho_o} \frac{\partial \rho'}{\partial t} \quad (6.27.5)$$

Differentiating Eq. (6.27.4) with respect to  $x_i$  and Eq. (6.27.5) with respect to  $t$ , we eliminate the velocity to obtain

$$\frac{\partial^2 p'}{\partial x_i \partial x_i} = \frac{\partial^2 \rho'}{\partial t^2} \quad (6.27.6)$$

We further assume that the flow is barotropic, i.e., the pressure depends explicitly on density only, so that the pressure  $p = p(\rho)$ . Expanding  $p(\rho)$  in a Taylor series about the rest value of pressure  $p_o$ , we have

$$p = p_o + \left( \frac{dp}{d\rho} \right)_{\rho_o} (\rho - \rho_o) + \dots \quad (6.27.7)$$

Neglecting higher-order terms

$$p' = c_o^2 \rho' \quad (6.27.8)$$

where

$$c_o^2 = \left( \frac{dp}{d\rho} \right)_{\rho_o} \quad (6.27.9)$$

Thus, for a barotropic fluid

$$c_o^2 \frac{\partial^2 p'}{\partial x_i \partial x_i} = \frac{\partial^2 p'}{\partial t^2} \quad (6.27.10)$$

and

$$c_o^2 \frac{\partial^2 \rho'}{\partial x_i \partial x_i} = \frac{\partial^2 \rho'}{\partial t^2} \quad (6.27.11)$$

These equations are exactly analogous (for one-dimensional waves) to the elastic wave equations of Chapter 5. Thus, we conclude that the pressure and density disturbances will propagate with a speed  $c_o = \sqrt{(dp/d\rho)_{\rho_o}}$ . We call  $c_o$  the **speed of sound** at stagnation, the local speed of sound is defined to be

$$c = \sqrt{\frac{dp}{d\rho}} \quad (6.27.12)$$

When the **isentropic** relation of  $p$  and  $\rho$  is used, i.e.,

$$p = \beta \rho^\gamma \quad (6.27.13)$$

where  $\gamma = c_p/c_v$  (ratio of specific heats) and  $\beta$  is a constant

$$\frac{dp}{d\rho} = \beta \gamma \rho^{\gamma-1} = \gamma \frac{p}{\rho} \quad (iii)$$

so that the speed of sound is

$$c = \sqrt{\gamma p / \rho} \quad (6.27.14)$$

#### Example 6.27.1

- Write an expression for a harmonic plane acoustic wave propagating in the  $\mathbf{e}_1$  direction.
- Find the velocity disturbance  $v_1$ .
- Compare  $\partial v_i / \partial t$  to the neglected  $v_j \partial v_i / \partial x_j$ .

*Solution.* In the following,  $p$ ,  $\rho$ ,  $v_1$  denote the disturbances, that is, we will drop the primes.

- Referring to the section on elastic waves, we have

$$p = \varepsilon \sin \left[ \frac{2\pi}{l} (x_1 - c_o t) \right] \quad (i)$$

- Using Eq. (6.27.4), we have

$$\frac{\partial v_1}{\partial t} = -\frac{\varepsilon}{\rho_o} \left( \frac{2\pi}{l} \right) \cos \left[ \frac{2\pi}{l} (x_1 - c_o t) \right] \quad (ii)$$

Therefore, the velocity disturbance

$$v_1 = \frac{\varepsilon}{\rho_o c_o} \sin \left[ \frac{2\pi}{l} (x_1 - c_o t) \right] \tag{iii}$$

is exactly the same form as the pressure wave.

(c) For the one-dimensional case, we have the following ratio of amplitudes

$$\left| \frac{v_1 \frac{\partial v_1}{\partial x_1}}{\frac{\partial v_1}{\partial t}} \right| = \frac{|v_1| (2\pi \varepsilon / \rho_o l c_o)}{\frac{\varepsilon}{\rho_o} \left( \frac{2\pi}{l} \right)} = \frac{|v_1|}{c_o}$$

Thus, the approximation is best when the disturbance has a velocity that is much smaller than the speed of sound.

### Example 6.27.2

Two fluids have a plane interface at  $x_1 = 0$ . Consider a plane acoustic wave that is normally incident on the interface and determine the amplitudes of the reflected and transmitted waves.

*Solution.* Let the fluid properties to the left of the interface ( $x_1 < 0$ ) be denoted by  $\rho_1, c_1$ , and to the right ( $x_1 > 0$ ) by  $\rho_2, c_2$ .

Now, let the incident pressure wave propagate to the right, as given by

$$p_I = \varepsilon_I \sin \frac{2\pi}{l_I} (x_1 - c_1 t) \quad (x_1 \leq 0) \tag{i}$$

This pressure wave results in a reflected wave

$$p_R = \varepsilon_R \sin \frac{2\pi}{l_R} (x_1 + c_1 t) \quad (x_1 \leq 0) \tag{ii}$$

and a transmitted wave

$$p_T = \varepsilon_T \sin \frac{2\pi}{l_T} (x_1 - c_2 t) \quad (x_1 \geq 0) \tag{iii}$$

We must now consider the conditions on the boundary  $x_1 = 0$ . First, the total pressure must be the same, so that

$$(p_I + p_R)|_{x_1 = 0} = (p_T)|_{x_1 = 0} \tag{iv}$$

or,

$$\varepsilon_I \sin \frac{2\pi c_1 t}{l_I} - \varepsilon_R \sin \frac{2\pi c_1 t}{l_R} = \varepsilon_T \sin \frac{2\pi c_2 t}{l_T} \tag{v}$$

This equation will be satisfied for all time if

$$l_I = l_R = \frac{c_1}{c_2} l_T \tag{vi}$$

and

$$\varepsilon_I - \varepsilon_R = \varepsilon_T \tag{vii}$$

In addition, we require the normal velocity be continuous at all time on  $x_1 = 0$ , so that  $\left(\frac{\partial v_1}{\partial t}\right)_{x_1=0}$ , is also continuous. Thus, by using Eq. (6.27.4),

$$-\left(\frac{\partial v_1}{\partial t}\right)_{x_1=0} = \frac{1}{\rho_1} \left(\frac{\partial p_I}{\partial x_1} + \frac{\partial p_R}{\partial x_1}\right)_{x_1=0} = \frac{1}{\rho_2} \left(\frac{\partial p_T}{\partial x_1}\right)_{x_1=0} \tag{viii}$$

Substituting for the pressure, we obtain

$$\frac{1}{\rho_1} \left(\frac{\varepsilon_I}{l_I} + \frac{\varepsilon_R}{l_R}\right) = \frac{1}{\rho_2} \left(\frac{\varepsilon_T}{l_T}\right) \tag{ix}$$

Combining Eqs. (vi) (vii) and (ix) we obtain

$$\varepsilon_T = \left(\frac{2}{1 + (\rho_1 c_1 / \rho_2 c_2)}\right) \varepsilon_I \tag{x}$$

$$\varepsilon_R = \left(\frac{(\rho_1 c_1 / \rho_2 c_2) - 1}{1 + (\rho_1 c_1 / \rho_2 c_2)}\right) \varepsilon_I \tag{xi}$$

Note that for the special case  $\rho_1 c_1 = \rho_2 c_2$ ,

$$\varepsilon_T = \varepsilon_I \text{ and } \varepsilon_R = 0 \tag{xii}$$

This product  $\rho c$  is referred to as the “fluid impedance”. This result shows that if the impedances match, there is no reflection.

### 6.28 Irrotational, Barotropic Flows of Inviscid Compressible Fluid

Consider an irrotational flow field given by

$$v_i = -\frac{\partial \varphi}{\partial x_i} \tag{6.28.1}$$

To satisfy the mass conservation principle, we must have

$$\frac{\partial \rho}{\partial t} + v_j \frac{\partial \rho}{\partial x_j} + \rho \frac{\partial v_j}{\partial x_j} = 0 \quad (6.28.2)$$

In terms of  $\varphi$  this equation becomes

$$\frac{\partial \rho}{\partial t} - \frac{\partial \varphi}{\partial x_j} \frac{\partial \rho}{\partial x_j} - \rho \frac{\partial^2 \varphi}{\partial x_j \partial x_j} = 0 \quad (6.28.3)$$

The equations of motion for an inviscid fluid are the Euler equations

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + B_i \quad (6.28.4)$$

We assume that the flow is **barotropic**, that is, the pressure is an explicit function of density only (such as in isentropic or isothermal flow). Thus, in a barotropic flow,

$$p = p(\rho) \quad \text{and} \quad \rho = \rho(p) \quad (6.28.5)$$

Now,

$$\frac{\partial}{\partial x_i} \left( \int \frac{1}{\rho} dp \right) = \left[ \frac{d}{dp} \int \frac{1}{\rho} dp \right] \frac{\partial p}{\partial x_i} = \frac{1}{\rho} \frac{\partial p}{\partial x_i} \quad (6.28.6)$$

Therefore, for barotropic flows of an inviscid fluid under conservative body forces (i.e.,  $B_i = -\frac{\partial \Omega}{\partial x_i}$ ), the equations of motion can be written

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{\partial}{\partial x_i} \left( \int \frac{dp}{\rho} + \Omega \right) \quad (6.28.7)$$

Comparing Eq. (6.28.7) with (6.21.6), we see immediately that under the conditions stated, irrotational flows are again always dynamically possible. In fact, the integration of Eq. (6.28.7) (in exactly the same way as was done in Section 6.21) gives the following Bernoulli equation

$$-\frac{\partial \varphi}{\partial t} + \int \frac{dp}{\rho} + \frac{v^2}{2} + \Omega = f(t) \quad (6.28.8)$$

which for steady flow, becomes

$$\int \frac{dp}{\rho} + \frac{v^2}{2} + \Omega = \text{constant} \quad (6.28.10)$$

For most problems in gas dynamics, the body force is small compared with other forces and often neglected. We then have

$$\int \frac{dp}{\rho} + \frac{v^2}{2} = \text{constant} \quad (6.28.11)$$

Example 6.28.1

Show that for steady isentropic irrotational flows of an inviscid compressible fluid (body force neglected)

$$\frac{\gamma}{\gamma-1} \frac{p}{\rho} + \frac{v^2}{2} = \text{constant} \tag{i}$$

*Solution.* For an isentropic flow

$$p = \beta \rho^\gamma, \quad dp = \beta \gamma \rho^{\gamma-1} d\rho$$

so that

$$\int \frac{dp}{\rho} = \beta \gamma \int \rho^{\gamma-2} d\rho = \beta \gamma \left( \frac{\rho^{\gamma-1}}{\gamma-1} \right) = \frac{\gamma}{\gamma-1} \frac{p}{\rho} \tag{ii}$$

Thus, the Bernoulli equation [Eq. (6.28.11)] becomes

$$\frac{\gamma}{\gamma-1} \frac{p}{\rho} + \frac{v^2}{2} = \text{constant} \tag{iii}$$

We note that this is the same result as that obtained in Example 26.1, Eq. (6.26.8), by the use of the energy equation. In other words, under the conditions stated (inviscid, non-heat conducting, initial homogeneous state), the Bernoulli equation and the energy equation are the same.

Example 6.28.2

Let  $p_o$  denote the pressure at zero speed (called **stagnation pressure** .) Show that for isentropic steady flow ( $p/\rho^\gamma = \text{constant}$ ) of an ideal gas,

$$p_o = p \left[ 1 + \frac{1}{2}(\gamma-1) \left( \frac{v}{c} \right)^2 \right]^{\gamma/(\gamma-1)} \tag{6.28.12}$$

where  $c$  is the local speed of sound.

*Solution.* Since (see the previous example)

$$\frac{\gamma}{\gamma-1} \frac{p}{\rho} + \frac{v^2}{2} = \text{constant} \tag{i}$$

therefore,

$$\frac{v^2}{2} = \frac{\gamma}{\gamma-1} \left( \frac{p_o}{\rho_o} - \frac{p}{\rho} \right) \tag{ii}$$

Now,  $c^2 = \gamma p / \rho$

$$\frac{v^2}{2c^2} = \frac{1}{\gamma-1} \left( \frac{p_o}{p} \right) \left( \frac{\rho}{\rho_o} \right) - \frac{1}{\gamma-1} \quad (\text{iii})$$

Since

$$\frac{\rho}{\rho_o} = \left( \frac{p}{p_o} \right)^{1/\gamma} = \left( \frac{p_o}{p} \right)^{-1/\gamma} \quad (\text{iv})$$

therefore, from (iii)

$$\left( \frac{\gamma-1}{2} \right) \left( \frac{v}{c} \right)^2 = \left( \frac{p_o}{p} \right)^{(\gamma-1)/\gamma} - 1 \quad (\text{v})$$

Thus,

$$\frac{p_o}{p} = \left[ 1 + \left( \frac{\gamma-1}{2} \right) \left( \frac{v}{c} \right)^2 \right]^{\gamma/(\gamma-1)} \quad (\text{vi})$$

For small Mach number  $M$ , (i.e.,  $v/c \ll 1$ ), we can use the binomial expansion to obtain from the above equation

$$\frac{p_o}{p} = 1 + \frac{\gamma}{2} \left( \frac{v}{c} \right)^2 + \frac{1}{8} \gamma \left( \frac{v}{c} \right)^4 + \dots \quad (\text{vii})$$

Noting that

$$\frac{p\gamma}{c^2} = \frac{p\gamma}{\gamma \left( \frac{p}{\rho} \right)} = \rho$$

we have, from (vi)

$$p_o = p + \frac{1}{2} \rho v^2 \left[ 1 + \frac{1}{4} \left( \frac{v}{c} \right)^2 + \dots \right] \quad (\text{viii})$$

For small Mach number  $M$ , the above equation is approximately

$$p_o = p + \frac{1}{2} \rho v^2 \quad (\text{ix})$$

which is the same as that for an incompressible fluid. In other words, for steady isentropic flow, the fluid may be considered as incompressible if the Mach number is small (say  $< 0.3$ .)

---

Example 6.28.3

For steady, barotropic irrotational flow, derive the equation for the velocity potential  $\varphi$ . Neglect body forces.

*Solution.* For steady flow, the equation of continuity is, with  $v_i = -\partial\varphi/\partial x_i$ ,

$$\frac{\partial\varphi}{\partial x_i} \frac{\partial\rho}{\partial x_i} + \rho \frac{\partial^2\varphi}{\partial x_i\partial x_i} = 0 \tag{i}$$

and the equation of motion is

$$\frac{\partial\varphi}{\partial x_j} \frac{\partial^2\varphi}{\partial x_j\partial x_i} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} = -\frac{1}{\rho} \frac{dp}{d\rho} \frac{\partial\rho}{\partial x_i} \tag{ii}$$

Let  $c^2 \equiv dp/d\rho$  (the local sound speed), then

$$-\frac{1}{\rho} \frac{\partial\rho}{\partial x_i} = \frac{1}{c^2} \left( \frac{\partial\varphi}{\partial x_j} \frac{\partial^2\varphi}{\partial x_j\partial x_i} \right) \tag{iii}$$

Substituting Eq. (iii) into Eq. (i), we obtain

$$-\frac{\partial\varphi}{\partial x_i} \frac{\partial\varphi}{\partial x_j} \frac{\partial^2\varphi}{\partial x_j\partial x_i} + c^2 \frac{\partial^2\varphi}{\partial x_i\partial x_i} = 0 \tag{iv}$$

or,

$$\left( c^2 \delta_{ij} - \frac{\partial\varphi}{\partial x_i} \frac{\partial\varphi}{\partial x_j} \right) \frac{\partial^2\varphi}{\partial x_i\partial x_j} = 0. \tag{v}$$

In long form, Eq. (v) reads

$$\begin{aligned} & \left[ c^2 - \left( \frac{\partial\varphi}{\partial x_1} \right)^2 \right] \frac{\partial^2\varphi}{\partial x_1^2} + \left[ c^2 - \left( \frac{\partial\varphi}{\partial x_2} \right)^2 \right] \frac{\partial^2\varphi}{\partial x_2^2} + \left[ c^2 - \left( \frac{\partial\varphi}{\partial x_3} \right)^2 \right] \frac{\partial^2\varphi}{\partial x_3^2} \\ & - 2 \left( \frac{\partial\varphi}{\partial x_1} \frac{\partial\varphi}{\partial x_2} \frac{\partial^2\varphi}{\partial x_1\partial x_2} + \frac{\partial\varphi}{\partial x_2} \frac{\partial\varphi}{\partial x_3} \frac{\partial^2\varphi}{\partial x_2\partial x_3} + \frac{\partial\varphi}{\partial x_3} \frac{\partial\varphi}{\partial x_1} \frac{\partial^2\varphi}{\partial x_3\partial x_1} \right) = 0 \end{aligned} \tag{vi}$$

6.29 One-Dimensional Flow of a Compressible Fluid

In this section, we discuss some internal flow problems of a compressible fluid. The fluid will be assumed to be an ideal gas. The flow will be assumed to be one-dimensional in the sense that the pressure, temperature, density, velocity, etc. are uniform over any cross-section of the channel or duct in which the fluid is flowing. The flow will also be assumed to be steady and adiabatic.

In steady flow, the rate of mass flow is constant for all cross-sections. With  $A$  denoting the variable cross-sectional area,  $\rho$  the density and  $v$  the velocity, we have

$$\rho Av = C \quad (\text{a constant}) \quad (6.29.1)$$

To see the effect of area variation on the flow, we take the total derivative of Eq. (6.29.1), i.e.,

$$d\rho(Av) + \rho(dA)v + \rho A(dv) = 0$$

Dividing the above equation by  $\rho Av$ , we obtain

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{dv}{v} = 0 \quad (6.29.2)$$

Thus,

$$\frac{dA}{A} = -\frac{d\rho}{\rho} - \frac{dv}{v} \quad (i)$$

Now, for barotropic flow of an ideal gas, we have [see Eq. (6.28.11)]

$$\frac{v^2}{2} + \int \frac{dp}{\rho} = \text{constant} \quad (6.29.3)$$

Thus,

$$v dv + \frac{dp}{\rho} = v dv + \frac{1}{\rho} \frac{dp}{dv} dv = 0 \quad (ii)$$

But  $\sqrt{(dp/d\rho)} = c$  (the speed of sound), thus,

$$\frac{dp}{\rho} = -\frac{v dv}{c^2} \quad (iii)$$

Combining Eqs. (i) and (iii), we get

$$\frac{dA}{A} = \frac{v dv}{c^2} - \frac{dv}{v} = \frac{dv}{v} \left( \frac{v^2}{c^2} - 1 \right) \quad (iv)$$

i.e.,

$$\frac{dA}{A} = \frac{dv}{v} (M^2 - 1) \quad (6.29.4)$$

Eq. (6.29.4) is sometimes known as **Hugoniot equation**. From this equation, we see that for subsonic flows ( $M < 1$ ), an increase in area produces a decrease in velocity, just as in the case of an incompressible fluid. On the other hand, for supersonic flows ( $M > 1$ ), an increase in area produces an increase in velocity. Furthermore, the critical velocity ( $M=1$ ) can only be obtained at the smallest cross-sectional area where  $dA = 0$ .

We now study the flow in a converging nozzle and the flow in a converging-diverging nozzle, using one-dimensional assumptions.

(i) *Flow in a Converging Nozzle*

Let us consider the adiabatic flow of an ideal gas from a large tank (inside which the pressure  $p_1$ , and the density  $\rho_1$  remain essentially unchanged) into a region of pressure  $p_R$ .

Application of the energy equation, using the conditions inside the tank and at the section (2) gives

$$\frac{v_2^2}{2} + \frac{\gamma}{\gamma-1} \frac{p_2}{\rho_2} = 0 + \frac{\gamma}{\gamma-1} \frac{p_1}{\rho_1} \tag{i}$$

where  $p_2$ ,  $\rho_2$ , and  $v_2$  are pressure, density, and velocity at section (2). Thus

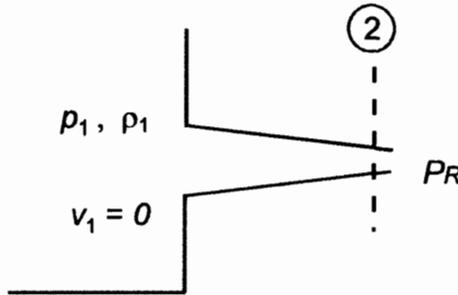


Fig. 6.15

$$v_2^2 = \frac{2\gamma}{\gamma-1} \left( \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} \right) \tag{ii}$$

For adiabatic flow,

$$\left( \frac{p_2}{p_1} \right)^{\frac{1}{\gamma}} = \left( \frac{\rho_2}{\rho_1} \right) \tag{iii}$$

Therefore,

$$v_2^2 = \frac{2\gamma}{\gamma-1} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{\frac{\gamma-1}{\gamma}} \right] = \frac{2\gamma}{\gamma-1} \frac{p_2}{\rho_2} \left[ \left( \frac{p_1}{p_2} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right] \tag{6.29.5}$$

Computing the rate of mass flow  $dm/dt$ , we have

$$\frac{dm}{dt} = A_2 \rho_2 v_2 = A_2 \left( \frac{p_2}{p_1} \right)^{1/\gamma} \rho_1 v_2 \tag{iv}$$

Thus,

$$\frac{dm}{dt} = A_2 \left( \frac{p_2}{p_1} \right)^{1/\gamma} \rho_1 \left( \frac{2\gamma}{\gamma-1} \frac{p_1}{\rho_1} \left[ 1 - \left( \frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma} \right] \right)^{1/2} \tag{v}$$

or

$$\frac{dm}{dt} = A_2 \left[ \frac{2\gamma}{\gamma-1} (p_1 \rho_1) \left( \left( \frac{p_2}{p_1} \right)^{2/\gamma} - \left( \frac{p_2}{p_1} \right)^{(1+\gamma)/\gamma} \right) \right]^{1/2} \tag{6.29.6}$$

For given  $p_1$ ,  $\rho_1$ , and  $A_2$  we see  $dm/dt$  depends only on  $p_2$ . When  $p_2 = 0$ ,  $dm/dt$  is zero and when  $p_2 = p_1$ ,  $dm/dt$  is also zero.

Figure 6.16 shows the curve of  $dm/dt$  versus  $p_2/p_1$ , according to Eq. (6.29.6). It can be easily established that  $(dm/dt)_{\max}$  occurs at

$$p = \left( \frac{2}{\gamma+1} \right)^{\gamma/(\gamma-1)} p_1 \tag{6.29.7}$$

and at this pressure  $p_2$ ,

$$v_2^2 = \frac{2\gamma}{\gamma-1} \frac{p_2}{\rho_2} \left( \frac{\gamma+1}{2} - 1 \right) = \gamma \frac{p_2}{\rho_2} = \text{speed of sound at section(2)} \tag{6.29.8}$$

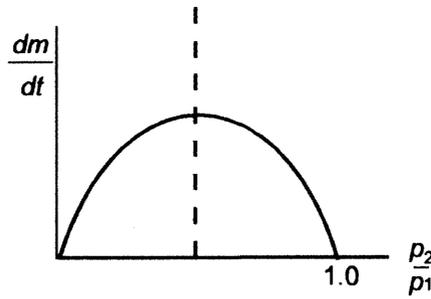


Fig. 6.16

The pressure  $p_2$  given by Eq. (6.29.8) is known as the critical pressure  $p_c$ . The pressure  $p_2$  at section (2) can never be less than  $p_c$  ( which depends only on  $p_1$ ) because otherwise the flow will become supersonic at section (2) which is impossible in view of the conclusion reached earlier that to have  $M = 1, dA$  must be zero, and to have  $M > 1, dA$  must be increasing (divergent nozzle). Thus, for the case of a convergent nozzle,  $p_2$  can never be less than  $p_R$ , the pressure surrounding the exit jet. When  $p_R \geq p_c, p_2 = p_R$ , and when  $p_R < p_c, p_2 = p_c$ . In other words, the relation between  $dm/dt$  and  $p_R/p_1$  is given as, for  $p_R \geq p_c$

$$\frac{dm}{dt} = A_2 \left[ \frac{2\gamma}{\gamma-1} (p_1 \rho_1) \right]^{1/2} \left[ \left( \frac{p_R}{p_1} \right)^{2/\gamma} - \left( \frac{p_R}{p_1} \right)^{(\gamma+1)/\gamma} \right]^{1/2} \tag{6.29.9}$$

and for  $p_R \leq p_c$

$$\frac{dm}{dt} = A_2 \left[ \frac{2\gamma}{\gamma-1} p_1 \rho_1 \right]^{1/2} \left[ \left( \frac{2}{\gamma+1} \right)^{2/(\gamma-1)} - \left( \frac{2}{\gamma+1} \right)^{(\gamma+1)/(\gamma-1)} \right]^{1/2} = \text{constant} \tag{6.29.10}$$

Figure 6.17 shows this relationship.

(ii) *Flow in a Convergent-Diverging Nozzle*

For a compressible fluid from a large supply tank, in order to increase the speed, a converging nozzle is needed. From (i), we have seen that the maximum attainable Mach number is unity in a converging passage. We have also concluded at the beginning of this section that in order to have the Mach number larger than unity, the cross-sectional area must increase in the direction of flow. Thus, in order to make supersonic flow possible from a supply

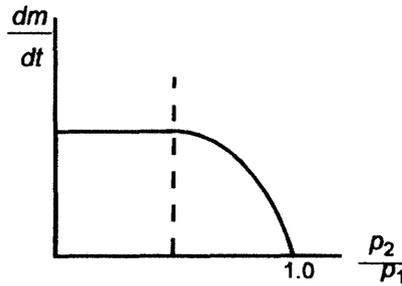


Fig. 6.17

tank, the fluid must flow in a converging-diverging nozzle as shown in Fig. 6.18. The flow in the converging part of the nozzle is always subsonic regardless of the receiver pressure  $p_R (< p_1)$ . The flow in the diverging passage is subsonic for certain range of  $p_R/p_1$  (see curves *a* and *b* in Fig. 6.18). There is a value of  $p_R$  at which the flow at the throat is sonic, the flow corresponding to this case is known as **choked flow** (curve *c*). Further reductions of  $p_R$  cannot affect the condition at the throat and produces no change in flow rate. There is one receiver pressure  $p_R$  for which the flow can expand isentropically to  $p_R$  (the solid curve *e*.)

If the receiver pressure  $p_R$  is between *c* and *e*, such as *d*, the flow following the throat for a short distance will be supersonic. This is then followed by a discontinuity<sup>†</sup> in pressure (compression shock) and flow becomes subsonic for the remaining distance to the exit. If the receiver pressure is below that indicated by *e* in the figure, a series of expansion waves and oblique shock waves occur outside the nozzle.

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† That is, the increase in pressure takes place in a very short distance

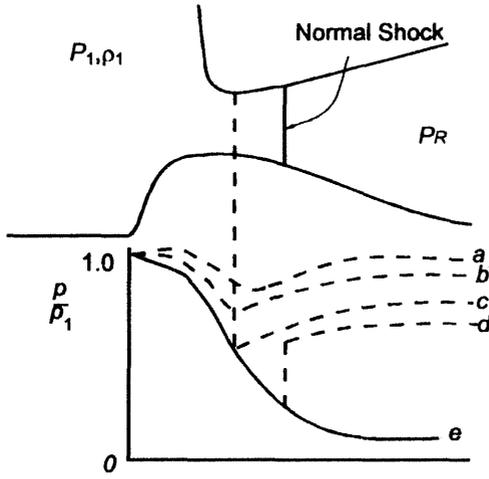


Fig. 6.18

## PROBLEMS

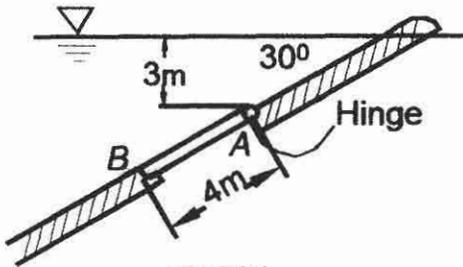


Fig.P6.1

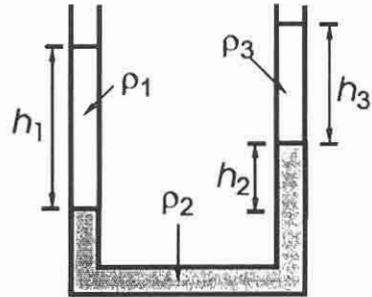


Fig.P6.3

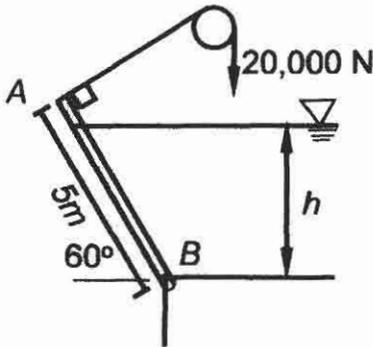


Fig.P6.2

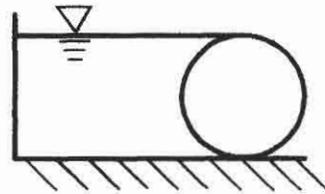


Fig.P6.4

6.1. In Fig.P6.1, the gate  $AB$  is rectangular, 60 cm wide and 4 m long. The gate is hinged at the upper edge  $A$ . Neglect the weight of the gate, find the reactional force at  $B$ . Take the specific weight of water to be  $9800 \text{ N/m}^3$  ( $62.4 \text{ lb/ft}^3$ )

6.2. The gate  $AB$  in Fig.P6.2 is 5 m long and 3 m wide. Neglect the weight of the gate, compute the water level  $h$  for which the gate will start to fall.

6.3. The liquids in the U-tube shown in Fig.P6.3 is in equilibrium. Find  $h_2$  as a function of  $\rho_1, \rho_2, \rho_3, h_1$  and  $h_3$ . The liquids are immiscible.

6.4. Referring to Fig.P6.4, (a) Find the buoyancy force on the cylinder and (b) find the resultant force on the cylindrical surface due to the water pressure. The centroid of a semi-circular area is  $4r/3\pi$  from the diameter, where  $r$  is the radius.

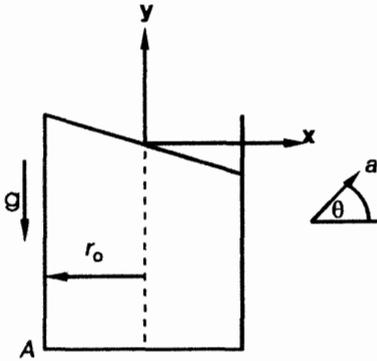


Fig.P6.5

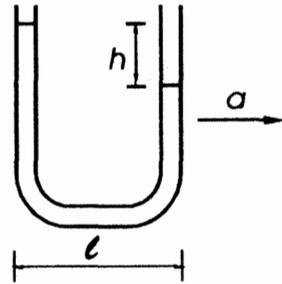


Fig.P6.6

- 6.5. A glass of water moves vertically upward with a constant acceleration  $a$ . Find the pressure at a point whose depth from the surface of the water is  $h$ .
- 6.6. A glass of water moves with a constant acceleration  $a$  in the direction shown in Fig.P6.5. Find the pressure at the point  $A$ . Take the atmospheric pressure to be  $p_a$ .
- 6.7. The slender U-tube shown in Fig.P6.6 is moving horizontally to the right with an acceleration  $a$ . Determine the relation between  $a$ ,  $l$  and  $h$ .
- 6.8. A liquid in a container rotates with a constant angular velocity  $\omega$  about a vertical axis. Find the shape of the liquid surface.
- 6.9. The slender U-tube rotates with an angular velocity  $\omega$  about the vertical axis shown in Fig.P6.7. Find the relation between  $\delta h (\equiv h_1 - h_2)$ ,  $\omega$ ,  $r_1$  and  $r_2$ .

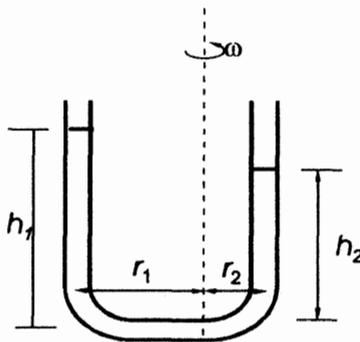


Fig.P6.7

**6.10.** In astrophysical applications, an atmosphere having the relation between the density  $\rho$  and pressure  $p$  given by

$$\frac{p}{p_0} = \left( \frac{\rho}{\rho_0} \right)^n$$

where  $p_0$  and  $\rho_0$  are some reference pressure and density, is known as a polytropic atmosphere. Find the distribution of pressure and density in a polytropic atmosphere.

**6.11.** For a steady parallel flow of an incompressible linearly viscous fluid, if we take the flow direction to be  $\mathbf{e}_3$ ,

(a) show that the velocity field is of the form

$$v_1 = 0, \quad v_2 = 0, \quad \text{and} \quad v_3 = v(x_1, x_2)$$

(b) If  $v(x_1, x_2) = kx_2$ , find the total normal stress on the plane whose normal is in the direction of  $\mathbf{e}_2 + \mathbf{e}_3$ , in terms of the viscosity  $\mu$  and pressure  $p$

(c) On what planes are the total normal stresses given by the so-called "pressure"?

**6.12.** Given the following velocity field (in m/s) for a Newtonian incompressible fluid with a viscosity  $\mu = 0.96$  mPa:

$$v_1 = x_1^2 - x_2^2, \quad v_2 = -2x_1x_2, \quad v_3 = 0.$$

At the point (1,2,1)m and on the plane whose normal is in the direction of  $\mathbf{e}_1$ ,

(a) find the excess of the total normal compressive stress over the pressure  $p$ ,

(b) find the magnitude of the shearing stress.

**6.13.** Do Problem 6.12 except that the plane has a normal in the direction of  $3\mathbf{e}_1 + 4\mathbf{e}_2$ .

**6.14.** Use the results of Sect. 2D2, Chapter 2 and the constitutive equations for the Newtonian viscous fluid, verify Eqs. (6.8.1).

**6.15.** Use the results of Sect. 2D3, Chapter 2 and the constitutive equations for the Newtonian viscous fluid, verify Eqs. (6.8.3).

**6.16.** Show that for a steady flow, the streamline containing a point  $P$  coincides with the pathline for a particle which passes through the point  $P$  at some time  $t$ .

**6.17.** For the two dimensional velocity field

$$v_1 = \frac{kx_1x_2}{1 + kx_2t}, \quad v_2 = 0$$

(a) Find the streamline at time  $t$ , which passes through the spatial point  $(\alpha_1, \alpha_2)$

(a) find the pathline for the particle which was at  $(X_1, X_2)$  at  $t = 0$ .

(c) Find the streakline at time  $t$ , formed by the particles which passed through the spatial position  $(\alpha_1, \alpha_2)$  at time  $\tau \leq t$ .

6.18. Do Prob. 6.17 for the following two dimensional velocity field

$$v_1 = \omega x_2, \quad v_2 = -\omega x_1$$

6.19. Do Prob. 6.17 for the following velocity field in polar coordinates  $(r, \theta)$

$$v_r = \frac{Q}{2\pi r}, \quad v_\theta = 0$$

6.20. Do Prob. 6.17 for the following velocity field in polar coordinates  $(r, \theta)$

$$v_r = 0, \quad v_\theta = \frac{C}{r}$$

6.21. From the Navier-Stokes equations, obtain Eq. (6.11.1) for the velocity distribution of the plane Couette flow.

6.22. For the plane Couette flow (see Section 6.11), if, in addition to the movement of the upper plate, there is also an applied negative pressure gradient  $\frac{\partial p}{\partial x_1}$ , obtain the velocity distribution. Also obtain the volume flow rate per unit width.

6.23. Obtain the steady uni-directional flow of an incompressible viscous fluid layer of uniform depth  $d$  flowing down an inclined plane which makes an angle  $\theta$  with the horizontal.

6.24. A layer of water ( $\rho g = 62.4 \text{ lb/ft}^3$ ) flows down an inclined plane ( $\theta = 30^\circ$ ) with a uniform thickness of 0.1 ft. Assuming the flow to be laminar, what is the pressure at any point on the inclined plane. Take the atmospheric pressure to be zero.

6.25. Two layers of liquids with viscosities  $\mu_1$  and  $\mu_2$ , density  $\rho_1$  and  $\rho_2$ , respectively, and with equal depths  $b$ , flow steadily between two fixed horizontal parallel plates. Find the velocity distribution for this steady uni-directional flow.

6.26. For the Hagen-Poiseuille flow in an inclined pipe, from the equations of motion show that if  $x_1$  is the direction of flow, then (a) the piezometric head depends only on  $x_1$ , i.e.,  $h = h(x_1)$  and (b)  $(dh/x_1) = \text{a constant}$ .

6.27. Verify the equation for the torque per unit length for the Couette flow, Eq. (6.15.5).

6.28. Consider the flow of an incompressible viscous fluid through the annular space between two concentric horizontal cylinders. The radii are  $a$  and  $b$ . (a) Find the flow field if there is no variation of pressure in the axial direction and if the inner and the outer cylinders have axial velocities  $v_a$  and  $v_b$  respectively and (b) find the flow field if there is a pressure gradient in the axial direction and both cylinders are fixed.

6.29. Show that for the velocity field

$$v_1 = v(y, z), \quad v_2 = v_3 = 0$$

the Navier-Stokes equations, with  $\rho \mathbf{B} = \mathbf{0}$ , reduces to

$$\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\mu} \frac{dp}{dx} = \beta = \text{constant.}$$

6.30. Referring to Problem 6.29, consider a pipe having an elliptic cross section given by  $y^2/a^2 + z^2/b^2 = 1$ . Assuming that

$$v = A \left[ \frac{y^2}{a^2} + \frac{z^2}{b^2} \right] + B,$$

find  $A$  and  $B$ .

6.31. Referring to Problem 6.29, consider an equilateral triangular cross-section defined by the planes  $z + b/(2\sqrt{3}) = 0$ ,  $z + \sqrt{3}y - b/\sqrt{3} = 0$ ,  $z - \sqrt{3}y - b/\sqrt{3} = 0$ . Assuming

$$v = A \left( z + \frac{b}{2\sqrt{3}} \right) \left( z + \sqrt{3}y - \frac{b}{\sqrt{3}} \right) \left( z - \sqrt{3}y - \frac{b}{\sqrt{3}} \right) + B,$$

find  $A$  and  $B$ .

6.32. For the steady-state, time dependent parallel flow of water (density  $\rho = 10^3 \text{ Kg/m}^3$ , viscosity  $\mu = 10^{-3} \text{ Ns/m}^2$ ) near an oscillating plate, calculate the wave length for  $\omega = 2\text{cps}$ .

6.33. The space between two concentric spherical shell is filled with an incompressible Newtonian fluid. The inner shell (radius  $r_i$ ) is fixed; the outer shell (radius  $r_o$ ) rotates with an angular velocity  $\Omega$  about a diameter. Find the velocity distribution. Assume the flow to be laminar without secondary flow.

6.34. Consider the following velocity field in cylindrical coordinates:

$$v_r = v(r), \quad v_\theta = v_z = 0.$$

(a) Show that  $v(r) = \frac{A}{\rho r}$ , where  $A$  is a constant so that the equation of conservation of mass is satisfied.

(b) If the rate of mass flow through a circular cylindrical surface of radius  $r$  and unit length is  $Q_m$ , determine the constant  $A$  in terms of  $Q_m$ .

6.35. Given the following velocity field in cylindrical coordinates

$$v_r = v(r, \theta), \quad v_\theta = 0, \quad v_z = 0$$

(a) Show from the continuity equation that

$$v_r = \frac{f(\theta)}{r}$$

(b) In the absence of body forces, show that

$$\frac{d^2 f}{d\theta^2} + 4f + \frac{\rho f^2}{\mu} + k = 0 \quad \text{and}$$

$$p = 2\mu \frac{f}{r^2} + \frac{k\mu}{2r^2} + C$$

where  $k$  and  $C$  are constants.

**6.36.** Determine the temperature distribution for the flow of Prob. 6.22 due to viscous dissipation when both plates are maintained at the same fixed temperature  $\theta_o$ . Assume constant physical properties.

**6.37.** Determine the temperature distribution in the plane Poiseuille flow where the bottom plate is kept at a constant temperature  $\theta_1$  and the top plate at  $\theta_2$ . Include the heat generated by viscous dissipation.

**6.38.** Determine the temperature distribution in the laminar flow between two coaxial cylinders (Couette flow) if the temperatures at the inner and the outer cylinders are kept at the same fixed temperature  $\theta_o$ .

**6.39.** Show that the dissipation function for a compressible fluid can be written as that given in Eq. (6.17.6b).

**6.40.** Given the velocity field of a linearly viscous fluid

$$v_1 = kx_1, \quad v_2 = -kx_2, \quad v_3 = 0$$

(a) Show that the velocity field is irrotational.

(b) Find the stress tensor.

(c) Find the acceleration field.

(d) Show that the velocity field satisfies the Navier-Stokes equations by finding the pressure distribution directly from the equations. Neglect body forces. Take  $p = p_o$  at the origin.

(e) Use the Bernoulli equation to find the pressure distribution.

(f) Find the rate of dissipation of mechanical energy into heat.

(g) If  $x_2 = 0$  is a fixed boundary, what condition is not satisfied by the velocity field?

**6.41.** Do Problem 6.40 for the following velocity field:

$$v_1 = k(x_1^2 - x_2^2), \quad v_2 = -2kx_1x_2, \quad v_3 = 0$$

**6.42.** Obtain the vorticity components for the plane Poiseuille flow.

**6.43.** Obtain the vorticity components for the Hagen-Poiseuille flow.

**6.44.** For two-dimensional flow of an incompressible fluid, we can express the velocity components in terms of a scalar function  $\psi$  (known as the Lagrange stream function) by the relation

$$v_1 = \frac{\partial \psi}{\partial y}, \quad v_2 = -\frac{\partial \psi}{\partial x}$$

(a) Show that the equation of conservation of mass is automatically satisfied for any  $\psi(x,y)$  which has continuous second partial derivatives.

(b) Show that for two-dimensional flow of an incompressible fluid,  $\psi = \text{constants}$  are streamlines, where  $\psi$  is the Lagrange stream function.

- (c) If the velocity field is irrotational, then  $v_i = -\partial\phi/\partial x_i$ , where  $\phi$  is known as the velocity potential. Show that the curves of constant velocity potential  $\phi = \text{constant}$  and the stream line  $\psi = \text{constant}$  are orthogonal to each other.
- (d) Obtain the only nonzero vorticity component in terms of  $\psi$ .

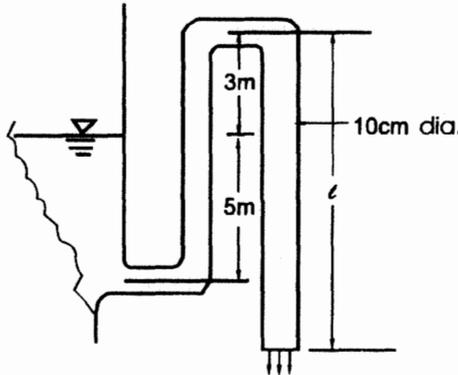


Fig.P6.8

6.45. Show that

$$\psi = V_0 y \left( 1 - \frac{a^2}{x^2 + y^2} \right)$$

represents a two-dimensional irrotational flow of an inviscid fluid. Sketch the stream lines in the region  $x^2 + y^2 \geq a^2$

6.46. Referring to Fig.P6.8, compute the maximum possible flow of water. Take the atmospheric pressure to be 93.1 kPa, the specific weight of water  $9800 \text{ N/m}^3$ , and the vapor pressure 17.2 kPa. Assume the fluid to be inviscid. Find the length  $l$  for this rate of discharge.

6.47. Water flows upward through a vertical pipeline which tapers from 25.4 cm to 15.2 cm diameter in a distance of 1.83 m. If the pressure at the beginning and end of the constriction are 207 kPa and 172 kPa respectively. What is the flow rate? Assume the fluid to be inviscid.

6.48. Verify that the equation of conservation of mass is automatically satisfied if the velocity components in cylindrical coordinates are given by

$$v_r = -\frac{1}{\rho r} \frac{\partial \psi}{\partial z}, \quad v_z = -\frac{1}{\rho r} \frac{\partial \psi}{\partial r}, \quad v_\theta = 0$$

where the density  $\rho$  is a constant and  $\psi$  is any function of  $r$  and  $z$  having continuous second partial derivatives.

6.49. Derive Eq. (6.25.6).

**6.50.** Show that for a one-dimensional, steady, adiabatic flow of an ideal gas, the ratio of temperature  $\theta_1/\theta_2$  at sections (1) and (2) is given by

$$\frac{\theta_1}{\theta_2} = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{1 + \frac{1}{2}(\gamma - 1)M_2^2}$$

where  $\gamma$  is the ratio of specific heat,  $M_1$  and  $M_2$  are local Mach numbers at section 1 and 2 respectively.

**6.51.** Show that for a compressible fluid in isothermal flow with no external work,

$$\frac{dM^2}{M^2} = 2\frac{dv}{v}$$

where  $M$  is the Mach number. (Assume perfect gas.)

**6.52.** Show that for a perfect gas flowing through a constant area duct at constant temperature conditions.

$$\frac{dp}{p} = -\frac{1}{2} \frac{dM^2}{M^2}.$$

**6.53.** For the flow of a compressible inviscid fluid around a thin body in a uniform stream of speed  $V_m$  in the  $x_1$  - direction, we let the velocity potential be

$$\varphi = -V_o(x_1 + \varphi_1),$$

where  $\varphi_1$  is assumed to be very small. Show that for steady flow the equation governing  $\varphi_1$  is, with  $M_o = V_o/c_o$

$$(1 - M_o^2) \frac{\partial^2 \varphi_1}{\partial x_1^2} + \frac{\partial^2 \varphi_1}{\partial x_2^2} + \frac{\partial^2 \varphi_1}{\partial x_3^2} = 0.$$

## Integral Formulation of General Principles

In Sections 3.15, 4.4, 4.7, 4.14, the field equations expressing the principles of conservation of mass, of linear momentum, of moment of momentum, and of energy were derived by the consideration of differential elements in the continuum. In the form of differential equations, the principles are sometimes referred to as **local principles**. In this chapter, we shall formulate the principles in terms of an arbitrary fixed part of the continuum. The principles are then in integral form, which is sometimes referred to as the **global principles**. Under the assumption of smoothness of functions involved, the two forms are completely equivalent and in fact the requirement that the global theorem be valid for each and every part of the continuum results in the differential form of the balance equations.

The purpose of the present chapter is twofold: (1) to provide an alternate approach to the formulation of field equations expressing the general principles, and (2) to apply the global theorems to obtain approximate solutions of some engineering problems, using the concept of control volumes, moving or fixed.

We shall begin by proving Green's theorem, from which the divergence theorem, which we shall need later in the chapter, will be introduced through a generalization (without proof).

### 7.1 Green's Theorem

Let  $P(x,y)$ ,  $\partial P/\partial x$  and  $\partial P/\partial y$  be continuous functions of  $x$  and  $y$  in a closed region  $R$  bounded by the closed curve  $C$ . Let  $\mathbf{n} = n_x \mathbf{e}_1 + n_y \mathbf{e}_2$  be the unit outward normal of  $C$ . Then **Green's theorem**<sup>†</sup> states that

$$\int_R \frac{\partial P}{\partial x} dA = \int_C P dy = \int_C P n_x ds \quad (7.1.1)$$

and

---

† The theorem is valid under less restrictive conditions on the first partial derivative.

$$\int_R \frac{\partial P}{\partial y} dA = - \int_C P dx = \int_C P n_y ds \quad (7.1.2)$$

where the subscript  $C$  denotes the line integral around the closed curve  $C$  in the counter-clockwise direction. For the proof, let us assume for simplicity that the region  $R$  is such that every straight line through an interior point and parallel to either axis cuts the boundary in exactly two points. Figure 7.1 shows one such region. Let  $a$  and  $b$  be the least and the greatest values of  $y$  on  $C$  (points  $G$  and  $H$  in the figure). Let  $x = x_1(y)$  and  $x = x_2(y)$  be equations for the boundaries  $HAG$  and  $GBH$  respectively. Then

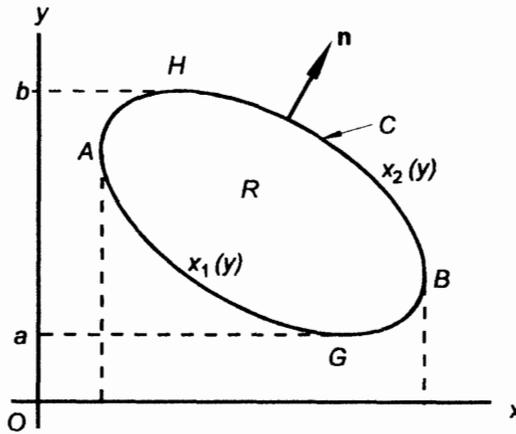


Fig. 7.1

$$\int_R \frac{\partial P}{\partial x} dA = \int_a^b \left[ \int_{x_1(y)}^{x_2(y)} \frac{\partial P}{\partial x} dx \right] dy \quad (i)$$

Now

$$\int_{x_1(y)}^{x_2(y)} \frac{\partial P}{\partial x} dx = P(x, y) \Big|_{x_1(y)}^{x_2(y)} = P[x_2(y), y] - P[x_1(y), y] \quad (ii)$$

Thus,

$$\begin{aligned} \int_R \frac{\partial P}{\partial x} dA &= \int_a^b P[x_2(y), y] dy - \int_a^b P[x_1(y), y] dy \\ &= \int_{GBH} P dy - \int_{GAH} P dy \end{aligned} \quad (iii)$$

Since

$$\int_{GAH} Pdy = - \int_{HAG} Pdy \quad (\text{iv})$$

Thus

$$\int_R \frac{\partial P}{\partial x} dA = \int_{GBH} Pdy + \int_{HAG} Pdy = \int_C Pdy \quad (\text{v})$$

Let  $s$  be the arc length measured along the boundary curve  $C$  in the counterclockwise direction and let  $x = x(s)$  and  $y = y(s)$  be the parametric equations for the boundary curve. Then,  $dy/ds = +n_x$ , Thus,

$$\int_R \frac{\partial P}{\partial x} dA = \int_C Pn_x ds$$

which is Eq. (7.1.1).

Equation (7.1.2) can be proven in a similar manner.

#### Example 7.1.1

For  $P(x,y) = xy^2$ , evaluate  $\int_C P(x,y)n_x ds$  along the closed path  $OABC$  (Fig. 7.2). Also, evaluate the area integral  $\int_R (\partial P/\partial x) dA$ . Compare the results.

*Solution.* We have

$$\begin{aligned} \int_C P(x,y)n_x ds &= \int_{OA} x(0)^2(0) ds + \int_{AB} by^2(1) dy + \int_{BC} xh^2(0) ds \\ &+ \int_{CO} (0)y^2(-1) ds = \int_0^h by^2 dy = \frac{bh^3}{3} \end{aligned} \quad (\text{i})$$

On the other hand,

$$\int_R \frac{\partial P}{\partial x} dA = \int_R y^2 dA = \int_0^h y^2 b dy = \frac{bh^3}{3} \quad (\text{ii})$$

Thus,

$$\int_C Pn_x ds = \int_R \frac{\partial P}{\partial x} dA \quad (\text{iii})$$

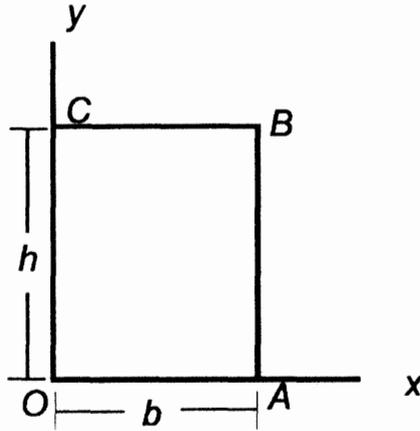


Fig. 7.2

**7.2 Divergence Theorem**

Let  $\mathbf{v} = v_1(x,y)\mathbf{e}_1 + v_2(x,y)\mathbf{e}_2$  be a vector field. Applying Eqs. (7.1.1) and (7.1.2) to  $v_1$  and  $v_2$  and adding, we have

$$\int_C (v_1 n_1 + v_2 n_2) ds = \int_R \left( \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} \right) dA \tag{7.2.1a}$$

In indicial notation, Eq. (7.2.1a) becomes

$$\int_C v_i n_i ds = \int_R \frac{\partial v_i}{\partial x_i} dA \tag{7.2.1b}$$

and in invariant notation,

$$\int_C \mathbf{v} \cdot \mathbf{n} ds = \int_R \text{div } \mathbf{v} dA \tag{7.2.1c}$$

The following generalization not only appears natural, but can indeed be proven (we omit the proof)

$$\int_S v_i n_i dS = \int_R \frac{\partial v_i}{\partial x_i} dV \tag{7.2.2a}$$

Or, in invariant notation,

$$\int_S \mathbf{v} \cdot \mathbf{n} dS = \int_R \operatorname{div} \mathbf{v} dV \quad (7.2.2b)$$

where  $S$  is a surface forming the complete boundary of a bounded closed region  $R$  in space and  $\mathbf{n}$  is the outward unit normal of  $S$ . Equation (7.2.2) is known as the **divergence theorem** (or **Gauss theorem**). The theorem is valid if the components of  $\mathbf{v}$  are continuous and have continuous first partial derivatives in  $R$ . It is also valid under less restrictive conditions on the derivatives.

Next, if  $T_{ij}$  are components of a tensor  $\mathbf{T}$ , then the application of Eq. (7.2.2a) gives

$$\int_S T_{ij} n_j dS = \int_R \frac{\partial T_{ij}}{\partial x_j} dV \quad (7.2.3a)$$

Or in invariant notation,

$$\int_S \mathbf{T} \mathbf{n} dS = \int_R \operatorname{div} \mathbf{T} dV \quad (7.2.3b)$$

Equation (7.2.3) is the divergence theorem for a tensor field. It is obvious that for tensor fields of higher order, Eq. (7.2.3b) is also valid provided the Cartesian components of  $\operatorname{div} \mathbf{T}$  are defined to be  $\partial T_{ijkl\dots s} / \partial x_s$ .

#### Example 7.2.1

Let  $\mathbf{T}$  be a stress tensor field and let  $S$  be a closed surface. Show that the resultant force of the distributive forces on  $S$  is given by

$$\int_V (\operatorname{div} \mathbf{T}) dV \quad (i)$$

*Solution.* Let  $\mathbf{f}$  be the resultant force, then

$$\mathbf{f} = \int_S \mathbf{t} dS \quad (ii)$$

where  $\mathbf{t}$  is the stress vector. But  $\mathbf{t} = \mathbf{T} \mathbf{n}$ , therefore from the divergence theorem, we have

$$\mathbf{f} = \int_S \mathbf{t} dS = \int_S \mathbf{T} \mathbf{n} dS = \int_V (\operatorname{div} \mathbf{T}) dV \quad (7.2.4a)$$

i.e.,

$$f_i = \int_V \frac{\partial T_{ij}}{\partial x_j} dV \quad (7.2.4b)$$

Example 7.2.2

Referring to Example 7.2.1, also show that the resultant moment, about a fixed point  $O$ , of the distributive forces on  $S$  is given by

$$\int_V [\mathbf{x} \times (\text{div} \mathbf{T}) + 2\mathbf{t}^A] dV \tag{i}$$

where  $\mathbf{x}$  is the position vector of the particle with volume  $dV$  from the fixed point  $O$  and  $\mathbf{t}^A$  is the axial (or dual) vector of the antisymmetric part of  $\mathbf{T}$  (see Sect. 2B16).

*Solution.* Let  $\mathbf{m}$  denote the resultant moment about  $O$ . Then

$$\mathbf{m} = \int_S \mathbf{x} \times \mathbf{t} dS \tag{ii}$$

Let  $m_i$  be the components of  $\mathbf{m}$ , then

$$m_i = \int_S \varepsilon_{ijk} x_j t_k dS = \int_S \varepsilon_{ijk} x_j T_{kp} n_p dS \tag{iii}$$

Using the divergence theorem, Eq. (7.2.3), we have

$$m_i = \int_V \frac{\partial}{\partial x_p} (\varepsilon_{ijk} x_j T_{kp}) dV \tag{iv}$$

Now,

$$\begin{aligned} \frac{\partial}{\partial x_p} (\varepsilon_{ijk} x_j T_{kp}) &= \varepsilon_{ijk} \left( \frac{\partial x_j}{\partial x_p} T_{kp} + x_j \frac{\partial T_{kp}}{\partial x_p} \right) \\ &= \varepsilon_{ijk} \left( \delta_{jp} T_{kp} + x_j \frac{\partial T_{kp}}{\partial x_p} \right) = \varepsilon_{ipk} T_{kp} + \varepsilon_{ijk} x_j \frac{\partial T_{kp}}{\partial x_p} = -\varepsilon_{ikp} T_{kp} + \varepsilon_{ijk} x_j \frac{\partial T_{kp}}{\partial x_p} \end{aligned} \tag{v}$$

Noting that  $-\varepsilon_{ikp} T_{kp}$  are components of twice the dual vector of the antisymmetric part of  $\mathbf{T}$  [see Eq. (2B16.2b)], and  $\varepsilon_{ijk} x_j (\frac{\partial T_{kp}}{\partial x_p})$  are components of  $[\mathbf{x} \times \text{div} \mathbf{T}]$ , we have

$$\mathbf{m} = \int_S \mathbf{x} \times \mathbf{t} dS = \int_V [\mathbf{x} \times (\text{div} \mathbf{T}) + 2\mathbf{t}^A] dV \tag{7.2.5}$$

Example 7.2.3

Referring to Example 7.2.2, show that the total power (rate of work done) by the stress vector on  $S$  is given by,

$$\int_V [(\operatorname{div} \mathbf{T}) \cdot \mathbf{v} + \operatorname{tr}(\mathbf{T}^T \nabla \mathbf{v})] dV \quad (\text{i})$$

where  $\mathbf{v}$  is the velocity field.

*Solution.* Let  $P$  be the total power, then

$$P = \int_S \mathbf{t} \cdot \mathbf{v} dS = \int_S \mathbf{T} \mathbf{n} \cdot \mathbf{v} dS \quad (\text{ii})$$

But  $\mathbf{T} \mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot \mathbf{T}^T \mathbf{v}$  (definition of transpose of a tensor). Thus,

$$P = \int_S \mathbf{n} \cdot (\mathbf{T}^T \mathbf{v}) dS \quad (\text{iii})$$

Application of the divergence theorem gives

$$P = \int_V \operatorname{div}(\mathbf{T}^T \mathbf{v}) dV \quad (\text{iv})$$

Now,

$$\operatorname{div}(\mathbf{T}^T \mathbf{v}) = \frac{\partial (T_{ji} v_j)}{\partial x_i} = \frac{\partial T_{ji}}{\partial x_i} v_j + T_{ji} \frac{\partial v_j}{\partial x_i} = (\operatorname{div} \mathbf{T}) \cdot \mathbf{v} + \operatorname{tr}(\mathbf{T}^T \nabla \mathbf{v}) \quad (\text{v})$$

Thus,

$$P = \int_S \mathbf{t} \cdot \mathbf{v} dS = \int_V [(\operatorname{div} \mathbf{T}) \cdot \mathbf{v} + \operatorname{tr}(\mathbf{T}^T \nabla \mathbf{v})] dV \quad (7.2.6)$$

### 7.3 Integrals over a Control Volume and Integrals over a Material Volume

Consider first a one-dimensional problem in which the motion of a continuum, in Cartesian coordinates, is given by

$$x = \hat{x}(X, t), \quad y = Y, \quad z = Z \quad (7.3.1)$$

and the density field is given by

$$\rho = \rho(x, t) \quad (7.3.2)$$

The integral

$$m(t, x^{(1)}, x^{(2)}) = \int_{x^{(1)}}^{x^{(2)}} \rho(x, t) A dx \quad (7.3.3)$$

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with fixed values of  $x^{(1)}$  and  $x^{(2)}$ , is an integral over a fixed **control volume**; it gives the total mass at time  $t$  within the spatially fixed cylindrical volume of constant cross-sectional area  $A$  and bounded by the end faces  $x = x^{(1)}$  and  $x = x^{(2)}$ .

Let  $X^{(1)}$  and  $X^{(2)}$  be the material coordinates for the particles which, at time  $t$  are at  $x^{(1)}$  and  $x^{(2)}$  respectively, i.e.,  $x^{(1)} = \hat{x}(X^{(1)}, t)$  and  $x^{(2)} = \hat{x}(X^{(2)}, t)$ , then the integral

$$M(t, X^{(1)}, X^{(2)}) = \int_{\hat{x}(X^{(1)}, t)}^{\hat{x}(X^{(2)}, t)} \rho(x, t) A dx \quad (7.3.4)$$

with its integration limits functions of time, (in accordance with the motion of the material particles which at time  $t$  are at  $x^{(1)}$  and  $x^{(2)}$ ), is an integral over a material volume; it gives the total mass at time  $t$ , of that part of material which is instantaneously (at time  $t$ ) coincidental with that inside the fixed boundary surface considered in Eq. (7.3.3). Obviously, at time  $t$ , both integrals, i.e., Eqs. (7.3.3) and (7.3.4), have the same value. At other times, say at  $t + dt$ , however, they have different values. Indeed,

$$\frac{\partial m}{\partial t} \equiv \left[ \frac{\partial}{\partial t} \int_{x^{(1)}}^{x^{(2)}} \rho A dx \right]_{x^{(1)}, x^{(2)} - \text{fixed}} \quad (7.3.5)$$

is different from

$$\frac{\partial M}{\partial t} = \left[ \frac{\partial}{\partial t} \int_{\hat{x}(X^{(1)}, t)}^{\hat{x}(X^{(2)}, t)} \rho(x, t) A dx \right]_{X^{(1)}, X^{(2)} - \text{fixed}} \equiv \frac{D}{Dt} \int_{\hat{x}(X^{(1)}, t)}^{\hat{x}(X^{(2)}, t)} \rho(x, t) A dx \quad (7.3.6)$$

We note that  $\partial m / \partial t$  in Eq. (7.3.5) gives the rate at which mass is increasing inside the fixed control volume bounded by the cylindrical lateral surface and the end faces  $x = x^{(1)}$  and  $x = x^{(2)}$ , whereas  $\partial M / \partial t$  in Eq. (7.3.6) gives the rate of increase of the mass of that part of material which at time  $t$  is coincidental with that in the fixed control volume. They should obviously be different. In fact, the principle of conservation of mass demands that the mass within a material volume should remain a constant, whereas the mass within the control volume in general changes with time.

The above one dimensional example serves to illustrate the two types of volume integrals which we shall employ in the following sections. We shall use  $V_c$  to indicate a fixed **control volume** and  $V_m$  to indicate a **material volume**. That is, for any tensor  $\mathbf{T}$  (including a scalar) the integral

$$\int_{V_c} \mathbf{T}(\mathbf{x}, t) dV$$

is over the fixed control volume  $V_c$  and the rate of change of this integral is denoted by

$$\frac{\partial}{\partial t} \int_{V_c} \mathbf{T}(\mathbf{x}, t) dV$$

whereas the integral

$$\int_{V_m} \mathbf{T}(\mathbf{x}, t) dV$$

is over the material volume  $V_m$  and the rate of change of this integral, is denoted by

$$\frac{D}{Dt} \int_{V_m} \mathbf{T}(\mathbf{x}, t) dV$$

We note that the integrals over the material volume is a special case of the more general integrals where the boundaries move in some prescribed manner which may or may not be in accordance with the motion of the material particles on the boundary. In this chapter, the control volume denoted by  $V_c$  will always denote a *fixed* control volume; they are either fixed with respect to an inertial frame or fixed with respect to a frame moving with respect to the inertial frame (see Section 7.7).

#### 7.4 Reynolds Transport Theorem

Let  $\mathbf{T}(\mathbf{x}, t)$  be a given scalar or tensor function of spatial coordinates  $(x_1, x_2, x_3)$  and time  $t$ . Examples of  $\mathbf{T}$  are: density  $\rho(\mathbf{x}, t)$ , linear momentum  $\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)$ , angular momentum  $\mathbf{r} \times [\rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)]$  etc.

Let

$$\int_{V_m(t)} \mathbf{T}(\mathbf{x}, t) dV \quad (i)$$

be an integral of  $\mathbf{T}(\mathbf{x}, t)$  over a material volume  $V_m(t)$ . As discussed in the last section, the material volume  $V_m(t)$  consists of the same material particles at all time and therefore has time-dependent boundary surface  $S_m(t)$  due to the movement of the material.

We wish to evaluate the rate of change of such integrals ( e.g., the rate of change of mass, of linear momentum etc., of a material volume ) and to relate them to physical laws (such as the conservation of mass, balance of linear momentum etc.)

The **Reynolds Transport Theorem** states that

$$\frac{D}{Dt} \int_{V_m(t)} \mathbf{T}(\mathbf{x}, t) dV = \int_{V_c} \frac{\partial \mathbf{T}(\mathbf{x}, t)}{\partial t} dV + \int_{S_c} \mathbf{T}(\mathbf{v} \cdot \mathbf{n}) dS \quad (7.4.1)$$

or

$$\frac{D}{Dt} \int_{V_m(t)} \mathbf{T}(\mathbf{x}, t) dV = \int_{V_c} \left( \frac{D\mathbf{T}}{Dt} + \mathbf{T} \text{div} \mathbf{v} \right) dV \quad (7.4.2)$$

where  $V_c$  is the control volume (fixed in space) which instantaneously coincides with the material volume  $V_m$  (moving with the continuum),  $S_c$  is the boundary surface of  $V_c$ ,  $\mathbf{n}$  is the outward unit normal vector. We note that the notation  $D/Dt$  in front of the integral at the left hand side of Eqs (7.4.2) emphasizes that the boundary surface of the integral moves with the material and we are calculating the rate of change by following the material.

Reynold’s theorem can be derived in the following two ways:

(A)

$$\frac{D}{Dt} \int_{V_{m(t)}} \mathbf{T}(\mathbf{x}, t) dV = \int_{V_m=V_c} \frac{D}{Dt} [\mathbf{T}(\mathbf{x}, t) dV] = \int_{V_c} \frac{D\mathbf{T}}{Dt} dV + \int_{V_c} \mathbf{T} \frac{D}{Dt}(dV) \tag{i}$$

Since [see Eq. (3.13.7) ]

$$\frac{D}{Dt}(dV) = (\text{div}\mathbf{v})dV \tag{ii}$$

therefore, Eq. (i) becomes

$$\frac{D}{Dt} \int_{V_{m(t)}} \mathbf{T}(\mathbf{x}, t) dV = \int_{V_c} \left[ \frac{D\mathbf{T}}{Dt} + \mathbf{T}(\text{div}\mathbf{v}) \right] dV \tag{iii}$$

This is Eq. (7.4.2).

In terms of Cartesian components, this equation reads, if  $T$  is a scalar

$$\frac{D}{Dt} \int_{V_{m(t)}} T(\mathbf{x}, t) dV = \int_{V_c} \left[ \frac{DT}{Dt} + T \left( \frac{\partial v_k}{\partial x_k} \right) \right] dV \tag{7.4.2a}$$

If  $\mathbf{T}$  is a vector, we replace  $T$  in Eq. (7.4.2a) with  $T_i$  and if  $\mathbf{T}$  is a second order tensor, we replace  $T$  with  $T_{ij}$  and so on.

Since

$$\frac{DT}{Dt} + T \left( \frac{\partial v_k}{\partial x_k} \right) = \frac{\partial T}{\partial t} + v_k \frac{\partial T}{\partial x_k} + T \frac{\partial v_k}{\partial x_k} = \frac{\partial T}{\partial t} + \frac{\partial}{\partial x_k} (T v_k) \tag{iv}$$

and from the Gauss theorem, we have

$$\int \frac{\partial}{\partial x_k} (T v_k) dV = \int T v_k n_k dS = \int T \mathbf{v} \cdot \mathbf{n} dS \tag{7.4.3}$$

so that, with  $\mathbf{T}$  denoting tensor of all orders (including scalars and vectors )

$$\frac{D}{Dt} \int_{V_m} \mathbf{T} dV = \int_{V_c} \frac{\partial \mathbf{T}}{\partial t} dV + \int_{V_c} \mathbf{T}(\mathbf{v} \cdot \mathbf{n}) dS \tag{v}$$

This is Eq. (7.4.1).

(B) Alternatively, we can derive Eq. (7.4.2) in the following way:

Since [see Eq. (3.29.3) ]

$$dV = (\det \mathbf{F}) dV_o \quad (\text{vi})$$

where  $\mathbf{F}$  is the deformation gradient and  $dV_o$  is the volume at the reference state, therefore

$$\int_{V_m} \mathbf{T}(\mathbf{x}, t) dV = \int_{V_o} \mathbf{T}(\mathbf{x}, t) (\det \mathbf{F}) dV_o \quad (\text{vii})$$

Thus,

$$\begin{aligned} \frac{D}{Dt} \int_{V_m} \mathbf{T}(\mathbf{x}, t) dV &= \frac{D}{Dt} \int_{V_o} \mathbf{T}(\det \mathbf{F}) dV_o = \int_{V_o} \left[ \frac{D}{Dt} (\mathbf{T} \det \mathbf{F}) \right] dV_o \\ &= \int_{V_o} \left[ \frac{D\mathbf{T}}{Dt} (\det \mathbf{F}) + \mathbf{T} \frac{D}{Dt} (\det \mathbf{F}) \right] dV_o \end{aligned} \quad (\text{viii})$$

But from Eq. (vi) and Eq. (ii), we have,

$$\frac{D}{Dt} (\det \mathbf{F}) = \frac{1}{dV_o} \left( \frac{D}{Dt} dV \right) = \frac{1}{dV_o} \text{div} \mathbf{v} dV = \text{div} (\det \mathbf{F}) \quad (\text{ix})$$

therefore,

$$\frac{D}{Dt} \int_{V_{m(t)}} \mathbf{T}(\mathbf{x}, t) dV = \int_{V_c} \left[ \frac{D\mathbf{T}}{Dt} + \mathbf{T} (\text{div} \mathbf{v}) \right] dV \quad (\text{x})$$

This is Eq. (7.4.2)

## 7.5 Principle of Conservation of Mass

The global principle of conservation of mass states that the total mass of a fixed part of material should remain constant at all times. That is

$$\frac{D}{Dt} \int_{V_m} \rho(\mathbf{x}, t) dV = 0 \quad (\text{7.5.1})$$

Using Reynolds Transport theorem (7.4.1), we obtain

$$\int_{V_c} \frac{\partial}{\partial t} \rho(\mathbf{x}, t) dV = - \int_{S_c} \rho(\mathbf{v} \cdot \mathbf{n}) dS \quad (\text{7.5.2a})$$

or,

$$\frac{\partial}{\partial t} \int_{V_c} \rho(\mathbf{x}, t) dV = - \int_{S_c} \rho(\mathbf{v} \cdot \mathbf{n}) dS \quad (7.5.2b)$$

This equation states that *the time rate at which mass is increasing inside a control volume = the mass influx (i.e., net rate of mass inflow) through the control surface.*

Substituting  $\rho$  for  $T$  in Eq. (7.4.3), we obtain from Eq. (7.5.2b)

$$\int_{V_c} \left[ \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) \right] dV = 0 \quad (7.5.3)$$

This equation is to be valid for all  $V_c$ , therefore, we must have

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0 \quad (7.5.4a)$$

This equation can also be written as

$$\frac{D\rho}{Dt} + \rho \text{div} \mathbf{v} = 0 \quad (7.5.4b)$$

This is the equation of continuity derived in Section 3.15.

#### Example 7.5.1

Given the motion

$$x_1 = (1+t)X_1, \quad x_2 = X_2, \quad x_3 = X_3 \quad (i)$$

and the density field

$$\rho = \frac{\rho_0}{1+t} \quad (\rho_0 = \text{constant}) \quad (ii)$$

- (a) Obtain the velocity field.
- (b) Check that the equation of continuity is satisfied.
- (c) Compute the total mass and the rate of increase of mass inside a cylindrical control volume of cross-sectional area  $A$  and having as its end faces the plane  $x_1 = 1$  and  $x_1 = 3$ .
- (d) Compute the net rate of inflow of mass into the control volume of part(c).
- (e) Find the total mass at time  $t$  of the material which at the reference time ( $t = 0$ ) was in the control volume of (c).
- (f) Compute the total linear momentum for the fixed part of material considered in part (e)

*Solution.* (a)

$$v_1 = \frac{Dx_1}{Dt} = X_1 = \frac{x_1}{1+t}, \quad v_2 = 0, \quad v_3 = 0 \quad (\text{iii})$$

(b)

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = \frac{\partial \rho}{\partial t} + v_1 \frac{\partial \rho}{\partial x_1} + \rho \frac{\partial v_1}{\partial x_1} = -\frac{\rho_0}{(1+t)^2} + \frac{x_1}{(1+t)}(0) + \frac{\rho_0}{(1+t)^2} = 0 \quad (\text{iv})$$

Thus, the equation of continuity is satisfied.

(c) The total mass inside the control volume at time  $t$  is

$$m(t) = \int_{V_c} \rho(x,t) dV = \int_{x_1=1}^{x_1=3} \rho(x,t) dV = \int_{x_1=1}^{x_1=3} \frac{\rho_0}{1+t} A dx_1 = \frac{2A\rho_0}{1+t} \quad (\text{v})$$

and the rate at which the mass is increasing inside the control volume at time  $t$  is

$$\frac{\partial m}{\partial t} = -\frac{2A\rho_0}{(1+t)^2} \quad (\text{vi})$$

i.e., the mass is decreasing.

(d) Since  $v_2 = v_3 = 0$ , there is neither inflow nor outflow through the lateral surface of the control volume. Through the end face  $x_1 = 1$ , the rate of inflow (mass influx) is

$$(\rho Av)_{x_1=1} = \frac{\rho_0 A}{(1+t)^2} \quad (\text{vii})$$

On the other hand, the mass outflux through the end face  $x_1 = 3$ , is

$$(\rho Av)_{x_1=3} = \frac{3\rho_0 A}{(1+t)^2}. \quad (\text{viii})$$

Thus, the net mass influx is

$$-\frac{2\rho_0 A}{(1+t)^2} \quad (\text{ix})$$

which is the same as Eq. (vi).

(e) The particles which were at  $x_1 = 1$  and  $x_1 = 3$  when  $t = 0$  have the material coordinate  $X_1 = 1$  and  $X_1 = 3$  respectively. Thus, the total mass at time  $t$  is

$$M = \int_{x_1=(1+t)}^{x_1=3(1+t)} \frac{\rho_0}{1+t} A dx_1 = \frac{\rho_0 A}{1+t} [3(1+t) - (1+t)] = 2\rho_0 A \quad (\text{x})$$

We see that this time-dependent integral turns out to be independent of time. This is because the chosen density and velocity field satisfy the equation of continuity so that, the total mass of a fixed part of material is a constant.

(f) Total linear momentum is, since  $v_2 = v_3 = 0$ ,

$$\begin{aligned} \mathbf{P} &= \int_{(1+t)}^{3(1+t)} \rho v_1 A dx_1 \mathbf{e}_1 = \frac{A\rho_0}{(1+t)^2} \int_{1+t}^{3(1+t)} x_1 dx_1 \mathbf{e}_1 \\ &= \frac{A\rho_0}{(1+t)^2} \left[ \frac{9(1+t)^2}{2} - \frac{(1+t)^2}{2} \right] \mathbf{e}_1 = 4A\rho_0 \mathbf{e}_1 \end{aligned} \tag{xi}$$

The fact that  $\mathbf{P}$  is also a constant is accidental. The given motion happens to be acceleration-less, which corresponds to no net force acting on the material volume. In general, the linear momentum for a fixed part of material is a function of time.

### 7.6 Principle of Linear Momentum

The **global principle of linear momentum** states that the total force (surface and body forces) acting on any fixed part of material is equal to the rate of change of linear momentum of the part. That is, with  $\rho$  denoting density,  $\mathbf{v}$  velocity,  $\mathbf{t}$  stress vector, and  $\mathbf{B}$  body force per unit mass, the principle states

$$\frac{D}{Dt} \int_{V_m} \rho \mathbf{v} dV = \int_{S_c} \mathbf{t} dS + \int_{V_c} \rho \mathbf{B} dV \tag{7.6.1}$$

Now, by using Reynolds Transport Theorem, Eq. (7.4.1), Eq. (7.6.1) can be written as

$$\int_{S_c} \mathbf{t} dS + \int_{V_c} \rho \mathbf{B} dV = \int_{V_c} \frac{\partial \rho \mathbf{v}}{\partial t} dV + \int_{S_c} \rho \mathbf{v} (\mathbf{v} \cdot \mathbf{n}) dS \tag{7.6.2}$$

In words, Eq. (7.6.2) states that

**Total force exerted on a fixed part of a material instantaneously in a control volume  $V_c$  = time rate of change of total linear momentum inside the control volume + net outflux of linear momentum through the control surface  $S_c$ .**

Equation (7.6.2) is very useful for obtaining approximate results in many engineering problems.

Using Eq. (7.4.2) (with  $\mathbf{T}$  replaced by  $\rho \mathbf{v}$ ), Eq. (7.6.1) can also be written as

$$\int_{V_c} \frac{D}{Dt} (\rho \mathbf{v}) dV + \int_{V_c} \rho \mathbf{v} \operatorname{div} \mathbf{v} dV = \int_{S_c} \mathbf{t} dS + \int_{V_c} \rho \mathbf{B} dV \tag{i}$$

But

$$\frac{D}{Dt} \rho \mathbf{v} = \frac{D\rho}{Dt} \mathbf{v} + \rho \frac{D\mathbf{v}}{Dt} \quad (\text{ii})$$

and

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = 0 \quad (\text{iii})$$

Therefore, Eq. (i) becomes

$$\int_{V_c} \rho \frac{D\mathbf{v}}{Dt} dV = \int_{S_c} \mathbf{t} dS + \int_{V_c} \rho \mathbf{B} dV \quad (7.6.3)$$

Since

$$\int_{V_c} \mathbf{t} dS = \int_{V_c} \mathbf{T} \mathbf{n} dS = \int_{V_c} \operatorname{div} \mathbf{T} dV \quad (\text{iv})$$

therefore, we have

$$\int_{V_c} \left( \rho \frac{D\mathbf{v}}{Dt} - \operatorname{div} \mathbf{T} - \rho \mathbf{B} \right) dV = 0 \quad (7.6.4)$$

from which the following field equation of motion is obtained:

$$\rho \frac{D\mathbf{v}}{Dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{B} \quad (7.6.5)$$

This is the same equation as Eq. (4.7.2).

We can also obtain the equation of motion in the reference state as follows:

Let  $\rho_o$ ,  $dS_o$ , and  $dV_o$  denote the density, surface area and volume respectively at the reference time  $t_o$  for the differential material having  $\rho$ ,  $dS$  and  $dV$  at time  $t$ , then the conservation of mass principle gives

$$\rho_o dV_o = \rho dV \quad (7.6.6)$$

and the definition of the stress vector  $\mathbf{t}_o$  associated with the first Piola-Kirchhoff stress tensor gives [see Section 4.10]

$$\mathbf{t}_o dS_o = \mathbf{t} dS \quad (7.6.7)$$

Now, using Eqs. (7.6.6) and (7.6.7), Equation (7.6.3) can be transformed to the reference configuration. That is

$$\int_{V_o} \rho_o \frac{D\mathbf{v}}{Dt} dV_o = \int_{S_o} \mathbf{t}_o dS_o + \int_{V_o} \rho_o \mathbf{B} dV_o = \int_{S_o} \mathbf{T}_o \mathbf{n}_o dS_o + \int_{V_o} \rho_o \mathbf{B} dV_o \quad (7.6.8)$$

In the above equation, everything is a function of the material coordinates  $X_i$  and  $t$ ,  $\mathbf{T}_o$  is the first Piola-Kirchhoff stress tensor and  $\mathbf{n}_o$  is the unit outward normal. Using the divergence theorem for the stress vector term, Eq. (7.6.8) becomes

$$\int_{V_o} \rho_o \frac{D\mathbf{v}}{Dt} dV_o = \int_{V_o} \text{Div} \mathbf{T}_o dV_o + \int_{V_o} \rho_o \mathbf{B} dV_o \quad (7.6.9)$$

where in Cartesian coordinates,  $\text{Div} \mathbf{T}_o = \frac{\partial T_{ij}}{\partial X_j}$ .

From Eq. (7.6.9), we obtain

$$\rho_o \frac{D\mathbf{v}}{Dt} = \text{Div} \mathbf{T}_o + \rho_o \mathbf{B} \quad (7.6.10)$$

This is the same equation derived in Chapter 4, Eq. (4.11.6).

#### Example 7.6.1

A homogeneous rope of total length  $l$  and total mass  $m$  slides down from the corner of a smooth table. Find the motion of the rope and the tension at the corner.

*Solution.* Let  $x$  denote the portion of rope that has slid down the corner at time  $t$ . Then the portion that remains on the table at time  $t$  is  $l-x$ . Consider the control volume shown as  $(V_c)_1$  in Figure 7.3. Then the momentum in the horizontal direction inside the control volume at any time  $t$  is, with  $\dot{x}$  denoting  $dx/dt$ :

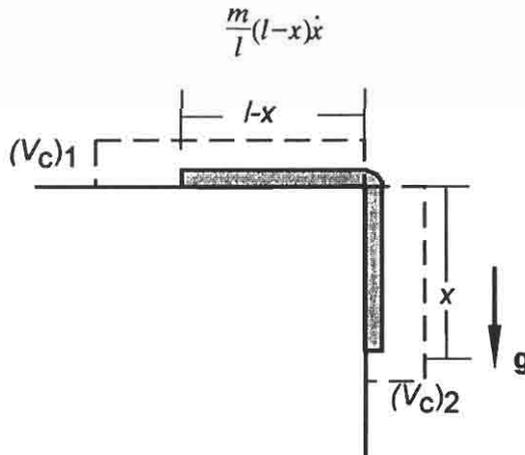


Fig. 7.3

and the net momentum outflux is

$$\left[ \frac{m \dot{x}}{l} \right] \dot{x}$$

Thus, if  $T$  denotes the tension at the corner point of the rope at time  $t$ , we have

$$T = \frac{d}{dt} \left[ \frac{m}{l} (l-x) \dot{x} \right] + \frac{m \dot{x}^2}{l} = \frac{m}{l} (-\dot{x}) \dot{x} + \frac{m}{l} (l-x) \ddot{x} + \frac{m \dot{x}^2}{l} \tag{i}$$

i.e.,

$$T = \frac{m}{l} (l-x) \ddot{x} \tag{ii}$$

as expected.

On the other hand, by considering the control volume  $(V_c)_2$  (see Fig. 7.3), we have, the momentum in the downward direction is  $(m/l)x\dot{x}$  and the momentum influx in the same direction is  $[(m/l)\dot{x}]\dot{x}$ . Thus,

$$-T + \left( \frac{m}{l} x \right) g = \frac{d}{dt} \left( \frac{m}{l} x \dot{x} \right) - \frac{m \dot{x}^2}{l} \tag{iii}$$

i.e.,

$$-T + \frac{m}{l} x g = \frac{m}{l} x \ddot{x} \tag{iv}$$

From Eqs. (ii) and (iv), we have

$$\frac{m}{l} (l-x) \ddot{x} = \frac{m}{l} x g - \frac{m}{l} x \ddot{x} \tag{v}$$

i.e.,

$$\ddot{x} - \frac{g}{l} x = 0 \tag{vi}$$

The general solution of Eq. (vi) is

$$x = C_1 \exp[\sqrt{g/l}t] + C_2 \exp[-\sqrt{g/l}t] \tag{vii}$$

Thus, if the rope starts from rest with an initial overhang of  $x_0$ , we have

$$x_0 = C_1 + C_2 \text{ and } 0 = C_1 - C_2 \tag{viii}$$

so that  $C_1 = C_2 = x_0/2$  and the solution is

$$x = \frac{x_0}{2} [\exp(\sqrt{g/l}t) + \exp(-\sqrt{g/l}t)] \tag{ix}$$

The tension at the corner is given by

$$T = \frac{m}{l}(l-x)\ddot{x} = \frac{m}{l}(l-x)\left(\frac{g}{l}x\right) \tag{x}$$

We note that the motion can also be obtained by considering the whole rope as a system. In fact, the total linear momentum of the rope at any time  $t$  is

$$\frac{m}{l}(l-x)\dot{x}\mathbf{e}_1 + \frac{m}{l}x\dot{x}\mathbf{e}_2 \tag{xi}$$

its rate of change is

$$\frac{m}{l}[(l-x)\ddot{x} - \dot{x}^2]\mathbf{e}_1 + \frac{m}{l}(x\ddot{x} + \dot{x}^2)\mathbf{e}_2 \tag{xii}$$

and the total resultant force on the rope is

$$\frac{m}{l}xg\mathbf{e}_2 \tag{xiii}$$

Thus, equating the force to the rate of change of momentum for the whole rope, we obtain

$$(l-x)\ddot{x} - \dot{x}^2 = 0 \tag{xiv}$$

and

$$x\ddot{x} + \dot{x}^2 = gx \tag{xv}$$

Eliminating  $\dot{x}^2$  from the above two equations, we arrive at Eq. (vi) again.

### Example 7.6.2

Figure 7.4 shows a steady jet of water impinging onto a curved vane in a tangential direction. Neglect the effect of weight and assume that the flow at the upstream region, section  $A$ , as well as at the downstream region, section  $B$  is a parallel flow with a uniform speed  $v_0$ . Find the resultant force (above that due to the atmospheric pressure) exerted on the vane by the jet. The volume flow rate is  $Q$ .

*Solution.* Let us take as control volume that portion of the jet bounded by the planes at  $A$  and  $B$ . Since the flow at  $A$  is assumed to be a parallel flow, therefore the stress vector on the plane  $A$  is normal to the plane with a magnitude equal to the atmospheric pressure which we take to be zero. [We recall that in the absence of gravity, the pressure is a constant along any direction which is perpendicular to the direction of a parallel flow (See Section 6.7)]. Thus, the only forces acting on the material in the control volume is that from the vane to the jet. Let  $\mathbf{F}$  be the resultant of these forces. Since the flow is steady, the rate of increase of momentum

inside the control volume is zero. The rate of outflow of linear momentum across  $B$  is  $\rho Q v_o (\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2)$  and the rate of inflow of linear momentum across  $A$  is  $\rho Q v_o \mathbf{e}_1$ . Thus

$$\mathbf{F} = \rho Q [v_o (\cos\theta - 1) \mathbf{e}_1 + v_o \sin\theta \mathbf{e}_2] \tag{i}$$

$$F_x = -\rho Q v_o (1 - \cos\theta) \tag{ia}$$

$$F_y = \rho Q v_o \sin\theta \tag{iib}$$

and the force components on the vane by the jet are equal and opposite to  $F_x$  and  $F_y$ .

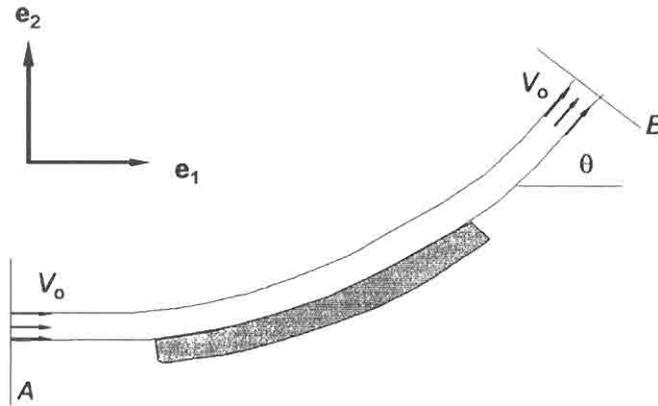


Fig. 7.4

Example 7.6.3

For boundary layer flow of water over a flat plate, if the velocity profile and that of the horizontal components at the leading and the trailing edges of the plate respectively are assumed to be those shown in Fig. 7.5, find the shear force acting on the fluid by the plate. Assume that the flow is steady and that the pressure is uniform in the whole flow.

*Solution.* Consider the control volume  $ABCD$ . Since the pressure is assumed to be uniform and since the flow outside of the boundary layer  $\delta$  is essentially uniform in horizontal velocity component in  $x$  direction with very small vertical velocity components (so that the shearing stress on  $BC$  is negligible), therefore, the net force on the control volume is the shearing force from the plate. Denoting this force (per unit width in  $z$  direction) by  $F \mathbf{e}_1$ , we have from the momentum principle, Eq. (7.6.2)

$$F = \text{net out flux of } x\text{-momentum through } ABCD$$

i.e.,

$$F = \int_{S_c} v_1(\rho \mathbf{v} \cdot \mathbf{n}) dS = - \int_0^\delta \bar{u}(\rho \bar{u}) dy + \int_{BC} \bar{u}(\rho v_2) dS + \int_0^\delta \left(\frac{\bar{u}y}{\delta}\right) \rho \left(\frac{\bar{u}y}{\delta}\right) dy + \int_{AD} (0) dS \tag{i}$$

where  $\bar{u}$  denotes the uniform horizontal velocity of the upstream flow and the uniform component of velocity at the trailing edge,  $v_1$ , and  $v_2$  are the velocity components of the fluid particles on the surface  $S_c$  and  $\delta$  is the thickness of the boundary layer. Thus,

$$F = -\rho \bar{u}^2 \delta + \frac{\rho \bar{u}^2 \delta}{3} + \bar{u} \int_{BC} (\rho v_2) dS \tag{ii}$$

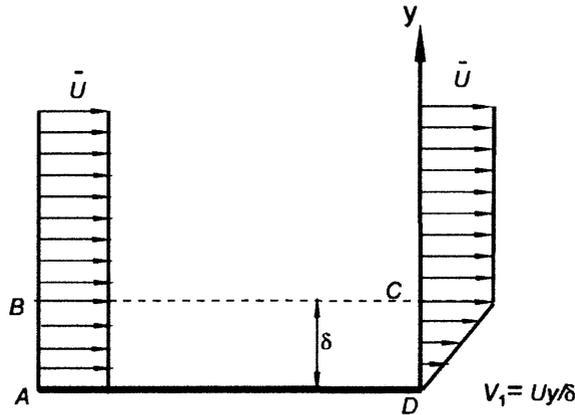


Fig. 7.5

From the principle of conservation of mass, we have

$$\int_{BC} (\rho v_2) dS - \int_0^\delta \rho \bar{u} dy + \int_0^\delta \rho \frac{\bar{u}}{\delta} y dy = 0 \tag{iii}$$

i.e.,

$$\int_{BC} (\rho v_2) dS = \rho \bar{u} \delta - \frac{\rho \bar{u} \delta}{2} = \frac{\rho \bar{u} \delta}{2} \tag{iv}$$

Thus,

$$F = -\rho \bar{u}^2 \delta + \frac{\rho \bar{u}^2 \delta}{3} + \frac{\rho \bar{u}^2 \delta}{2} = -\frac{\rho \bar{u}^2 \delta}{6} \tag{v}$$

That is, the force per unit width on the fluid by the plate is acting to the left with a magnitude of  $\frac{\rho \bar{u}^2 \delta}{6}$ .

### 7.7 Moving Frames

There are certain problems, for which the use of a control volume fixed with respect to a frame moving relative to an inertial frame, is advantageous. For this purpose, we derive the momentum principle valid for a frame moving relative to an inertial frame.

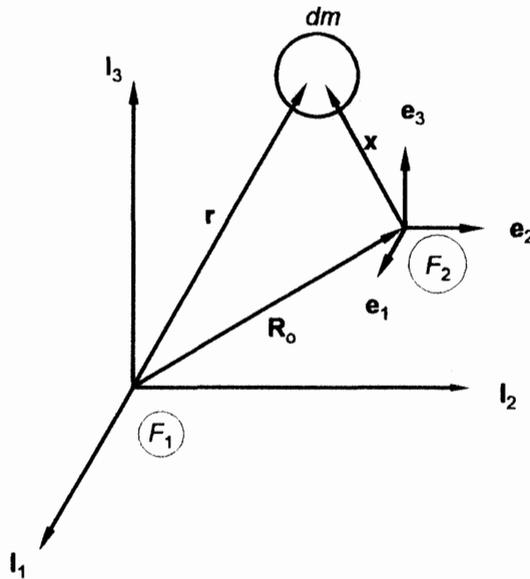


Fig. 7.6

Let \$F\_1\$ and \$F\_2\$ be two frames of references. Let \$\mathbf{r}\$ denote the position vector of a differential mass \$dm\$ in a continuum relative to \$F\_1\$ and let \$\mathbf{x}\$ denote the position vector relative to \$F\_2\$ (see Fig. 7.6). Then the velocity of \$dm\$ relative to \$F\_1\$ is

$$\left(\frac{d\mathbf{r}}{dt}\right)_{F_1} \equiv \mathbf{v}_{F_1} \tag{7.1}$$

and the velocity relative to \$F\_2\$ is

$$\left(\frac{dx}{dt}\right)_{F_2} \equiv v_{F_2} \tag{7.7.2}$$

Since

$$\mathbf{r} = \mathbf{R}_o + \mathbf{x} \tag{7.7.3}$$

thus,

$$\left(\frac{d\mathbf{r}}{dt}\right)_{F_1} = \left(\frac{d\mathbf{R}_o}{dt}\right)_{F_1} + \left(\frac{d\mathbf{x}}{dt}\right)_{F_1} \tag{i}$$

i.e.,

$$v_{F_1} = (v_o)_{F_1} + \left(\frac{d\mathbf{x}}{dt}\right)_{F_1} \tag{ii}$$

But, from a course in rigid body dynamics, we learned that for any vector  $\mathbf{b}$ ,

$$\left(\frac{d\mathbf{b}}{dt}\right)_{F_1} = \left(\frac{d\mathbf{b}}{dt}\right)_{F_2} + \boldsymbol{\omega} \times \mathbf{b} \tag{iii}$$

Where  $\boldsymbol{\omega}$  is the angular velocity of  $F_2$  relative to  $F_1$ . Thus,

$$\left(\frac{d\mathbf{x}}{dt}\right)_{F_1} = \left(\frac{d\mathbf{x}}{dt}\right)_{F_2} + \boldsymbol{\omega} \times \mathbf{x} = (v)_{F_2} + \boldsymbol{\omega} \times \mathbf{x} \tag{iv}$$

Therefore,

$$v_{F_1} = (v_o)_{F_1} + v_{F_2} + \boldsymbol{\omega} \times \mathbf{x} \tag{7.7.4}$$

Now, the linear momentum relative to  $F_1$  is  $\int v_{F_1} dm$  and that relative to  $F_2$  is  $\int v_{F_2} dm$ . These rates of change of linear momentum are related in the following way: (for simplicity, we drop the subscript of the integral  $V_m$ )

$$\begin{aligned} \left(\frac{D}{Dt}\right)_{F_1} \int v_{F_1} dm &= \left(\frac{D}{Dt}\right)_{F_1} [ (v_o)_{F_1} \int dm + \int v_{F_2} dm + \boldsymbol{\omega} \times \int \mathbf{x} dm ] \\ &= (\mathbf{a}_o)_{F_1} \int dm + \left(\frac{D}{Dt}\right)_{F_1} \int v_{F_2} dm + \left(\frac{D}{Dt}\right)_{F_1} (\boldsymbol{\omega} \times \int \mathbf{x} dm) \end{aligned} \tag{7.7.5}$$

Now, again, using Eq. (iii) for the vector  $\int v_{F_2} dm$ , we have

$$\left(\frac{D}{Dt}\right)_{F_1} \int v_{F_2} dm = \left(\frac{D}{Dt}\right)_{F_2} \int v_{F_2} dm + \boldsymbol{\omega} \times \int v_{F_2} dm \tag{v}$$

and

$$\begin{aligned} \left(\frac{D}{Dt}\right)_{F_1} (\boldsymbol{\omega} \times \int \mathbf{x} dm) &= \dot{\boldsymbol{\omega}} \times \int \mathbf{x} dm + \boldsymbol{\omega} \times \left(\frac{D}{Dt}\right)_{F_1} \left(\int \mathbf{x} dm\right) \\ &= \dot{\boldsymbol{\omega}} \times \int \mathbf{x} dm + \boldsymbol{\omega} \times \int \mathbf{v}_{F_2} dm + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int \mathbf{x} dm) \end{aligned} \quad (vi)$$

Thus,

$$\begin{aligned} \left(\frac{D}{Dt}\right)_{F_1} \left(\int \mathbf{v}_{F_1} dm\right) &= \left(\frac{D}{Dt}\right)_{F_2} \int \mathbf{v}_{F_2} dm + (\mathbf{a}_o)_{F_1} \int dm + 2\boldsymbol{\omega} \times \int \mathbf{v}_{F_2} dm \\ &\quad + \dot{\boldsymbol{\omega}} \times \int \mathbf{x} dm + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int \mathbf{x} dm) \end{aligned} \quad (7.7.6)$$

Now, let  $F_1$  be an inertial frame so that the momentum principle reads

$$\left(\frac{D}{Dt}\right)_{F_1} \int \mathbf{v}_{F_1} dm = \int \mathbf{t} dS + \int \rho \mathbf{B} dV \quad (7.7.7)$$

Using Eq. (7.7.6), the momentum principle [Eq. (7.7.7)] becomes

$$\begin{aligned} \left(\frac{D}{Dt}\right)_{F_2} \int_{V_m} \mathbf{v}_{F_2} dm &= \int_{S_c} \mathbf{t} dS + \int_{V_c} \rho \mathbf{B} dV + \\ &[-m\mathbf{a}_o - 2\boldsymbol{\omega} \times \int \mathbf{v}_{F_2} dm - \dot{\boldsymbol{\omega}} \times \int \mathbf{x} dm - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int \mathbf{x} dm)] \end{aligned} \quad (7.7.8)$$

where  $m = \int dm$ .

Equation (7.7.8) shows that when a moving frame is used to compute momentum and its time rate of change, the same momentum principle for an inertial frame can be used provided we add those terms given inside the bracket in the right-hand side of Eq. (7.7.8) to the surface and body force terms.

## 7.8 Control Volume Fixed with respect to a Moving Frame

If a control volume is chosen to be fixed with respect to a frame of reference which moves relative to an inertial frame with an acceleration  $\mathbf{a}_o$ , an angular velocity  $\boldsymbol{\omega}$  and angular acceleration  $\dot{\boldsymbol{\omega}}$ , the momentum equation is given by Eq. (7.7.8). If we now use the Reynold's transport theorem for the left-hand side of Eq. (7.7.8), we obtain

$$\begin{aligned} \int_{V_c} \frac{\partial}{\partial t} (\rho \mathbf{v}_{F_2}) dV + \int_{S_c} \rho \mathbf{v}_{F_2} (\mathbf{v}_{F_2} \cdot \mathbf{n}) dS &= \int_{S_c} \mathbf{t} dS + \int_{V_c} \rho \mathbf{B} dV + \\ &[-m\mathbf{a}_o - 2\boldsymbol{\omega} \times \int \mathbf{v}_{F_2} dm - \dot{\boldsymbol{\omega}} \times \int \mathbf{x} dm - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int \mathbf{x} dm)] \end{aligned} \quad (7.8.1)$$

In particular, if the control volume has only translation (acceleration =  $\mathbf{a}_o$ ) with respect to the inertial frame and no rotations, then we have

$$\int_{V_c} \frac{\partial}{\partial t} (\rho \mathbf{v}_{F_2}) dV + \int_{S_c} \rho \mathbf{v}_{F_2} (\mathbf{v}_{F_2} \cdot \mathbf{n}) dS = \int_{S_c} \mathbf{t} dS + \int_{V_c} \rho \mathbf{B} dV - m \mathbf{a}_o \quad (7.8.2)$$

Example 7.8.1

A rocket of initial total mass  $M_o$  moves upward while ejecting a jet of gases at the rate of  $\gamma$  unit of mass per unit time. The exhaust velocity of the jet relative to the rocket is  $v_r$  and the gage pressure in the jet of area  $A$  is  $p$ . Derive the differential equation governing the motion of the rocket and find the velocity as a function of time. Neglect drag forces.

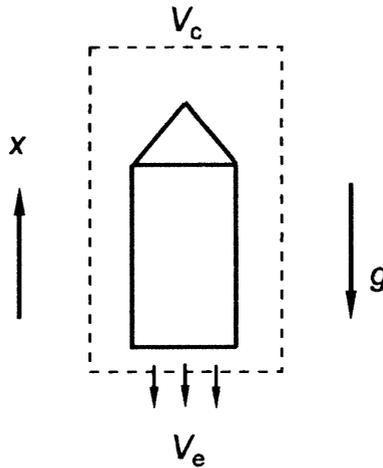


Fig. 7.7

*Solution.* Let  $V_r$  be a control volume which moves upward with the rocket. Then relative to  $V_r$  the net  $x$  momentum outflux is  $-\gamma v_e$ . The motion of gases due to internal combustion does not produce any net momentum change relative to the rocket, therefore, there is no rate of change of momentum inside the control volume. The net surface force is an upward force of  $pA$  and the body force is  $(M_o - \gamma t)g$  downward. However, since the control volume is moving with the rocket which has an acceleration  $\ddot{x}$ , therefore, the term  $\ddot{x}(M - \gamma t)$  is to be added to the other force terms [See Eq. (7.8.2)]. Thus,

$$-\gamma v_e = pA - (M_o - \gamma t)g - (M_o - \gamma t)\ddot{x} \quad (i)$$

i.e.,

$$(M_o - \gamma t)x'' = \gamma v_e + pA - (M_o - \gamma t)g \tag{ii}$$

This equation can be written

$$dx' = \frac{\gamma v_e + pA}{M_o - \gamma t} dt - g dt \tag{ii}$$

If at  $t = 0$ ,  $\dot{x} = 0$ , then

$$\dot{x} = \frac{\gamma v_e + pA}{\gamma} \ln \left( \frac{M_o}{M_o - \gamma t} \right) - gt \tag{iii}.$$

### 7.9 Principle of Moment of Momentum

The global principle of moment of momentum states that the total moment about a fixed point, of surface and body forces on a fixed part of material is equal to the time rate of change of total moment of momentum of the part about the same point. That is,

$$\frac{D}{Dt} \int_{V_m} \mathbf{x} \times \rho \mathbf{v} dV = \int_{S_c} (\mathbf{x} \times \mathbf{t}) dS + \int_{V_c} \mathbf{x} \times \rho \mathbf{B} dV \tag{7.9.1}$$

where  $\mathbf{x}$  is the position vector for a particle.

Using the Reynold's transport theorem, Eq. (7.4.2), the left side of Eq. (7.9.1) becomes

$$\frac{D}{Dt} \int_{V_m} \mathbf{x} \times \rho \mathbf{v} dV = \int_{V_c} \left[ \frac{D}{Dt} (\mathbf{x} \times \rho \mathbf{v}) + (\mathbf{x} \times \rho \mathbf{v})(\text{div } \mathbf{v}) \right] dV \tag{i}$$

Since

$$\frac{D}{Dt} (\mathbf{x} \times \rho \mathbf{v}) = \mathbf{v} \times \rho \mathbf{v} + \mathbf{x} \times \frac{D\rho}{Dt} \mathbf{v} + \mathbf{x} \times \rho \frac{D\mathbf{v}}{Dt} = \mathbf{x} \times \frac{D\rho}{Dt} \mathbf{v} + \mathbf{x} \times \rho \frac{D\mathbf{v}}{Dt} \tag{ii}$$

therefore the integrand in Eq. (i) becomes

$$\mathbf{x} \times \left( \frac{D\rho}{Dt} + \rho \text{div } \mathbf{v} \right) \mathbf{v} + \mathbf{x} \times \rho \frac{D\mathbf{v}}{Dt} = \mathbf{x} \times \rho \frac{D\mathbf{v}}{Dt} \tag{iii}$$

Thus

$$\frac{D}{Dt} \int_{V_m} \mathbf{x} \times \rho \mathbf{v} dV = \int_{V_c} \left[ \mathbf{x} \times \rho \frac{D\mathbf{v}}{Dt} \right] dV \tag{iv}$$

Also, from Example 7.2.2 of this chapter, we have

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$$\int_{S_c} (\mathbf{x} \times \mathbf{t}) dS = \int_{V_c} [\mathbf{x} \times (\text{div} \mathbf{T}) + 2 \boldsymbol{\tau}^A] dV \quad (v)$$

Therefore, Eq. (7.9.1) becomes

$$\mathbf{x} \times \int_{V_c} [\rho \frac{D\mathbf{v}}{Dt} - \text{div} \mathbf{T} - \rho \mathbf{B}] dV - 2 \int_{V_c} \boldsymbol{\tau}^A dV = 0 \quad (7.9.2)$$

where  $\boldsymbol{\tau}^A$  is the axial vector of the antisymmetric part of the stress tensor  $\mathbf{T}$ . Now the first term in Eq. (7.9.2) vanishes because of Eq. (7.6.5), therefore,  $\boldsymbol{\tau}^A = \mathbf{0}$  and the symmetry of the stress tensor

$$\mathbf{T} = \mathbf{T}^T \quad (7.9.4)$$

is obtained.

On the other hand, if we use the Reynold's transport theorem, Eq. (7.4.1), for the left side of Eq. (7.9.1), we obtain

$$\int_{S_c} \mathbf{x} \times \mathbf{t} dS + \int_{V_c} \mathbf{x} \times \rho \mathbf{B} dV = \int_{V_c} \frac{\partial}{\partial t} (\mathbf{x} \times \rho \mathbf{v}) dV + \int_{S_c} (\mathbf{x} \times \rho \mathbf{v}) (\mathbf{v} \cdot \mathbf{n}) dS \quad (7.9.4)$$

**That is, the total moment about a fixed point due to surface and body forces acting on the material instantaneously inside a control volume = total rate of change of moment of momenta inside the control volume + total net rate of outflow of moment of momenta across the control surface**

If the control volume is fixed in a moving frame, then the following terms should be added to the left side of Eq. (7.9.4)

$$- (\int \mathbf{x} dm) \times \mathbf{a}_o - \int \mathbf{x} \times (\dot{\boldsymbol{\omega}} \times \mathbf{x}) dm - \int \mathbf{x} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})] dm - 2 \int \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{v}) dm \quad (7.9.5)$$

where  $\boldsymbol{\omega}$  and  $\dot{\boldsymbol{\omega}}$  are absolute angular velocity and acceleration of the moving frame (and of the control volume), the vector  $\mathbf{x}$  of  $(dm)$  is measured from the arbitrary chosen point  $O$  in the control volume,  $\mathbf{a}_o$  is the absolute acceleration of point  $O$  and  $\mathbf{v}$  is the velocity of  $(dm)$  relative to the control volume.

Example 7.9.1

Each sprinkler arm in Fig. 7.8 discharges a constant volume of water  $Q$  and is free to rotate around the vertical center axis. Determine its constant speed of rotation.

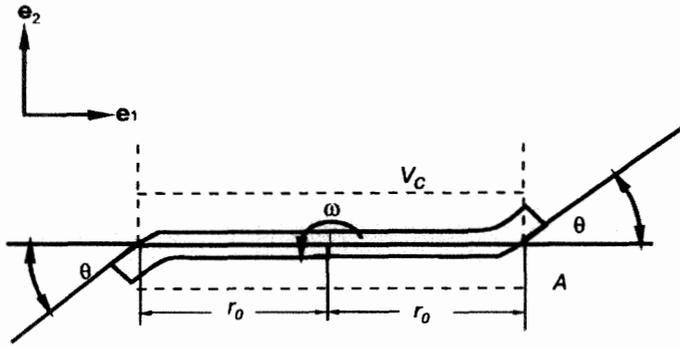


Fig. 7.8

*Solution.* Let  $V_c$  be a control volume that rotates with the sprinkler arms. The velocity of water particles relative to the sprinkler is  $(Q/A)\mathbf{e}_1$  inside the right arm and  $(Q/A)(-\mathbf{e}_1)$  inside the left arm. If  $\rho$  is density, then the total net outflux of moment of momentum about point  $O$  is  $2\rho Q(Q/A)\sin\theta r_0\mathbf{e}_3$ . The moment of momentum about  $O$  due to weight is zero. Since the pressure in the water jet is the same as the atmospheric pressure, taken to be zero gage pressure, there is no contribution due to surface force on the control volume. Now, since the control volume is rotating with the sprinkler, therefore, we need to add those terms given in Eq. (7.9.5) to the moment of forces. With  $\mathbf{x}$  measured from  $O$ , the first term of Eq. (7.9.5) is zero, with  $\boldsymbol{\omega}$  a constant, the second term of Eq. (7.9.5) is zero, with  $\mathbf{x} = x\mathbf{e}_1$  and  $\boldsymbol{\omega} = \omega\mathbf{e}_3$ , the third term of Eq. (7.9.5) is zero. Thus, the only nonzero term is

$$-2\int \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{v}) dm \quad (\text{i})$$

which is the moment due to the Coriolis forces. Now, for the right arm,  $\mathbf{v} = (Q/A)\mathbf{e}_1$ , therefore,

$$\boldsymbol{\omega} \times \mathbf{v} = \omega\mathbf{e}_3 \times \frac{Q}{A}\mathbf{e}_1 = \frac{\omega Q}{A}\mathbf{e}_2 \quad (\text{ii})$$

and

$$\mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{v}) = x\mathbf{e}_1 \times \frac{\omega Q}{A}\mathbf{e}_2 = \frac{x\omega Q}{A}\mathbf{e}_3 \quad (\text{iii})$$

Thus, the contribution from the fluid in the right arm to the integral in Eq. (i) is

$$-\frac{2\omega Q}{A}\mathbf{e}_3 \int_0^{r_0} x(\rho A dx) = -\omega Q \rho r_0^2 \mathbf{e}_3 \quad (\text{iv})$$

Including that due to the left arm, the integral has the value of  $-2\omega Q\rho r_o^2\mathbf{e}_3$ . Therefore, from the moment of momentum principle for a moving control volume, we have

$$2\rho Q\left(\frac{Q}{A}\right)\sin\theta r_o = -2\omega Q\rho r_o^2 \tag{v}$$

from which

$$\omega = -\frac{Q\sin\theta}{A r_o} \tag{vi}$$

### 7.10 Principle of Conservation of Energy

The principle of conservation of energy states that the time rate of change of the kinetic energy and internal energy for a fixed part of material is equal to the sum of the rate of work done by the surface and body forces and the heat energy entering the boundary surface. That is, if  $v^2$  denotes  $\mathbf{v}\cdot\mathbf{v}$ ,  $u$  the internal energy per unit mass, and  $\mathbf{q}$  the rate of heat flow vector across a unit area, then the principle states:

$$\frac{D}{Dt} \int_{V_m} (\rho \frac{v^2}{2} + \rho u) dV = \int_{S_c} \mathbf{t}\cdot\mathbf{v} dS + \int_{V_m} \rho \mathbf{B}\cdot\mathbf{v} dV - \int_{S_c} \mathbf{q}\cdot\mathbf{n} dS \tag{7.10.1}$$

the minus sign in the last term is due to the convention that  $\mathbf{n}$  is an outward unit normal vector and therefore  $-\mathbf{q}\cdot\mathbf{n}$  represents inflow.

Again, using the Reynold's transport theorem Eq. (7.4.2), we have

$$\begin{aligned} \frac{D}{Dt} \int_{V_m} \rho (\frac{v^2}{2} + u) dV &= \int_{V_c} \left[ \frac{D}{Dt} \rho (\frac{v^2}{2} + u) + \rho (\frac{v^2}{2} + u) \text{div}\mathbf{v} \right] dV \\ &= \int_{V_c} \left[ \rho \frac{D}{Dt} (\frac{v^2}{2} + u) + (\frac{v^2}{2} + u) (\frac{D\rho}{Dt} + \rho \text{div}\mathbf{v}) \right] dV = \int_{V_c} \left[ \rho \frac{D}{Dt} (\frac{v^2}{2} + u) \right] dV \end{aligned} \tag{i}$$

In Example 7.2.3 we obtained that

$$\int_{S_c} \mathbf{t}\cdot\mathbf{v} dS = \int_{V_c} [(\text{div}\mathbf{T})\cdot\mathbf{v} + \text{tr}(\mathbf{T}^T\nabla\mathbf{v})] dV \tag{ii}$$

Also, the divergence theorem gives

$$\int_{S_c} \mathbf{q}\cdot\mathbf{n} dS = \int_{V_c} \text{div}\mathbf{q} dV \tag{iii}$$

Thus, using Eqs. (i)(ii) and (iii), Eq. (7.10.1) becomes

$$\int_{V_c} \rho \frac{D}{Dt} \left( \frac{v^2}{2} + u \right) dV = \int_{V_c} [(\text{div} \mathbf{T} + \rho \mathbf{B}) \cdot \mathbf{v} + \text{tr}(\mathbf{T}^T \nabla \mathbf{v}) - \text{div} \mathbf{q}] dV \quad (7.10.2)$$

Since

$$(\text{div} \mathbf{T} + \rho \mathbf{B}) \cdot \mathbf{v} = \rho \frac{D\mathbf{v}}{Dt} \cdot \mathbf{v} = \frac{1}{2} \rho \frac{Dv^2}{Dt} \quad (\text{iv})$$

Therefore, Eq. (7.10.2) becomes

$$\int_{V_c} \rho \frac{Du}{Dt} dV = \int_{V_c} [\text{tr}(\mathbf{T}^T \nabla \mathbf{v}) - \text{div} \mathbf{q}] dV \quad (\text{v})$$

Thus, at every point, we have

$$\rho \frac{Du}{Dt} = \text{tr}(\mathbf{T}^T \nabla \mathbf{v}) - \text{div} \mathbf{q} \quad (7.10.3a)$$

For a symmetry tensor  $\mathbf{T}$ , this equation can also be written

$$\rho \frac{Du}{Dt} = \text{tr}(\mathbf{T} \nabla \mathbf{v}) - \text{div} \mathbf{q} \quad (7.10.3b)$$

Equations (7.10.3a) or (7.10.3b) is the energy equation. A slightly different form of Eq. (7.10.3b) can be obtained if we recall that  $\nabla \mathbf{v} = \mathbf{D} + \mathbf{W}$ , where  $\mathbf{D}$ , the symmetric part of  $\nabla \mathbf{v}$  is the rate of deformation tensor, and  $\mathbf{W}$ , the antisymmetric part of  $\nabla \mathbf{v}$ , is the spin tensor. We have

$$\text{tr}(\mathbf{T} \nabla \mathbf{v}) = \text{tr}(\mathbf{T} \mathbf{D} + \mathbf{T} \mathbf{W}) = \text{tr}(\mathbf{T} \mathbf{D}) + \text{tr}(\mathbf{T} \mathbf{W}) \quad (\text{vi})$$

But  $\text{tr}(\mathbf{T} \mathbf{W}) = T_{ij} W_{ji} = T_{ji} W_{ji} = T_{ij} W_{ij} = -T_{ij} W_{ji}$ , so that

$$\text{tr}(\mathbf{T} \mathbf{W}) = 0 \quad (\text{vii})$$

therefore, we rediscover the energy equation in the following form:

$$\rho \frac{Du}{Dt} = \text{tr}(\mathbf{T} \mathbf{D}) - \text{div} \mathbf{q} \quad (7.10.4)$$

On the other hand, if we use the Reynold's theorem in the form of Eq. (7.4.1), we obtain

$$\int_{S_c} \mathbf{t} \cdot \mathbf{v} dS + \int_{V_m} \rho \mathbf{B} \cdot \mathbf{v} dV - \int_{S_c} \mathbf{q} \cdot \mathbf{n} dS = \int_{V_m} \frac{\partial}{\partial t} \rho \left( \frac{v^2}{2} + u \right) dV + \int_{V_m} \rho \left( \frac{v^2}{2} + u \right) (\mathbf{v} \cdot \mathbf{n}) dS \quad (10.5)$$

Equation (10.5) states that the time rate of work done by surface and body forces in a control volume + rate of heat input = total rate of increase of internal and kinetic energy of the material inside the control volume + rate of outflow of the internal and kinetic energy across the control surface

## Example 7.10.1

A supersonic one-dimensional flow in an insulating duct suffers a normal compression shock. Assuming ideal gas, find the pressure after the shock in terms of the pressure and velocity before the shock.

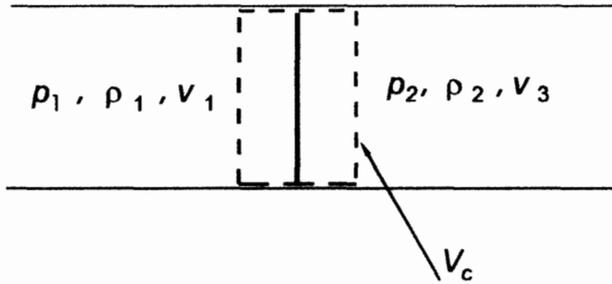


Fig. 7.9

*Solution* For the control volume shown, we have, for steady flow:

(1) Mass outflux = mass influx, that is

$$\rho_1 A v_1 = \rho_2 A v_2 \quad (\text{i})$$

i.e.,

$$\rho_1 v_1 = \rho_2 v_2 \quad (\text{ii})$$

(2) Force in  $x$  direction = net momentum outflux in  $x$  direction

$$p_1 A - p_2 A = (\rho_2 A v_2) v_2 - (\rho_1 A v_1) v_1 \quad (\text{iii})$$

i.e.,

$$(p_1 - p_2) = \rho_2 v_2^2 - \rho_1 v_1^2 \quad (\text{iv})$$

(3) Rate of work done by surface force = net energy (internal and kinetic) outflux. That is

$$p_1 A v_1 - p_2 A v_2 = [(\rho_2 A v_2) u_2 - (\rho_1 A v_1) u_1] + \left[ \frac{1}{2} (\rho_2 A v_2) v_2^2 - \frac{1}{2} (\rho_1 A v_1) v_1^2 \right] \quad (\text{v})$$

For ideal gas, the internal energy per unit mass is given by, [see Eq. (6.26.10)]

$$u = \frac{p}{\rho} \left( \frac{1}{\gamma - 1} \right) \quad (\text{vi})$$

where  $\gamma = c_p / c_v$  is the ratio of specific heats. Thus, Eq. (v) becomes

$$p_1 v_1 - p_2 v_2 = p_2 v_2 \left( \frac{1}{\gamma - 1} \right) - p_1 v_1 \left( \frac{1}{\gamma - 1} \right) + \frac{1}{2} \rho_2 v_2^3 - \frac{1}{2} \rho_1 v_1^3 \quad (\text{vii})$$

Or,

$$\frac{\gamma}{\gamma - 1} p_1 v_1 + \frac{1}{2} \rho_1 v_1^3 = \frac{\gamma}{\gamma - 1} p_2 v_2 + \frac{1}{2} \rho_2 v_2^3 \quad (\text{viii})$$

that is

$$\rho_1 v_1 \left[ \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + \frac{1}{2} v_1^2 \right] = \rho_2 v_2 \left[ \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} + \frac{1}{2} v_2^2 \right] \quad (\text{ix})$$

In view of Eq. (ii), this equation becomes

$$\frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} + \frac{1}{2} v_1^2 = \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} + \frac{1}{2} v_2^2 \quad (\text{x})$$

We note here that this is the same energy equation derived in Chapter 6 (Example 6.28.1) using differential forms of energy equation for an inviscid nonheat-conducting fluid. From Eqs. (ii)(iv) and (x) we obtain

$$p_2 = \frac{1}{\gamma + 1} [2\rho_1 v_1^2 - (\gamma - 1)p_1] \quad (\text{xi})$$

PROBLEMS

7.1. Verify the divergence theorem for the vector field  $\mathbf{v} = 2x\mathbf{e}_1 + z\mathbf{e}_2$ , by considering the region bounded by  $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$ .

7.2. Show that

$$\int_S \mathbf{x} \cdot \mathbf{n} dS = 3V$$

where  $V$  is the volume enclosed by the boundary  $S$ .

7.3. (a) Consider the vector field  $\mathbf{v} = \varphi \mathbf{a}$ , where  $\varphi$  is a given scalar field and  $\mathbf{a}$  is an arbitrary constant vector (independent of position). Using the divergence theorem, prove that

$$\int_V \nabla \varphi dV = \int_S \varphi \mathbf{n} dS$$

(b) Show that for any closed surface  $S$  that

$$\int_S \mathbf{n} dS = 0$$

7.4. A stress field  $\mathbf{T}$  is in equilibrium with a body force  $\rho \mathbf{B}$ . Using the divergence theorem, show that for any volume  $V$ , and boundary surface  $S$ , that

$$\int_S \mathbf{t} dS + \int_V \rho \mathbf{B} dV = 0$$

That is, the total resultant force is equipollent to zero.

7.5. Let  $\mathbf{u}^*$  define an infinitesimal strain field  $\mathbf{E}^* = \frac{1}{2}(\nabla \mathbf{u}^* + (\nabla \mathbf{u}^*)^T)$  and let  $\mathbf{T}^{**}$  be in equilibrium with a body force  $\rho \mathbf{B}^{**}$  and a surface traction  $\mathbf{t}^{**}$ . Using the divergence theorem, verify the identity (theorem of virtual work)

$$\int_S \mathbf{t}^{**} \cdot \mathbf{u}^* dS + \int_V \rho \mathbf{B}^{**} \cdot \mathbf{u}^* dV = \int_V T_{ij}^{**} E_{ij}^* dV$$

7.6. Using the equations of motion and the divergence theorem, verify the following rate of work identity

$$\int_V \rho \mathbf{B} \cdot \mathbf{v} dV + \int_S \mathbf{t} \cdot \mathbf{v} dS = \int_V \rho \frac{D}{Dt} \left( \frac{v^2}{2} \right) dV + \int_V T_{ij} D_{ij} dV$$

7.7. Consider the velocity and density fields

$$\mathbf{v} = x_1 \mathbf{e}_1, \quad \rho = \rho_0 e^{-t}$$

(a) Check the equation of mass conservation.

(b) Compute the mass and rate of increase of mass in the cylindrical control volume of cross-section  $A$  and bounded by  $x_1 = 0$  and  $x_1 = 3$ .

(c) Compute the net mass inflow into the control volume of part (b). Does the net mass inflow equal the rate of mass increase?

7.8. (a) Check that the motion

$$x_1 = X_1 e^{t-t_0}, \quad x_2 = X_2, \quad x_3 = X_3$$

corresponds to the velocity field of Prob. 7.7.

(b) For a density field  $\rho = \rho_0 e^{-(t-t_0)}$ , verify that the mass contained in the material volume that was coincident with the control volume of Prob. 7.7 at time  $t_0$ , remain a constant.

(c) Compute the total linear momentum for the material volume of part (b).

7.9. Do Problem 7.7 for the velocity field  $\mathbf{v} = x_1 \mathbf{e}_1$  and the density field  $\rho = \frac{\rho_0}{x_1}$  and for the cylindrical control volume bounded by  $x_1 = 1$  and  $x_1 = 3$ .

7.10. The center of mass  $\mathbf{x}_{c,m}$  of a material volume is defined by the equation

$$m \mathbf{x}_{c,m} = \int_{V_m} \mathbf{x} \rho dV, \quad \text{where } m = \int_{V_m} \rho dV$$

Demonstrate that the linear momentum principle may be written in the form

$$\int_{S_c} \mathbf{t} dS + \int_{V_c} \rho \mathbf{B} dV = m \mathbf{a}_{c,m}$$

where  $\mathbf{a}_{c,m}$  is the acceleration of the mass center.

7.11. Consider the following velocity field and density field

$$\mathbf{v} = \left( \frac{x_1}{1+t} \right) \mathbf{e}_1, \quad \rho = \frac{\rho_0}{1+t}$$

(a) Compute the total linear momentum and rate of increase of linear momentum in a cylindrical control volume of cross-sectional area  $A$  and bounded by the plane  $x_1 = 1$  and  $x_1 = 3$ .

(b) Compute the net rate of outflow of linear momentum from the control volume of part (a).

(c) Compute the total force on the material in the control volume.

(d) Compute the total kinetic energy and rate of increase of kinetic energy for the control volume of part (a).

(e) Compute the net rate of outflow of kinetic energy from the control volume.

7.12. Consider the velocity and density fields

$$\mathbf{v} = x_1 \mathbf{e}_1, \quad \rho = \rho_0 e^{-t}$$

For an arbitrary time  $t$ , consider the material contained in the cylindrical control volume bounded by  $x_1 = 0$  and  $x_1 = 3$ .

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- (a) Determine the linear momentum and rate of increase of linear momentum in this control volume.
- (b) Determine the outflux of linear momentum.
- (c) Determine the net resultant force that is acting on the material contained in the control volume.

7.13. Do Problem 7.12 for the same velocity field, with  $\rho = \frac{\rho_0}{x_1}$  and the cylindrical control volume bounded by  $x_1 = 1$  and  $x_1 = 3$ .

7.14. Consider the flow field  $\mathbf{v} = x\mathbf{e}_1 - y\mathbf{e}_2$  with  $\rho = \text{constant}$ . For a control volume defined by  $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$  determine the net resultant force and couple that is acting on the material contained in this volume.

7.15. Do Problem 7.14 for the control volume defined by  $x = 2, y = 2, xy = 2, z = 0, z = 2$ .

7.16. For Hagen-Poiseuille flow in a pipe

$$\mathbf{v} = C(r_0^2 - r^2)\mathbf{e}_1$$

calculate the momentum flux across a cross-section. For the same flow rate, if the velocity is assumed to be uniform, what is the momentum flux across a cross section? Compare the two results.

7.17. A pile of chain on a table falls through a hole from the table under the action of gravity. Derive the differential equation governing the hanging length  $x$ .

7.18. A water jet of 5 cm. diameter moves at 12 m/sec, impinges on a curved vane which deflects it  $60^\circ$  from its direction. Neglect the weight. Obtain the force exerted by the liquid on the vane.

7.19. A horizontal pipeline of 10 cm. diameter bends through  $90^\circ$ , and while bending, changes its diameter to 5 cm. The pressure in the 10 cm. pipe is 140 kPa. Estimate the resultant force on the bends when  $0.05 \text{ m}^3/\text{sec}$  of water is flowing in the pipeline.

7.20. Figure P7.1 shows a steady water jet of area  $A$  impinging onto the flat wall. Find the force exerted on the wall. Neglect weight and viscosity of water.

7.21. Frequently in open channel flow, a high speed flow "jumps" to a low speed flow with an abrupt rise in the water surface. This is known as a hydraulic jump. Referring to Fig.P7.2, if the flow rate is  $Q$  per unit width, find the relation between  $y_1$  and  $y_2$ . Assume the flow before and after the jump is uniform and the pressure distribution is hydrostatic.

7.22. If the curved vane of Example. 7.6.2 moves with a velocity  $v < v_0$  in the same direction as the oncoming jet, find the resultant force exerted on the vane by the jet.

7.23. For the half-arm sprinkler shown in Fig.P7.3, find the angular speed if  $Q = 0.566 \text{ m}^3/\text{sec}$ . Neglect friction.

7.24. The tank car shown in Fig.P7.4 contains water and compressed air which is regulated to force a water jet out of the nozzle at a constant rate of  $Q \text{ m}^3/\text{s}$ . The diameter of the jet is  $d \text{ cm}$ . the initial total mass of the tank car is  $M_0$ . Neglecting frictional forces, find the velocity of the car as a function of time.

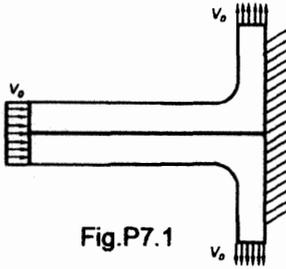


Fig.P7.1

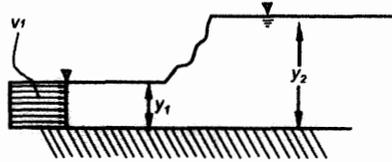


Fig.P7.2

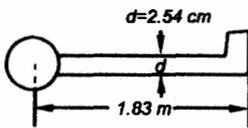


Fig.P7.3

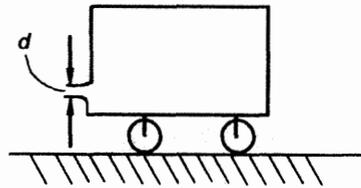


Fig.P7.4

## Non-Newtonian Fluids

In Chapter 6, the linear viscous fluid was discussed as an example of a constitutive equation of an idealized fluid. The mechanical behavior of many real fluids appears to be adequately described under a wide range of circumstances by this constitutive equation which is referred to as the constitutive equation of a Newtonian fluid. Many other real fluids exhibit behaviors which are not accounted for by the theory of Newtonian fluid. Examples of such substances include polymeric solutions, paints, molasses, etc.

For a steady unidirectional laminar flow of water in a circular pipe, the theory of Newtonian fluid gives the experimentally confirmed result that the volume discharge  $Q$  is proportional to the (constant) pressure gradient in the axial direction and to the fourth power of the diameter  $d$  of the pipe, that is [see Eq. (6.13.6)]

$$Q = \frac{\pi d^4}{128\mu} \left| \frac{dp}{dx} \right| \quad (8.0.1)$$

However, for many polymeric solutions, it was observed that the above equation does not hold. For a fixed  $d$ , the  $Q$  versus  $|dp/dx|$  relation is nonlinear as sketched in Fig. 8.1.

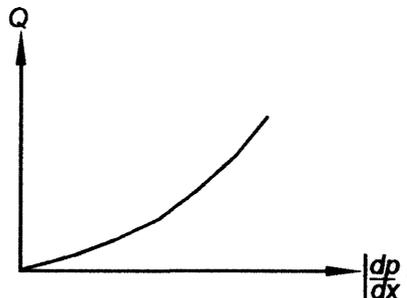


Fig. 8.1

For a steady laminar flow of water placed between two very long coaxial cylinders of radii  $r_1$  and  $r_2$ , if the inner cylinder is at rest while the outer one is rotating with an angular velocity  $\Omega$ , the theory of Newtonian fluid gives the result agreeing with experimental observations that the torque per unit height which must be applied to the cylinders to maintain the flow is proportional to  $\Omega$ . In fact [see Eq. (6.15.5)]

$$M = \frac{4\pi\mu r_1^2 r_2^2 \Omega}{r_2^2 - r_1^2} \quad (8.0.2)$$

However, for those fluids which do not obey Eq. (8.0.1), it is found that they do not obey Eq. (8.0.2) either. Furthermore, for water in this flow, the normal stress exerted on the outer cylinder is always larger than that on the inner cylinder due to the effect of centrifugal forces. However, for those fluids which do not obey Eq. (8.0.1), the compressive normal stress on the inner cylinder can be larger than that on the outer cylinder. Fig. 8.2 is a schematic diagram showing a higher fluid level in the center tube than in the outer tube for a non-Newtonian fluid in spite of the centrifugal forces due to the rotations of the cylinders. Other manifestations of the non-Newtonian behaviors include the ability of the fluids to store elastic energy and the occurrence of non-zero stress relaxation time when the fluid is suddenly given a constant shear deformation. (For Newtonian fluids, relaxation is instantaneous).

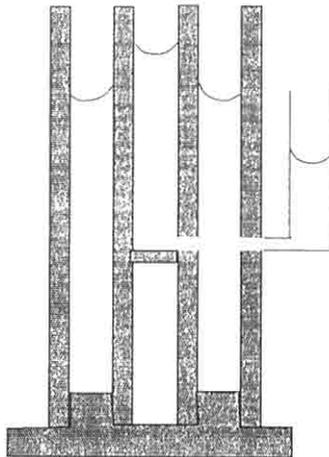


Fig. 8.2

In this chapter, we shall discuss several special constitutive equations and one general one which define idealized viscoelastic fluids exhibiting various characteristics of Non-Newtonian behaviors.

**Part A Linear Viscoelastic Fluid**

**8.1 Linear Maxwell Fluid**

The linear Maxwell fluid is defined by the following constitutive equation:

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau} \tag{8.1.1a}$$

$$\boldsymbol{\tau} + \lambda \frac{\partial \boldsymbol{\tau}}{\partial t} = 2\mu \mathbf{D} \tag{8.1.1b}$$

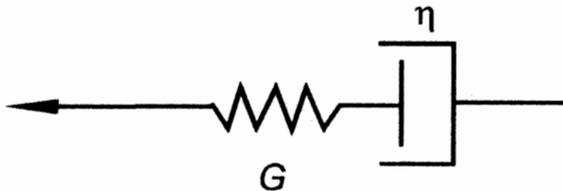
where  $-p\mathbf{I}$  is the isotropic pressure which is constitutively indeterminate due to the incompressibility property of the fluid,  $\boldsymbol{\tau}$  is called the “extra stress” which is related to the rate of deformation  $\mathbf{D}$  by Eq. (8.1.1b).

In the following example, we show, with the help of a mechanical analogy, that the linear Maxwell fluid possesses elasticity.

**Example 8.1.1**

Figure 8.3 shows the so-called linear Maxwell element which consists of a spring (an elastic element) with spring constant  $G$ , connected in series to a viscous dashpot (viscous element) with a damping coefficient  $\eta$ . The elongation (or strain) of the Maxwell element can be divided into an elastic portion  $\epsilon_e$  and a viscous portion  $\epsilon_v$ , i.e.,

$$\epsilon = \epsilon_e + \epsilon_v \tag{i}$$



**Fig. 8.3**

Since the spring and the dashpot are connected in series, the force  $\tau$  in each is the same (inertia effects are neglected). That is

$$\tau = G \varepsilon_e = \eta \frac{d \varepsilon_v}{dt} \quad (\text{ii})$$

Thus,

$$\frac{d \varepsilon_e}{dt} = \frac{1}{G} \frac{d \tau}{dt} \quad (\text{iii})$$

and

$$\frac{d \varepsilon_v}{dt} = \frac{1}{\eta} \tau \quad (\text{iv})$$

Taking time derivative of Eq. (i) and using Eqs. (iii) and (iv), we obtain the relation between the rate of strain of the Maxwell element with the force  $\tau$  as follows:

$$\frac{d \varepsilon}{dt} = \frac{1}{G} \frac{d \tau}{dt} + \frac{1}{\eta} \tau \quad (8.1.2)$$

or

$$\tau + \lambda \frac{d \tau}{dt} = \eta \frac{d \varepsilon}{dt} \quad (8.1.3)$$

where

$$\lambda = \frac{\eta}{G} \quad (8.1.4)$$

has the dimension of time, the physical meaning of which will be discussed below. Equation (8.1.3) is of the same form as Eq. (8.1.1b). Indeed both  $\mathbf{D}$  and  $d \varepsilon / dt$  (in the right side of these equations) describe rates of deformation. (We note that in a simple shearing flow in the  $xy$  plane, the rate of change of shearing strain is given by  $2D_{xy}$ ). Thus, by analogy, we see that the constitutive equation, Eq. (8.1.1b) endows the fluid with "elasticity" through the term  $\lambda d \tau / dt$  with an equivalent elastic modulus  $G$  given by Eq. (8.1.4).

Let us consider the following experiment performed on the Maxwell element: Starting at time  $t=0$ , a constant force  $\tau_o$  is applied to the element. We are interested in how, for  $t > 0$ , the strain changes with time. This is the so-called **creep experiment**. From Eq. (8.1.3), we have, since  $d \tau / dt = 0$ , for  $t > 0$ ,

$$\frac{d \varepsilon}{d \tau} = \frac{1}{\eta} \tau_o \quad \text{for } t > 0 \quad (\text{v})$$

which yields

$$\varepsilon = \frac{\tau_o}{\eta} t + \varepsilon_o \quad (\text{vi})$$

The integration constant  $\varepsilon_0$  is the instantaneous strain  $\varepsilon$  of the element at  $t = 0+$  from the elastic response of the spring and is therefore given by  $\tau_0/G$ . Thus

$$\varepsilon = \frac{\tau_0}{\eta} t + \frac{\tau_0}{G} \quad (8.1.5)$$

We see from Eq. (8.1.5) that under the action of a constant force  $\tau_0$  in creep experiment, the strain of the Maxwell element first has an instantaneous jump from 0 to  $\tau_0/G$  and then continues to increase with time ( i.e. flow ) without limit.

We note that there is no contribution to the instantaneous strain from the dashpot because, with  $d\varepsilon/dt \rightarrow \infty$ , an infinitely large force is required for the dashpot to do that. On the other hand, there is no contribution to the rate of elongation from the spring because the elastic response is a constant under a constant load.

We may write Eq. (8.1.5) as

$$\frac{\varepsilon}{\tau_0} = \frac{1}{\eta} t + \frac{1}{G} \equiv J \quad (8.1.6)$$

The function  $J(t)$  gives the creep history per unit force. It is known as the **creep compliance function** for the linear Maxwell element.

In another experiment, the Maxwell element is given a strain  $\varepsilon_0$  at  $t=0$  which is then maintained at all time. We are interested in how the force  $\tau$  changes with time. This is the so-called **stress relaxation experiment**. From Eq. (8.1.3), with  $d\varepsilon/dt = 0$ , for  $t > 0$ , we have

$$\tau + \lambda \frac{d\tau}{dt} = 0 \quad \text{for } t > 0 \quad (\text{vii})$$

which yields

$$\tau = \tau_0 e^{-t/\lambda} \quad (\text{viii})$$

The integration constant  $\tau_0$  is the instantaneous elastic force which is required to produce the strain  $\varepsilon_0$  at  $t = 0$ . That is,  $\tau_0 = G \varepsilon_0$ . Thus,

$$\tau = G \varepsilon_0 e^{-t/\lambda} \quad (8.1.7)$$

Eq. (8.1.7) is the force history for the stress relaxation experiment for the Maxwell element. We may write Eq. (8.1.7) as

$$\phi(t) \equiv \frac{\tau}{\varepsilon_0} = G e^{-t/\lambda} = \frac{\eta}{\lambda} e^{-t/\lambda} \quad (8.1.8)$$

The function  $\phi(t)$  gives the stress history per unit strain. It is called the **stress relaxation function**, and the constant  $\lambda$  is known as the relaxation time which is the time for the force to relax to  $1/e$  of the initial value of  $\tau$ .

It is interesting to consider the limiting cases of the Maxwell element. If  $G = \infty$ , then the spring element becomes a rigid bar and the element no longer possesses elasticity. That is, it is a purely viscous element. In creep experiment, there will be no instantaneous elongation, the element simply creeps linearly with time (see Eq. (8.1.6)) from the unstretched initial position. In the stress relaxation experiment, an infinitely large force is needed at  $t=0$  to produce the finite jump in elongation (from 0 to 1). The force however is instantaneously returned to zero ( i.e., the relaxation time  $\lambda = \eta/G \rightarrow 0$  ). We can write the relaxation function for the purely viscous element in the following way

$$\tau = \eta\delta(t) \quad \text{viscous element only} \quad (8.1.9)$$

where  $\delta(t)$  is known as the Dirac delta function which may be defined to be the derivative of the unit step function  $H(t)$  defined by:

$$H(t) = \begin{cases} 0 & -\infty < t < 0 \\ 1 & 0 \leq t < \infty \end{cases} \quad (8.1.10)$$

Thus,

$$\delta(t) \equiv \frac{dH(t)}{dt} \quad (8.1.11)$$

and

$$\int^t \delta(t)dt = H(t) \quad (8.1.12)$$

### Example 8.1.2

Consider a linear Maxwell fluid, defined by Eq. (8.1.1), in steady simple shearing flow:

$$v_1 = kx_2, \quad v_2 = v_3 = 0 \quad (i)$$

Find the stress components.

*Solution.* Since the given velocity field is steady, all field variables are independent of time. Thus,  $\frac{\partial \tau}{\partial t} = 0$  and we have

$$\tau = 2\mu \mathbf{D} \quad (ii)$$

Thus, the stress field is exactly the same as that of a Newtonian incompressible fluid and the viscosity is independent of the rate of shear for this fluid.

## Example 8.1.3

For a Maxwell fluid, consider the stress relaxation experiment with the displacement field given by

$$u_1 = \varepsilon_0 H(t) x_2, \quad u_2 = u_3 = 0 \quad (\text{i})$$

where  $H(t)$  is the unit step function defined in Eq. (8.1.10). Neglect inertia effects,

(i) obtain the components of the rate of deformation tensor.

(ii) obtain  $\tau_{12}$  at  $t = 0$ .

(iii) obtain the history of the shear stress  $\tau_{12}$ .

*Solution.* Differentiate Eq. (i) with respect to time, we get

$$v_1 = \varepsilon_0 \delta(t) x_2, \quad v_2 = v_3 = 0 \quad (\text{ii})$$

where  $\delta(t)$  is the Dirac delta function defined in Eq. (8.1.11). The only non-zero rate of deformation component is  $D_{12} = \frac{\varepsilon_0 \delta(t)}{2}$ . Thus, from the constitutive equation for the linear Maxwell fluid, Eq. (8.1.1b), we obtain

$$\tau_{12} + \lambda \frac{d\tau_{12}}{dt} = \mu \varepsilon_0 \delta(t) \quad (\text{iii})$$

Integrating the above equation from  $t=0-\varepsilon$  to  $t=0+\varepsilon$ , we have

$$\int_{0-\varepsilon}^{0+\varepsilon} \tau_{12} dt + \lambda \int_{0-\varepsilon}^{0+\varepsilon} \frac{d\tau_{12}}{dt} dt = \mu \varepsilon_0 \int_{0-\varepsilon}^{0+\varepsilon} \delta(t) dt \quad (\text{iv})$$

The integral on the right side of the above equation is equal to 1 [see Eq. (8.1.12)]. As  $\varepsilon \rightarrow 0$ , the first integral on the left side of the above equation approaches zero whereas the second integral becomes:

$$\lambda [\tau_{12}(0+) - \tau_{12}(0-)] \quad (\text{v})$$

Thus, since  $\tau_{12}(0-) = 0$ , from Eq. (iv), we have

$$\tau_{12}(0+) = \frac{\mu \varepsilon_0}{\lambda}$$

For  $t > 0$ ,  $\delta(t) = 0$  so that Eq. (iii) becomes

$$\tau_{12} + \lambda \frac{d\tau_{12}}{dt} = 0 \quad t > 0 \quad (\text{vi})$$

The solution of the above equation with the initial condition  $\tau_{12}(0+) = \frac{\mu \varepsilon_0}{\lambda}$  is

$$\frac{\tau_{12}}{\varepsilon_0} = \frac{\mu}{\lambda} e^{-t/\lambda} \quad (8.1.13)$$

This is the same relaxation function which we obtained for the spring-dashpot model in Eq.(8.1.7). In arriving at Eq. (8.1.7), we made use of the initial condition  $\tau_0 = G \varepsilon_0$ , which was obtained from considerations of the responses of the elastic element. Here in the present example, the initial condition is obtained by integrating the differential equation, Eq. (iii), over an infinitesimal time interval (from  $t=0^-$  to  $t=0^+$ ). By comparing Eq. (8.1.13) here with Eq. (8.1.8) of the mechanical model, we see that  $\frac{\mu}{\lambda}$  is the equivalent of the spring constant  $G$  of the mechanical model. It gives a measure of the elasticity of the linear Maxwell fluid.

#### Example 8.1.4

A linear Maxwell fluid is confined between two infinitely large parallel plates. The bottom plate is fixed. The top plate undergoes a one-dimensional oscillation of small amplitude  $u_0$  in its own plane. Neglect the inertia effects, find the response of the shear stress.

*Solution.* The boundary conditions for the displacement components may be written:

$$\text{At } y = h: \quad u_x = u_0 e^{i\omega t}, \quad u_y = u_z = 0 \quad (i)$$

$$\text{At } y = 0: \quad u_x = u_y = u_z = 0 \quad (ii)$$

where  $i = \sqrt{-1}$  and  $e^{i\omega t} = \cos\omega t + i \sin\omega t$ . We may take the real part of  $u_x$  to correspond to our physical problem. That is, in the physical problem,  $u_x = u_0 \cos\omega t$ .

Consider the following displacement field:

$$u_x(y) = u_0 e^{i\omega t} \left( \frac{y}{h} \right), \quad u_y = u_z = 0 \quad (iii)$$

Clearly, this displacement field satisfies the boundary conditions (i) and (ii). The velocity field corresponding to Eq. (iii) is given by:

$$v_x(y) = i\omega u_0 e^{i\omega t} \left( \frac{y}{h} \right), \quad v_y = v_z = 0 \quad (iii)$$

Thus, the components of the rate of deformation tensor  $\mathbf{D}$  are:

$$D_{12} = \frac{1}{2} i \omega u_0 e^{i\omega t} \left( \frac{1}{h} \right) \quad \text{and all other } D_{ij} = 0 \quad (iv)$$

This is a homogeneous field and it corresponds to a homogeneous stress field. In the absence of inertia forces, every homogeneous stress field satisfies all the momentum equations and is therefore a physically acceptable solution. Let the homogeneous stress component  $\tau_{12}$  be given by

$$\tau_{12} = \tau_o e^{i\omega t} \tag{v}$$

We wish to obtain the complex number  $\tau_o$ . Substituting  $\tau_{12} = \tau_o e^{i\omega t}$  into the constitutive equation for  $\tau_{12}$ :

$$\tau_{12} + \lambda \frac{d\tau_{12}}{dt} = 2\mu D_{12} = \mu i \omega \left( \frac{u_o}{h} \right) e^{i\omega t} \tag{vi}$$

one obtains

$$\frac{\tau_o}{\left( \frac{u_o}{h} \right)} = \frac{\mu i \omega}{(1 + \lambda i \omega)} = \frac{\lambda \mu \omega^2}{(1 + \lambda^2 \omega^2)} + i \frac{\mu \omega}{(1 + \lambda^2 \omega^2)} \tag{vii}$$

The ratio  $\frac{\tau_o}{\left( \frac{u_o}{h} \right)} \equiv G^*$  is known as the **complex shear modulus**, which can be written as

$$G^* = G'(\omega) + iG''(\omega) \tag{8.1.14a}$$

The real part of this complex modulus is

$$G' = \frac{\lambda \mu \omega^2}{(1 + \lambda^2 \omega^2)} \tag{8.1.14b}$$

and the imaginary part is

$$G'' = \frac{\mu \omega}{(1 + \lambda^2 \omega^2)} \tag{8.1.14c}$$

If we write  $\frac{\mu}{\lambda}$  as  $G$ , the spring constant in the spring-dashpot model, we have

$$G' = \frac{\mu^2 \omega^2 G}{(G^2 + \mu^2 \omega^2)} \tag{viii a}$$

and

$$G'' = \frac{\mu \omega G^2}{(G^2 + \mu^2 \omega^2)} \tag{viii b}$$

We note that as limiting cases of the Maxwell model, a purely elastic element has  $\mu \rightarrow \infty$  so that  $G' = G$  and  $G'' = 0$  and a purely viscous element has  $G \rightarrow \infty$  so that  $G' = 0$  and  $G'' = \mu \omega$ . Thus,  $G'$  characterizes the extent of elasticity of the fluid which is capable of storing elastic energy whereas  $G''$  characterizes the extent of loss of energy due to

viscous dissipation of the fluid. Thus,  $G'$  is called the **storage modulus** and  $G''$  is called the **loss modulus**.

writing

$$G^* = G' + iG'' = |G^*|e^{i\delta} \tag{8.1.15a}$$

where

$$|G^*| = (G'^2 + G''^2)^{1/2} \tag{8.1.15b}$$

and

$$\tan\delta = \frac{G''}{G'} \tag{8.1.15c}$$

we have,

$$G^* e^{i\omega t} = |G^*| e^{i(\omega t + \delta)} \tag{ix}$$

Therefore, taking the real part of Eq. (v), we obtain, with Eq. (ix)

$$\tau_{12} = |G^*| \cos(\omega t + \delta) \left( \frac{u_0}{h} \right) \tag{x}$$

Thus, for a Maxwell fluid, the shear stress response in a sinusoidal oscillatory experiment under the condition that the inertia effects are negligible is

$$T_{12} = \tau_{12} = \left( \frac{u_0}{h} \right) \frac{\mu \omega}{\sqrt{(1 + \lambda^2 \omega^2)}} \cos(\omega t + \tan^{-1} \frac{1}{\lambda \omega}) \tag{xi}$$

The angle  $\delta$  is known as the **phase angle**. For a purely elastic material in a sinusoidally oscillation, the stress and the strain are oscillating in the same phase ( $\delta = 0$ ) whereas for a purely viscous fluid, the stress is  $90^\circ$  ahead of the strain.

### 8.2 Generalized Linear Maxwell Fluid with Discrete Relaxation Spectra

A linear Maxwell fluid with  $N$  discrete relaxation spectra is defined by the following constitutive equation:

$$\boldsymbol{\tau} = \sum_1^N \boldsymbol{\tau}_n \tag{8.2.1a}$$

where

$$\boldsymbol{\tau}_n + \lambda_n \frac{\partial \boldsymbol{\tau}_n}{\partial t} = 2\mu_n \mathbf{D} \tag{8.2.1b}$$

The mechanical analog for this constitutive equation may be represented by  $N$  Maxwell elements connected in parallel. The shear relaxation function is the sum of the  $N$  relaxation functions each with a different relaxation time  $\lambda_n$ :

$$\phi(t) = \sum_1^N \frac{\mu_n}{\lambda_n} e^{-t/\lambda_n} \tag{8.2.2}$$

It can be shown that Eqs. (8.2.1) is equivalent to the following constitutive equation

$$\boldsymbol{\tau} + \sum_1^N a_n \frac{\partial^n \boldsymbol{\tau}}{\partial t^n} = b_o \mathbf{D} + \sum_1^{N-1} b_n \frac{\partial^n \mathbf{D}}{\partial t^n} \tag{8.2.3}$$

We demonstrate this equivalence for the case of  $N = 2$  as follows: When  $N = 2$ ,

$$\boldsymbol{\tau} = \boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 \tag{8.2.4a}$$

and

$$\boldsymbol{\tau}_1 + \lambda_1 \frac{\partial \boldsymbol{\tau}_1}{\partial t} = 2\mu_1 \mathbf{D} \quad \text{and} \quad \boldsymbol{\tau}_2 + \lambda_2 \frac{\partial \boldsymbol{\tau}_2}{\partial t} = 2\mu_2 \mathbf{D} \tag{8.2.4b}$$

Thus

$$\begin{aligned} (\lambda_1 + \lambda_2) \frac{\partial \boldsymbol{\tau}}{\partial t} &= \lambda_1 \frac{\partial \boldsymbol{\tau}_1}{\partial t} + \lambda_2 \frac{\partial \boldsymbol{\tau}_2}{\partial t} + \lambda_2 \frac{\partial \boldsymbol{\tau}_1}{\partial t} + \lambda_1 \frac{\partial \boldsymbol{\tau}_2}{\partial t} = 2(\mu_1 + \mu_2) \mathbf{D} - (\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2) \\ &+ \lambda_2 \frac{\partial \boldsymbol{\tau}_1}{\partial t} + \lambda_1 \frac{\partial \boldsymbol{\tau}_2}{\partial t} = (2\mu_1 + 2\mu_2) \mathbf{D} - \boldsymbol{\tau} + \lambda_2 \frac{\partial \boldsymbol{\tau}_1}{\partial t} + \lambda_1 \frac{\partial \boldsymbol{\tau}_2}{\partial t} \end{aligned} \tag{i}$$

and

$$\lambda_1 \lambda_2 \frac{\partial^2 \boldsymbol{\tau}}{\partial t^2} = \lambda_2 \lambda_1 \frac{\partial^2 \boldsymbol{\tau}_1}{\partial t^2} + \lambda_1 \lambda_2 \frac{\partial^2 \boldsymbol{\tau}_2}{\partial t^2} = 2(\lambda_2 \mu_1 + \lambda_1 \mu_2) \frac{\partial \mathbf{D}}{\partial t} - \lambda_2 \frac{\partial \boldsymbol{\tau}_1}{\partial t} - \lambda_1 \frac{\partial \boldsymbol{\tau}_2}{\partial t} \tag{ii}$$

Adding Eqs. (i) and (ii), we obtain

$$\boldsymbol{\tau} + (\lambda_1 + \lambda_2) \frac{\partial \boldsymbol{\tau}}{\partial t} + \lambda_1 \lambda_2 \frac{\partial^2 \boldsymbol{\tau}}{\partial t^2} = 2(\mu_1 + \mu_2) \mathbf{D} + 2(\lambda_2 \mu_1 + \lambda_1 \mu_2) \frac{\partial \mathbf{D}}{\partial t} \tag{iii}$$

Let

$$a_1 = \lambda_1 + \lambda_2, a_2 = \lambda_1 \lambda_2, b_o = 2(\mu_1 + \mu_2), \text{ and } b_1 = 2(\lambda_2 \mu_1 + \lambda_1 \mu_2) \tag{iv}$$

we have

$$\boldsymbol{\tau} + a_1 \frac{\partial \boldsymbol{\tau}}{\partial t} + a_2 \frac{\partial^2 \boldsymbol{\tau}}{\partial t^2} = b_o \mathbf{D} + b_1 \frac{\partial \mathbf{D}}{\partial t} \tag{8.2.5}$$

In the above equation, if  $a_2 = 0$ , the equation is sometimes called the **Jeffrey's model**.

### 8.3 Integral Form of the Linear Maxwell Fluid and of the Generalized Linear Maxwell Fluid with Discrete Relaxation Spectra

Consider the following integral form of constitutive equation:

$$\boldsymbol{\tau} = 2 \int_{-\infty}^t \phi(t-t') \mathbf{D}(t') dt' \tag{8.3.1a}$$

where

$$\phi(t) = \frac{\mu}{\lambda} e^{-t/\lambda} \tag{8.3.1b}$$

is the shear relaxation function for the linear Maxwell fluid defined by Eq. (8.1.1b). If we differentiate Eq. (8.3.1) with respect to time  $t$ , we obtain (note that  $t$  appears in both the integrand and the integration limit, we need to use the Leibnitz rule of differentiation)

$$\begin{aligned} \frac{\partial \boldsymbol{\tau}}{\partial t} &= \frac{2\mu}{\lambda} \left[ \int_{-\infty}^t \left(-\frac{1}{\lambda}\right) \exp[-(t-t')/\lambda] \mathbf{D}(t') dt' + \mathbf{D}(t) \right] \\ &= -\left(\frac{1}{\lambda}\right) \boldsymbol{\tau} + \frac{2\mu}{\lambda} \mathbf{D} \end{aligned} \tag{i}$$

That is,

$$\boldsymbol{\tau} + \lambda \frac{\partial \boldsymbol{\tau}}{\partial t} = 2\mu \mathbf{D} \tag{8.1.1b}$$

Thus, the integral form constitutive equation, Eqs. (8.3.1) is the same as the rate form constitutive equation, Eq. (8.1.1b). Of course, Eq. (8.3.1) is nothing but the solution of the linear non-homogeneous ordinary differential equation, Eq. (8.1.1b). [See Prob. 8.6]

It is not difficult to show that the constitutive equation for the generalized linear Maxwell equation with  $N$  discrete relaxation spectra, Eq. (8.2.1) is equivalent to the following integral form

$$\boldsymbol{\tau} = 2 \int_{-\infty}^t \sum_1^N \frac{\mu_n}{\lambda_n} \exp[-(t-t')/\lambda_n] \mathbf{D}(t') dt' \tag{8.3.2}$$

We may write the above equation in the following form:

$$\boldsymbol{\tau} = 2 \int_{-\infty}^t \phi(t-t') \mathbf{D}(t') dt' \tag{8.3.3}$$

where the shear relaxation function  $\phi(t)$  is given by

$$\phi = \sum_1^N \frac{\mu_n}{\lambda_n} e^{-t/\lambda_n} \tag{8.3.4}$$

**8.4 Generalized Linear Maxwell Fluid with a Continuous Relaxation Spectrum.**

The linear Maxwell fluid with a continuous relaxation spectrum is defined by the constitutive equation:

$$\boldsymbol{\tau} = 2 \int_{-\infty}^t \phi(t-t') \mathbf{D}(t') dt' \tag{8.4.1}$$

where the relaxation function  $\phi(t)$  is given by

$$\phi(t) = \int_0^{\infty} \frac{H(\lambda)}{\lambda} e^{-t/\lambda} d\lambda \tag{8.4.2a}$$

The function  $H(\lambda)/\lambda$  is the relaxation spectrum. Eq. (8.4.2a) can also be written

$$\phi(t) = \int_0^{\infty} H(\lambda) e^{-t/\lambda} d \ln \lambda \tag{8.4.2b}$$

As we shall see later that the linear Maxwell models considered so far are physically acceptable models only if the motion is such that the components of the relative deformation gradient (i.e., deformation gradient measured from the configuration at the current time  $t$ , see Section 8.5 ) are small. When this is the case, the components of rate of deformation tensor  $\mathbf{D}$  are also small so that [see Eq. (v), Example 5.2.1]

$$\mathbf{D} \approx \frac{\partial \mathbf{E}}{\partial t} \tag{8.4.3}$$

where  $\mathbf{E}$  is the infinitesimal strain measured with respect to the current configuration. Substituting the above approximation in Eq. (8.4.1) and integrating the right hand side by parts, we obtain

$$\boldsymbol{\tau} = 2 \int_{-\infty}^t \phi(t-t') \frac{\partial \mathbf{E}}{\partial t'} dt' = 2\phi(t-t') \mathbf{E}(t') \Big|_{-\infty}^t - 2 \int_{-\infty}^t \mathbf{E}(t') \frac{d\phi(t-t')}{dt'} dt' \tag{8.4.4}$$

The first term in the right hand side is zero because  $\phi(\infty) = 0$  for a fluid and  $\mathbf{E}(t)=0$  because the deformation is measured relative to the configuration at time  $t$ . Thus,

$$\boldsymbol{\tau} = -2 \int_{-\infty}^t \frac{d\phi(t-t')}{dt'} \mathbf{E}(t') dt' \tag{8.4.5}$$

Or, letting  $t-t' = s$ , we can write the above equation as

$$\tau = 2 \int_0^{\infty} \frac{d\phi(s)}{ds} \mathbf{E}(t-s) ds \quad (8.4.6)$$

Let

$$\frac{d\phi(s)}{ds} \equiv f(s) \quad (8.4.7)$$

we can write Eq. (8.4.6) as

$$\tau = 2 \int_0^{\infty} f(s) \mathbf{E}(t-s) ds \quad (8.4.8a)$$

or

$$\tau = 2 \int_{-\infty}^t f(t-t') \mathbf{E}(t') dt' \quad (8.4.8b)$$

The above equation is the integral form of constitutive equation for the linear Maxwell fluid written in terms of the infinitesimal strain tensor  $\mathbf{E}$  (instead of the rate of deformation tensor  $\mathbf{D}$ ). The function  $f(s)$  in this equation is known as the **memory function**. The relation between the memory function and the relaxation function is given by Eq. (8.4.7).

The constitutive equation given in Eq. (8.4.8) can be viewed as the superposition of all the stresses, weighted by the memory function  $f(s)$ , caused by the deformation of the fluid particle (relative to the current time) at all the past time ( $t' = -\infty$  to the current time  $t$ ).

For the linear Maxwell fluid with one relaxation time, the memory function is given by

$$f(s) = \frac{d}{ds} \phi(s) = \frac{d}{ds} \left( \frac{\mu}{\lambda} e^{-s/\lambda} \right) = -\frac{\mu}{\lambda^2} e^{-s/\lambda} \quad (8.4.9)$$

For the linear Maxwell fluid with discrete relaxation spectra, the memory function is:

$$f(s) = - \sum_1^N \frac{\mu_N}{\lambda_N^2} e^{-s/\lambda_N} \quad (8.4.10)$$

and for the Maxwell fluid with a continuous spectrum

$$f(s) = - \int_0^{\infty} \frac{H(\lambda)}{\lambda^2} e^{-s/\lambda} d\lambda \quad (8.4.11)$$

**Part B Nonlinear Viscoelastic Fluid**

**8.5 Current Configuration as Reference Configuration**

Let  $\mathbf{x}$  be the position vector of a particle at current time  $t$ , and let  $\mathbf{x}'$  be the position vector of the same particle at time  $\tau$ . Then the equation

$$\mathbf{x}' = \mathbf{x}'_t(\mathbf{x}, \tau) \quad \text{with} \quad \mathbf{x} = \mathbf{x}'_t(\mathbf{x}, t) \tag{8.5.1}$$

defines the motion of a continuum using the current time  $t$  as the reference time. The subscript  $t$  in the function  $\mathbf{x}'_t(\mathbf{x}, \tau)$  indicates that the current  $t$  is the reference time and as such  $\mathbf{x}'_t(\mathbf{x}, \tau)$  is also a function of  $t$

For a given velocity field  $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ , the velocity at the position  $\mathbf{x}'$  at time  $\tau$  is  $\mathbf{v} = \mathbf{v}(\mathbf{x}', \tau)$ . On the other hand, for a particular particle (i.e., for fixed  $\mathbf{x}$  and  $t$ ), the velocity at time  $\tau$  is given by  $\left(\frac{\partial \mathbf{x}'_t}{\partial \tau}\right)_{\mathbf{x}, t - \text{fixed}}$ . Thus,

$$\mathbf{v}(\mathbf{x}', \tau) = \left(\frac{\partial \mathbf{x}'_t}{\partial \tau}\right) \tag{8.5.2}$$

Equation (8.5.2) allows one to obtain the pathline equations from a given velocity field, using the current time  $t$  as the reference time.

**Example 8.5.1**

Given the velocity field of the steady unidirectional flow

$$v_1 = v(x_2), \quad v_2 = 0, \quad v_3 = 0 \tag{i}$$

describe the motion of the particles by using the current time  $t$  as the reference time.

*Solution.* From the given velocity field, we have, the velocity components at the position  $(x'_1, x'_2, x'_3)$  at time  $\tau$ :

$$v_1 = v(x'_2), \quad v_2 = 0, \quad v_3 = 0 \tag{ii}$$

Thus, with  $\mathbf{x}' = x'_i \mathbf{e}_i$ , Eq. (8.5.2) gives

$$\frac{\partial x'_1}{\partial \tau} = v(x'_2), \quad \frac{\partial x'_2}{\partial \tau} = 0, \quad \frac{\partial x'_3}{\partial \tau} = 0 \tag{iii}$$

From  $\frac{\partial x'_2}{\partial \tau} = 0$ , we have

$$x'_2 = f(x_1, x_2, x_3, t)$$

But, at  $\tau = t, x'_2 = x_2$ , therefore, for all  $\tau$

$$x'_2 = x_2 \tag{iv}$$

Similarly, for all  $\tau$

$$x_3' = x_3 \tag{v}$$

Since  $x_2' = x_2$  for all  $\tau$ , therefore, from Eq. (ii)

$$\frac{\partial x_1'}{\partial \tau} = v(x_2)$$

Thus

$$x_1' = v(x_2)\tau + g(x_1, x_2, x_3, t)$$

At  $\tau = t$ ,  $x_1' = x_1$ , therefore

$$x_1 = v(x_2)t + g(x_1, x_2, x_3, t)$$

from which

$$g(x_1, x_2, x_3, t) = x_1 - v(x_2)t$$

and

$$x_1' = v(x_2)\tau + x_1 - v(x_2)t \tag{vi}$$

Thus,

$$x_1' = x_1 + v(x_2)(\tau - t)$$

$$x_2' = x_2$$

$$x_3' = x_3$$

### 8.6 Relative Deformation Gradient

Let  $dx$  and  $dx'$  be the differential vectors representing the same material element at time  $t$  and  $\tau$ , respectively. Then they are related by

$$dx' = x'_i(x + dx, \tau) - x'_i(x, \tau) = (\nabla x'_i) dx \tag{i}$$

That is

$$dx' = F_t dx \tag{8.6.1}$$

The tensor

$$F_t \equiv \nabla x'_i \tag{8.6.2}$$

is known as the **relative deformation gradient**. Here, the adjective “relative ” indicates that the deformation gradient is relative to the configuration at the current time. We note that for  $\tau = t$ ,  $dx' = dx$  so that

$$F_t(t) = I \tag{8.6.3}$$

In rectangular Cartesian coordinates,

$$[F_t] \equiv [\nabla \mathbf{x}'_t] = \begin{bmatrix} \frac{\partial x'_1}{\partial x_1} & \frac{\partial x'_1}{\partial x_2} & \frac{\partial x'_1}{\partial x_3} \\ \frac{\partial x'_2}{\partial x_1} & \frac{\partial x'_2}{\partial x_2} & \frac{\partial x'_2}{\partial x_3} \\ \frac{\partial x'_3}{\partial x_1} & \frac{\partial x'_3}{\partial x_2} & \frac{\partial x'_3}{\partial x_3} \end{bmatrix} \tag{8.6.4}$$

In cylindrical coordinates, with pathline equations given by

$$r' = r'(r, \theta, z, \tau), \quad \theta' = \theta'(r, \theta, z, \tau), \quad z' = z'(r, \theta, z, \tau) \tag{8.6.5}$$

the two point components of  $F_t$ , with respect to  $(\mathbf{e}'_r, \mathbf{e}'_\theta, \mathbf{e}'_z)$  at time  $\tau$  and  $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$  at the current time  $t$  are given by the matrix

$$[F_t] = \begin{bmatrix} \frac{\partial r'}{\partial r} & \frac{1}{r} \frac{\partial r'}{\partial \theta} & \frac{\partial r'}{\partial z} \\ r' \frac{\partial \theta'}{\partial r} & r' \frac{\partial \theta'}{\partial \theta} & r' \frac{\partial \theta'}{\partial z} \\ \frac{\partial z'}{\partial r} & \frac{1}{r} \frac{\partial z'}{\partial \theta} & \frac{\partial z'}{\partial z} \end{bmatrix}_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z, \mathbf{e}'_r, \mathbf{e}'_\theta, \mathbf{e}'_z} \tag{8.6.6}$$

In spherical coordinates, with pathline equations given by

$$r' = r'(r, \theta, \phi, \tau), \quad \theta' = \theta'(r, \theta, \phi, \tau), \quad \phi' = \phi'(r, \theta, \phi, \tau) \tag{8.6.7}$$

the two point components of  $F_t$ , with respect to  $(\mathbf{e}'_r, \mathbf{e}'_\theta, \mathbf{e}'_\phi)$  at time  $\tau$  and  $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$  at the current time  $t$  are given by the matrix

$$[F_t] = \begin{bmatrix} \frac{\partial r'}{\partial r} & \frac{1}{r} \frac{\partial r'}{\partial \theta} & \frac{\partial r'}{r \sin \theta \partial \phi} \\ r' \frac{\partial \theta'}{\partial r} & r' \frac{\partial \theta'}{\partial \theta} & r' \frac{\partial \theta'}{r \sin \theta \partial \phi} \\ r' \sin \theta' \frac{\partial \phi'}{\partial r} & r' \sin \theta' \frac{\partial \phi'}{r \partial \theta} & \frac{\partial z'}{r \sin \theta \partial \phi} \end{bmatrix}_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi, \mathbf{e}'_r, \mathbf{e}'_\theta, \mathbf{e}'_\phi} \tag{8.6.8}$$

### 8.7 Relative Deformation Tensors

The descriptions of the relative deformation tensors (using the current time  $t$  as reference time) are similar to those of the deformation tensors using a fixed reference time. [See Chapter 3, Section 3. 18 to 3.29]. Indeed by polar decomposition theorem (Section 3.21)

$$F_t = \mathbf{R}_t \mathbf{U}_t = \mathbf{V}_t \mathbf{R}_t \tag{8.7.1}$$

where  $\mathbf{U}_t$  and  $\mathbf{V}_t$  are relative right and left stretch tensor respectively and  $\mathbf{R}_t$  is the relative rotation tensor. Note

$$\mathbf{F}_t(t) = \mathbf{U}_t(t) = \mathbf{V}_t(t) = \mathbf{R}_t(t) = \mathbf{I} \tag{8.7.2}$$

From Eq. (8.7.1), we clearly also have

$$\mathbf{V}_t = \mathbf{R}_t \mathbf{U}_t \mathbf{R}_t^T \tag{8.7.3}$$

and

$$\mathbf{U}_t = \mathbf{R}_t^T \mathbf{V}_t \mathbf{R}_t \tag{8.7.4}$$

The relative right Cauchy-Green deformation tensor  $\mathbf{C}_t$  is defined by

$$\mathbf{C}_t = \mathbf{U}_t^2 = \mathbf{F}_t^T \mathbf{F}_t \tag{8.7.5}$$

and the relative left Cauchy-Green deformation tensor  $\mathbf{B}_t$  is defined by

$$\mathbf{B}_t = \mathbf{V}_t^2 = \mathbf{F}_t \mathbf{F}_t^T \tag{8.7.6}$$

and these two tensors are related by

$$\mathbf{B}_t = \mathbf{R}_t \mathbf{C}_t \mathbf{R}_t^T \text{ and } \mathbf{C}_t = \mathbf{R}_t^T \mathbf{B}_t \mathbf{R}_t \tag{8.7.7}$$

The tensors  $\mathbf{C}_t^{-1}$  and  $\mathbf{B}_t^{-1}$  are often encountered in the literature. They are known as the **relative Finger deformation tensor** and the **relative Piola deformation tensor** respectively.

We note that

$$\mathbf{C}_t(\mathbf{x},t) = \mathbf{B}_t(\mathbf{x},t) = \mathbf{C}_t^{-1}(\mathbf{x},t) = \mathbf{B}_t^{-1}(\mathbf{x},t) = \mathbf{I}$$

Example 8.7.1

Show that if  $d\mathbf{x}^{(1)}$  and  $d\mathbf{x}^{(2)}$  are two material elements emanating from a point  $P$  at time  $t$  and  $d\mathbf{x}'^{(1)}$  and  $d\mathbf{x}'^{(2)}$  are the corresponding elements at time  $\tau$ , then

$$d\mathbf{x}'^{(1)} \cdot d\mathbf{x}'^{(2)} = d\mathbf{x}^{(1)} \cdot \mathbf{C}_t d\mathbf{x}^{(2)} \tag{8.7.8}$$

and

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = d\mathbf{x}'^{(1)} \cdot \mathbf{B}_t^{-1} d\mathbf{x}'^{(2)} \tag{8.7.9}$$

*Solution.* From Eq. (8.6.1), we have

$$d\mathbf{x}'^{(1)} \cdot d\mathbf{x}'^{(2)} = [(\mathbf{F}_t) d\mathbf{x}^{(1)}] \cdot [(\mathbf{F}_t) d\mathbf{x}^{(2)}] \tag{i}$$

By the definition of the transpose

$$d\mathbf{x}'^{(1)} \cdot d\mathbf{x}'^{(2)} = d\mathbf{x}^{(1)} \cdot (\mathbf{F}_t)^T (\mathbf{F}_t) d\mathbf{x}^{(2)} \tag{ii}$$

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Thus,

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = d\mathbf{x}^{(1)} \cdot \mathbf{C}_t d\mathbf{x}^{(2)} \quad (\text{iii})$$

Also, since

$$d\mathbf{x} = \mathbf{F}_t^{-1} d\mathbf{x}' \quad (8.7.10)$$

therefore,

$$d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = [(\mathbf{F}_t^{-1} d\mathbf{x}'^{(1)})] \cdot [(\mathbf{F}_t^{-1} d\mathbf{x}'^{(2)})] = d\mathbf{x}'^{(1)} \cdot \mathbf{B}_t^{-1} d\mathbf{x}'^{(2)} \quad (\text{iv})$$

Let  $d\mathbf{x} = ds\mathbf{e}_1$  be a material element at the current time  $t$  and  $d\mathbf{x}' = ds'\mathbf{n}$  be the same material element at time  $\tau$ , (where  $\mathbf{e}_1$  is a unit vector in a coordinate direction and  $\mathbf{n}$  is a unit vector), then Eq. (8.7.8) gives

$$(C_t)_{11} = \mathbf{e}_1 \cdot \mathbf{C}_t \mathbf{e}_1 = \left( \frac{ds'}{ds} \right)^2 \quad (8.7.11)$$

On the other hand, if  $d\mathbf{x}' = ds'\mathbf{e}_1$  is a material element at time  $\tau$  and  $d\mathbf{x} = ds\mathbf{n}$  is the same material element at current time  $t$ , then Eq. (8.7.9) gives

$$(B_t^{-1})_{11} = \mathbf{e}_1 \cdot \mathbf{B}_t^{-1} \mathbf{e}_1 = \left( \frac{ds}{ds'} \right)^2 \quad (8.7.12)$$

The meaning of the other components can be obtained using Eq. (8.7.8) and (8.7.9). [See also Sections 3.23 to 3.26 on finite deformation tensors in Chapter 3. However, care must be taken in comparing equations in those sections with those in this chapter because of the difference in reference configurations.]

We note that

$$\mathbf{C}_t(\mathbf{x}, t) = \mathbf{B}_t(\mathbf{x}, t) = \mathbf{C}_t^{-1}(\mathbf{x}, t) = \mathbf{B}_t^{-1}(\mathbf{x}, t) = \mathbf{I} \quad (8.7.13)$$

## 8.8 Calculations of the Relative Deformation Tensor

### (A) Rectangular Coordinates

With the motion given by:

$$x_1' = x_1'(x_1, x_2, x_3, \tau), \quad x_2' = x_2'(x_1, x_2, x_3, \tau), \quad x_3' = x_3'(x_1, x_2, x_3, \tau) \quad (8.8.1)$$

Equations (8.7.5) and (8.6.4) give

$$(C_t)_{11} = \left( \frac{\partial x_1'}{\partial x_1} \right)^2 + \left( \frac{\partial x_2'}{\partial x_1} \right)^2 + \left( \frac{\partial x_3'}{\partial x_1} \right)^2 \quad (8.8.2a)$$

$$(C_t)_{22} = \left( \frac{\partial x_1'}{\partial x_2} \right)^2 + \left( \frac{\partial x_2'}{\partial x_2} \right)^2 + \left( \frac{\partial x_3'}{\partial x_2} \right)^2 \quad (8.8.2.b)$$

$$(C_t)_{33} = \left( \frac{\partial x_1'}{\partial x_3} \right)^2 + \left( \frac{\partial x_2'}{\partial x_3} \right)^2 + \left( \frac{\partial x_3'}{\partial x_3} \right)^2 \quad (8.8.2.c)$$

$$(C_t)_{12} = \left( \frac{\partial x_1'}{\partial x_1} \right) \left( \frac{\partial x_1'}{\partial x_2} \right) + \left( \frac{\partial x_2'}{\partial x_1} \right) \left( \frac{\partial x_2'}{\partial x_2} \right) + \left( \frac{\partial x_3'}{\partial x_1} \right) \left( \frac{\partial x_3'}{\partial x_2} \right) \quad (8.8.2.d)$$

$$(C_t)_{23} = \left( \frac{\partial x_1'}{\partial x_2} \right) \left( \frac{\partial x_1'}{\partial x_3} \right) + \left( \frac{\partial x_2'}{\partial x_2} \right) \left( \frac{\partial x_2'}{\partial x_3} \right) + \left( \frac{\partial x_3'}{\partial x_2} \right) \left( \frac{\partial x_3'}{\partial x_3} \right) \quad (8.8.2.e)$$

$$(C_t)_{13} = \left( \frac{\partial x_1'}{\partial x_1} \right) \left( \frac{\partial x_1'}{\partial x_3} \right) + \left( \frac{\partial x_2'}{\partial x_1} \right) \left( \frac{\partial x_2'}{\partial x_3} \right) + \left( \frac{\partial x_3'}{\partial x_1} \right) \left( \frac{\partial x_3'}{\partial x_3} \right) \quad (8.8.2.f)$$

To obtain the components of  $C_t^{-1}$ , one can either invert the symmetric matrix whose components are given by Eqs. (8.8.2), or one can obtain them from the inverse functions of Eq. (8.8.1), i.e.,

$$x_1 = x_1(x_1', x_2', x_3', \tau), \quad x_2 = x_2(x_1', x_2', x_3', \tau), \quad x_3 = x_3(x_1', x_2', x_3', \tau) \quad (8.8.3)$$

Indeed, it can be obtained

$$(C_t^{-1})_{11} = \left( \frac{\partial x_1}{\partial x_1'} \right)^2 + \left( \frac{\partial x_1}{\partial x_2'} \right)^2 + \left( \frac{\partial x_1}{\partial x_3'} \right)^2 \quad (8.8.4.a)$$

$$(C_t^{-1})_{22} = \left( \frac{\partial x_2}{\partial x_1'} \right)^2 + \left( \frac{\partial x_2}{\partial x_2'} \right)^2 + \left( \frac{\partial x_2}{\partial x_3'} \right)^2 \quad (8.8.4.b)$$

$$(C_t^{-1})_{12} = \left( \frac{\partial x_1}{\partial x_1'} \right) \left( \frac{\partial x_2}{\partial x_1'} \right) + \left( \frac{\partial x_1}{\partial x_2'} \right) \left( \frac{\partial x_2}{\partial x_2'} \right) + \left( \frac{\partial x_1}{\partial x_3'} \right) \left( \frac{\partial x_2}{\partial x_3'} \right) \quad (8.8.4.c)$$

etc.

### Example 8.8.1

Find the relative right Cauchy-Green deformation tensor and its inverse for the velocity field given in Example 8.5.1

*Solution.* Since

$$x_1' = x_1 + v(x_2)(\tau - t), \quad x_2' = x_2, \quad x_3' = x_3, \quad (i)$$

we have, with  $k \equiv dv/dx_2$ ,

$$[\mathbf{F}_t] = \begin{bmatrix} 1 & k(\tau-t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{ii}$$

and

$$[\mathbf{C}_t] = \begin{bmatrix} 1 & 0 & 0 \\ k(\tau-t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & k(\tau-t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k(\tau-t) & 0 \\ k(\tau-t) & k^2(\tau-t)^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{iii}$$

The inverse of Eqs.(i) are

$$x_1 = x_1' - v(x_2)(\tau-t), \quad x_2 = x_2', \quad x_3 = x_3', \tag{iv}$$

$$[\mathbf{F}_t^{-1}] = \begin{bmatrix} 1 & -k(\tau-t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{vii}$$

$$[\mathbf{C}_t^{-1}] = \begin{bmatrix} 1 & -k(\tau-t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -k(\tau-t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+k^2(\tau-t)^2 & -k(\tau-t) & 0 \\ -k(\tau-t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{viii}$$

(B) Cylindrical Coordinates

The procedures described below for obtaining the formulas for computing the components for the relative right Cauchy-Green tensors, are the same as those used in Section 3.30 of Chapter 3.

We have

$$dx' = dr'e'_r + r'd\theta'e'_\theta + dz'e'_z, \quad dx = dre_r + rd\theta e_\theta + dze_z \tag{8.8.5}$$

Thus, from  $dx' = \mathbf{F}_t dx$ ,

$$dr' = dr(\mathbf{e}'_r \cdot \mathbf{F}_t \mathbf{e}_r) + r d\theta(\mathbf{e}'_r \cdot \mathbf{F}_t \mathbf{e}_\theta) + dz(\mathbf{e}'_r \cdot \mathbf{F}_t \mathbf{e}_z) \tag{i}$$

But from calculus

$$dr' = \frac{\partial r'}{\partial r} dr + \frac{\partial r'}{\partial \theta} d\theta + \frac{\partial r'}{\partial z} dz \tag{ii}$$

Thus, we have

$$\mathbf{e}'_r \cdot \mathbf{F}_t \mathbf{e}_r = \frac{\partial r'}{\partial r} \quad \mathbf{e}'_r \cdot \mathbf{F}_t \mathbf{e}_\theta = \frac{1}{r} \frac{\partial r'}{\partial \theta} \quad \mathbf{e}'_r \cdot \mathbf{F}_t \mathbf{e}_z = \frac{\partial r'}{\partial z} \tag{iii}$$

Similarly, one can obtain

$$\mathbf{e}'_\theta \cdot \mathbf{F}_t \mathbf{e}_r = r' \frac{\partial \theta'}{\partial r} \quad \mathbf{e}'_\theta \cdot \mathbf{F}_t \mathbf{e}_\theta = \frac{r'}{r} \frac{\partial \theta'}{\partial \theta} \quad \mathbf{e}'_\theta \cdot \mathbf{F}_t \mathbf{e}_z = r' \frac{\partial \theta'}{\partial z} \tag{iv}$$

and

$$\mathbf{e}'_z \cdot \mathbf{F}_t \mathbf{e}_r = \frac{\partial z'}{\partial r} \quad \mathbf{e}'_z \cdot \mathbf{F}_t \mathbf{e}_\theta = \frac{1}{r} \frac{\partial z'}{\partial \theta} \quad \mathbf{e}'_z \cdot \mathbf{F}_t \mathbf{e}'_z = \frac{\partial z'}{\partial z} \tag{v}$$

Equations (iii) to (v) are equivalent to the following equations:

$$\mathbf{F}_t \mathbf{e}_r = \frac{\partial r'}{\partial r} \mathbf{e}'_r + r' \frac{\partial \theta'}{\partial r} \mathbf{e}'_\theta + \frac{\partial z'}{\partial r} \mathbf{e}'_z \tag{8.8.6a}$$

$$\mathbf{F}_t \mathbf{e}_\theta = \frac{1}{r} \frac{\partial r'}{\partial \theta} \mathbf{e}'_r + \frac{r'}{r} \frac{\partial \theta'}{\partial \theta} \mathbf{e}'_\theta + \frac{1}{r} \frac{\partial z'}{\partial \theta} \mathbf{e}'_z \tag{8.8.6b}$$

$$\mathbf{F}_t \mathbf{e}_z = \frac{\partial r'}{\partial z} \mathbf{e}'_r + r' \frac{\partial \theta'}{\partial z} \mathbf{e}'_\theta + \frac{\partial z'}{\partial z} \mathbf{e}'_z \tag{8.8.6c}$$

As already noted in the previous section, the matrix

$$[\mathbf{F}_t] = \begin{bmatrix} \frac{\partial r'}{\partial r} & \frac{1}{r} \frac{\partial r'}{\partial \theta} & \frac{\partial r'}{\partial z} \\ r' \frac{\partial \theta'}{\partial r} & \frac{r'}{r} \frac{\partial \theta'}{\partial \theta} & r' \frac{\partial \theta'}{\partial z} \\ \frac{\partial z'}{\partial r} & \frac{1}{r} \frac{\partial z'}{\partial \theta} & \frac{\partial z'}{\partial z} \end{bmatrix}_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z, \mathbf{e}'_r, \mathbf{e}'_\theta, \mathbf{e}'_z} \tag{8.8.7}$$

being obtained using bases at two different points, give the **two point components** of the tensor  $\mathbf{F}_t$ . Now, from

$$\mathbf{C}_t = \mathbf{F}_t^T \mathbf{F}_t$$

we have

$$\begin{aligned} (\mathbf{C}_t)_{rr} &= \mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{F}_t \mathbf{e}_r = \mathbf{e}_r \cdot \mathbf{F}_t^T \left[ \frac{\partial r'}{\partial r} \mathbf{e}'_r + r' \frac{\partial \theta'}{\partial r} \mathbf{e}'_\theta + \frac{\partial z'}{\partial r} \mathbf{e}'_z \right] \\ &= \frac{\partial r'}{\partial r} (\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_r) + r' \frac{\partial \theta'}{\partial r} (\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_\theta) + \frac{\partial z'}{\partial r} (\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_z) \\ &= \frac{\partial r'}{\partial r} (\mathbf{e}'_r \cdot \mathbf{F}_t \mathbf{e}_r) + r' \frac{\partial \theta'}{\partial r} (\mathbf{e}'_\theta \cdot \mathbf{F}_t \mathbf{e}_r) + \frac{\partial z'}{\partial r} (\mathbf{e}'_z \cdot \mathbf{F}_t \mathbf{e}_r) \\ &= \left( \frac{\partial r'}{\partial r} \right)^2 + \left( r' \frac{\partial \theta'}{\partial r} \right)^2 + \left( \frac{\partial z'}{\partial r} \right)^2 \tag{vi} \\ (\mathbf{C}_t)_{r\theta} &= \mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{F}_t \mathbf{e}_\theta = \mathbf{e}_r \cdot \mathbf{F}_t^T \left[ \frac{1}{r} \frac{\partial r'}{\partial \theta} \mathbf{e}'_r + \frac{r'}{r} \frac{\partial \theta'}{\partial \theta} \mathbf{e}'_\theta + \frac{1}{r} \frac{\partial z'}{\partial \theta} \mathbf{e}'_z \right] \\ &= \frac{1}{r} \frac{\partial r'}{\partial \theta} (\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_r) + \frac{r'}{r} \frac{\partial \theta'}{\partial \theta} (\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_\theta) + \frac{1}{r} \frac{\partial z'}{\partial \theta} (\mathbf{e}_r \cdot \mathbf{F}_t^T \mathbf{e}'_z) \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{r} \frac{\partial r'}{\partial \theta'} (\mathbf{e}'_r \cdot \mathbf{F}_t \mathbf{e}_r) + \frac{r'}{r} \frac{\partial \theta'}{\partial \theta'} (\mathbf{e}'_\theta \cdot \mathbf{F}_t \mathbf{e}_r) + \frac{1}{r} \frac{\partial z'}{\partial \theta'} (\mathbf{e}'_z \cdot \mathbf{F}_t \mathbf{e}_r) \\
 &= \left( \frac{1}{r} \frac{\partial r'}{\partial \theta'} \right) \left( \frac{\partial r'}{\partial r} \right) + \frac{r'}{r} \left( \frac{\partial \theta'}{\partial \theta'} \right) \left( \frac{r' \partial \theta'}{\partial r} \right) + \left( \frac{1}{r} \frac{\partial z'}{\partial \theta'} \right) \left( \frac{\partial z'}{\partial r} \right)
 \end{aligned} \tag{vii}$$

Other components can be derived similarly. Thus, with the pathline equations given by

$$r' = r'(r, \theta, z, \tau), \quad \theta' = \theta'(r, \theta, z, \tau), \quad z' = z'(r, \theta, z, \tau) \tag{8.8.8}$$

the components of  $\mathbf{C}_t$  with respect to the bases  $\mathbf{e}_r, \mathbf{e}_\theta$  and  $\mathbf{e}_z$  are:

$$(\mathbf{C}_t)_{rr} = \left( \frac{\partial r'}{\partial r} \right)^2 + \left( r' \frac{\partial \theta'}{\partial r} \right)^2 + \left( \frac{\partial z'}{\partial r} \right)^2 \tag{8.8.9a}$$

$$(\mathbf{C}_t)_{\theta\theta} = \frac{1}{r^2} \left[ \left( \frac{\partial r'}{\partial \theta} \right)^2 + \left( r' \frac{\partial \theta'}{\partial \theta} \right)^2 + \left( \frac{\partial z'}{\partial \theta} \right)^2 \right] \tag{8.8.9b}$$

$$(\mathbf{C}_t)_{zz} = \left( \frac{\partial r'}{\partial z} \right)^2 + \left( r' \frac{\partial \theta'}{\partial z} \right)^2 + \left( \frac{\partial z'}{\partial z} \right)^2 \tag{8.8.9c}$$

$$(\mathbf{C}_t)_{r\theta} = \frac{1}{r} \left[ \left( \frac{\partial r'}{\partial r} \right) \left( \frac{\partial r'}{\partial \theta} \right) + r'^2 \left( \frac{\partial \theta'}{\partial r} \right) \left( \frac{\partial \theta'}{\partial \theta} \right) + \left( \frac{\partial z'}{\partial r} \right) \left( \frac{\partial z'}{\partial \theta} \right) \right] \tag{8.8.9d}$$

$$(\mathbf{C}_t)_{rz} = \left[ \left( \frac{\partial r'}{\partial r} \right) \left( \frac{\partial r'}{\partial z} \right) + r'^2 \left( \frac{\partial \theta'}{\partial r} \right) \left( \frac{\partial \theta'}{\partial z} \right) + \left( \frac{\partial z'}{\partial r} \right) \left( \frac{\partial z'}{\partial z} \right) \right] \tag{8.8.9e}$$

$$(\mathbf{C}_t)_{\theta z} = \frac{1}{r} \left[ \left( \frac{\partial r'}{\partial \theta} \right) \left( \frac{\partial r'}{\partial z} \right) + r'^2 \left( \frac{\partial \theta'}{\partial \theta} \right) \left( \frac{\partial \theta'}{\partial z} \right) + \left( \frac{\partial z'}{\partial \theta} \right) \left( \frac{\partial z'}{\partial z} \right) \right] \tag{8.8.9f}$$

To obtain the components of  $\mathbf{C}_t^{-1}$ , one can either invert the symmetric matrix whose components are given by Eqs. (8.8.9), or one can obtain them from the inverse functions of Eq. (8.8.8), i.e.,

$$r = r(r', \theta', z', \tau), \quad \theta = \theta(r', \theta', z', \tau), \quad z = z(r', \theta', z', \tau) \tag{8.8.10}$$

In fact, from  $d\mathbf{x} = \mathbf{F}_t^{-1} d\mathbf{x}'$ , we obtain

$$dr = \mathbf{e}_r \cdot d\mathbf{x} = dr' (\mathbf{e}_r \cdot \mathbf{F}_t^{-1} \mathbf{e}'_r) + r' d\theta' (\mathbf{e}_r \cdot \mathbf{F}_t^{-1} \mathbf{e}'_\theta) + dz' (\mathbf{e}_r \cdot \mathbf{F}_t^{-1} \mathbf{e}'_z) \tag{viii}$$

$$r d\theta = \mathbf{e}_\theta \cdot d\mathbf{x} = dr' (\mathbf{e}_\theta \cdot \mathbf{F}_t^{-1} \mathbf{e}'_r) + r' d\theta' (\mathbf{e}_\theta \cdot \mathbf{F}_t^{-1} \mathbf{e}'_\theta) + dz' (\mathbf{e}_\theta \cdot \mathbf{F}_t^{-1} \mathbf{e}'_z) \tag{ix}$$

etc. Thus,

$$\mathbf{e}_r \cdot \mathbf{F}_t^{-1} \mathbf{e}'_r = \mathbf{e}'_r \cdot \mathbf{F}_t^{-1T} \mathbf{e}_r = \frac{\partial r}{\partial r'}, \quad \mathbf{e}_r \cdot \mathbf{F}_t^{-1} \mathbf{e}'_\theta = \mathbf{e}'_\theta \cdot \mathbf{F}_t^{-1T} \mathbf{e}_r = \frac{\partial r}{r' \partial \theta'} \tag{x}$$

$$\mathbf{e}_\theta \cdot \mathbf{F}_t^{-1} \mathbf{e}'_r = \mathbf{e}'_r \cdot \mathbf{F}_t^{-1T} \mathbf{e}_\theta = \frac{r\partial\theta}{\partial r'}, \quad \mathbf{e}_\theta \cdot \mathbf{F}_t^{-1} \mathbf{e}'_\theta = \mathbf{e}'_\theta \cdot \mathbf{F}_t^{-1T} \mathbf{e}_\theta = \frac{r\partial\theta}{r'\partial\theta'} \quad (xi)$$

etc. These equations are equivalent to the following equations:

$$\mathbf{F}_t^{-1} \mathbf{e}'_r = \frac{\partial r}{\partial r'} \mathbf{e}_r + \frac{r\partial\theta}{\partial r'} \mathbf{e}_\theta + \frac{\partial z}{\partial r'} \mathbf{e}_z \quad (8.8.11a)$$

$$\mathbf{F}_t^{-1} \mathbf{e}'_\theta = \frac{\partial r}{r'\partial\theta'} \mathbf{e}_r + \frac{r\partial\theta}{r'\partial\theta'} \mathbf{e}_\theta + \frac{\partial z}{r'\partial\theta'} \mathbf{e}_z \quad (8.8.11b)$$

$$\mathbf{F}_t^{-1} \mathbf{e}'_z = \frac{\partial r}{\partial z'} \mathbf{e}_r + \frac{r\partial\theta}{\partial z'} \mathbf{e}_\theta + \frac{\partial z}{\partial z'} \mathbf{e}_z \quad (8.8.11c)$$

and

$$\mathbf{F}_t^{-1T} \mathbf{e}_r = \frac{\partial r}{\partial r'} \mathbf{e}'_r + \frac{\partial r}{r'\partial\theta'} \mathbf{e}'_\theta + \frac{\partial r}{\partial z'} \mathbf{e}'_z \quad (8.8.12a)$$

$$\mathbf{F}_t^{-1T} \mathbf{e}_\theta = \frac{r\partial\theta}{\partial r'} \mathbf{e}'_r + \frac{r\partial\theta}{r'\partial\theta'} \mathbf{e}'_\theta + \frac{r\partial\theta}{\partial z'} \mathbf{e}'_z \quad (8.8.12b)$$

$$\mathbf{F}_t^{-1T} \mathbf{e}_z = \frac{\partial z}{\partial r'} \mathbf{e}'_r + \frac{\partial z}{r'\partial\theta'} \mathbf{e}'_\theta + \frac{\partial z}{\partial z'} \mathbf{e}'_z \quad (8.8.12c)$$

From

$$(\mathbf{C}_t^{-1})_{rr} = \mathbf{e}_r \cdot \mathbf{F}_t^{-1} \mathbf{F}_t^{-1T} \mathbf{e}_r, \quad (\mathbf{C}_t^{-1})_{r\theta} = \mathbf{e}_r \cdot \mathbf{F}_t^{-1} \mathbf{F}_t^{-1T} \mathbf{e}_\theta \quad (8.8.13)$$

etc., we obtain, with the help of Eqs. (8.8.11) and (8.8.12),

$$(\mathbf{C}_t^{-1})_{rr} = \left(\frac{\partial r}{\partial r'}\right)^2 + \left(\frac{\partial r}{r'\partial\theta'}\right)^2 + \left(\frac{\partial r}{\partial z'}\right)^2 \quad (8.8.14a)$$

$$(\mathbf{C}_t^{-1})_{\theta\theta} = \left[ \left(\frac{r\partial\theta}{\partial r'}\right)^2 + \left(\frac{r\partial\theta}{r'\partial\theta'}\right)^2 + \left(\frac{r\partial\theta}{\partial z'}\right)^2 \right] \quad (8.8.14b)$$

$$(\mathbf{C}_t^{-1})_{r\theta} = \left[ \left(\frac{\partial r}{\partial r'}\right) \left(\frac{r\partial\theta}{\partial r'}\right) + \left(\frac{\partial r}{r'\partial\theta'}\right) \left(\frac{r\partial\theta}{r'\partial\theta'}\right) + \left(\frac{\partial r}{\partial z'}\right) \left(\frac{r\partial\theta}{\partial z'}\right) \right] \quad (8.8.14c)$$

The other components can be easily written down following the patterns given in the above equations..

(C) Spherical coordinates

With path line equations given by

$$r' = r'(\tau, r, \theta, \phi, t), \quad \theta' = \theta'(\tau, r, \theta, \phi, t), \quad \phi' = \phi'(\tau, r, \theta, \phi, t) \quad (8.8.15)$$

the components of  $\mathbf{C}_t$  with respect to  $\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi$  can be obtained to be

$$(C_t)_{rr} = \left(\frac{\partial r'}{\partial r}\right)^2 + \left(r' \frac{\partial \theta'}{\partial r}\right)^2 + \left(\frac{r' \sin \theta' \partial \phi'}{\partial r}\right)^2 \tag{8.8.16a}$$

$$(C_t)_{\theta\theta} = \frac{1}{r'^2} \left[ \left(\frac{\partial r'}{\partial \theta}\right)^2 + \left(r' \frac{\partial \theta'}{\partial \theta}\right)^2 + \left(\frac{r' \sin \theta' \partial \phi'}{\partial \theta}\right)^2 \right] \tag{8.8.16b}$$

$$(C_t)_{\phi\phi} = \frac{1}{(r \sin \theta)^2} \left[ \left(\frac{\partial r'}{\partial \phi}\right)^2 + \left(r' \frac{\partial \theta'}{\partial \phi}\right)^2 + \left(\frac{r' \sin \theta' \partial \phi'}{\partial \phi}\right)^2 \right] \tag{8.8.16c}$$

$$(C_t)_{r\theta} = \frac{1}{r} \left[ \left(\frac{\partial r'}{\partial r}\right) \left(\frac{\partial r'}{\partial \theta}\right) + r'^2 \left(\frac{\partial \theta'}{\partial r}\right) \left(\frac{\partial \theta'}{\partial \theta}\right) + (r' \sin \theta')^2 \left(\frac{\partial \phi'}{\partial r}\right) \left(\frac{\partial \phi'}{\partial \theta}\right) \right] \tag{8.8.16d}$$

$$(C_t)_{\theta\phi} = \frac{1}{r^2 \sin \theta} \left[ \left(\frac{\partial r'}{\partial \theta}\right) \left(\frac{\partial r'}{\partial \phi}\right) + r'^2 \left(\frac{\partial \theta'}{\partial \theta}\right) \left(\frac{\partial \theta'}{\partial \phi}\right) + (r' \sin \theta')^2 \left(\frac{\partial \phi'}{\partial \theta}\right) \left(\frac{\partial \phi'}{\partial \phi}\right) \right] \tag{8.8.16e}$$

$$(C_t)_{\phi r} = \frac{1}{r \sin \theta} \left[ \left(\frac{\partial r'}{\partial r}\right) \left(\frac{\partial r'}{\partial \phi}\right) + r'^2 \left(\frac{\partial \theta'}{\partial r}\right) \left(\frac{\partial \theta'}{\partial \phi}\right) + (r' \sin \theta')^2 \left(\frac{\partial \phi'}{\partial r}\right) \left(\frac{\partial \phi'}{\partial \phi}\right) \right] \tag{8.8.16f}$$

Again, using the inverse functions of Eqs. (8.8.15), we can obtain the following components

$$(C_t^{-1})_{rr} = \left(\frac{\partial r}{\partial r'}\right)^2 + \left(\frac{\partial r}{r' \partial \theta'}\right)^2 + \left(\frac{\partial r}{r' \sin \theta' \partial \phi'}\right)^2 \tag{8.8.17a}$$

$$(C_t^{-1})_{\theta\theta} = \left(\frac{r \partial \theta}{\partial r'}\right)^2 + \left(\frac{r \partial \theta}{r' \partial \theta'}\right)^2 + \left(\frac{r \partial \theta}{r' \sin \theta' \partial \phi'}\right)^2 \tag{8.8.17b}$$

$$(C_t^{-1})_{r\theta} = \left(\frac{\partial r}{\partial r'}\right) \left(\frac{r \partial \theta}{\partial r'}\right) + \left(\frac{\partial r}{r' \partial \theta'}\right) \left(\frac{r \partial \theta}{r' \partial \theta'}\right) + \left(\frac{\partial r}{r' \sin \theta' \partial \phi'}\right) \left(\frac{r \partial \theta}{r' \sin \theta' \partial \phi'}\right) \tag{8.8.17c}$$

The other components can be easily written down following the patterns given in the above equations.

### 8.9 History of the Relative Deformation Tensor. Rivlin-Ericksen Tensors

The tensor  $C_t(\mathbf{x}, \tau)$  describes the deformation at time  $\tau$  of the element which is at  $\mathbf{x}$  at time  $t$ . Thus, as one varies  $\tau$  from  $\tau = -\infty$  to  $\tau = t$  in the function  $C_t(\mathbf{x}, \tau)$ , one gets the whole history of the deformation from infinitely long time ago to the present time  $t$ .

If we assume that we can expand the components of  $C_t$  in Taylor series about  $\tau = t$ , we have,

$$C_t(\mathbf{x}, \tau) = C_t(\mathbf{x}, t) + \left(\frac{\partial C_t}{\partial \tau}\right)_{\tau=t} (\tau-t) + \frac{1}{2} \left(\frac{\partial^2 C_t}{\partial \tau^2}\right)_{\tau=t} (\tau-t)^2 + \dots \tag{8.9.1}$$

Let

$$\mathbf{A}_1 = \left( \frac{\partial \mathbf{C}_t}{\partial \tau} \right)_{\tau=t} \tag{8.9.2a}$$

$$\mathbf{A}_2 = \left( \frac{\partial^2 \mathbf{C}_t}{\partial \tau^2} \right)_{\tau=t} \tag{8.9.2b}$$

and

$$\mathbf{A}_N = \left( \frac{\partial^N \mathbf{C}_t}{\partial \tau^N} \right)_{\tau=t}, \quad N=3,4,\dots \tag{8.9.2c}$$

We have, (note,  $\mathbf{C}_t(\mathbf{x},t) = \mathbf{I}$ )

$$\mathbf{C}_t(\mathbf{x},\tau) = \mathbf{I} + (\tau-t)\mathbf{A}_1 + \frac{(\tau-t)^2}{2}\mathbf{A}_2 + \dots \tag{8.9.3}$$

The tensor  $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$  are known as **Rivlin-Ericksen tensors**.

We see from the above equation that provided the Taylor series expansion is valid, the  $\mathbf{A}_n$ 's determine the history of relative deformation. It should be noted however, that not all histories of relative deformation can be expanded in Taylor series; For example, the stress relaxation test in which a sudden jump in deformation is imposed on the fluid, has a history of deformation which is not representable by a Taylor series.

### Example 8.9.1

Find the Rivlin-Ericksen tensor for the uni-directional flows of Example 8.8.1.

*Solution.* We have, from Example 8.8.1

$$\begin{aligned} [\mathbf{C}_t(\mathbf{x},\tau)] &= \begin{bmatrix} 1 & k(\tau-t) & 0 \\ k(\tau-t) & 1+k^2(\tau-t)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\tau-t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{(\tau-t)^2}{2} \end{aligned} \tag{i}$$

Thus, (see Eq. (8.9.3))

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

$$[A_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

$$[A_n] = 0 \text{ for all } n \geq 3$$

where  $k = dv/dx_2$ .

Example 8.9.2

Given an axisymmetric velocity field in cylindrical coordinates:

$$v_r = 0, \quad v_\theta = 0, \quad v_z = v(r) \tag{i}$$

- (a) Obtain the motion using current time  $t$  as reference
- (b) Compute the relative deformation tensor  $C_t$
- (c) Find the Rivlin-Ericksen tensors.

*Solution.* (a) Let the motion be

$$r' = r'(r, \theta, z, \tau), \quad \theta' = \theta'(r, \theta, z, \tau), \quad z' = z'(r, \theta, z, \tau) \tag{ii}$$

then, from the given velocity field, we have

$$\frac{dr'}{d\tau} = 0, \quad \frac{d\theta'}{d\tau} = 0, \quad \frac{dz'}{d\tau} = v(r') \tag{iii}$$

Integration of these equations with the conditions that at  $\tau = t$ ,  $r' = r$ ,  $\theta' = \theta$  and  $z' = z$ , we obtain

$$r' = r, \quad \theta' = \theta, \quad z' = z + v(r)(\tau - t) \tag{iv}$$

(b) Using Eq. (8.8.9), we obtain, with  $k(r) \equiv dv/dr$

$$[C_t] = \begin{bmatrix} 1+k^2(\tau-t)^2 & 0 & k(\tau-t) \\ 0 & 1 & 0 \\ k(\tau-t) & 0 & 1 \end{bmatrix} \tag{v}$$

(c) From the result of part (b), we have

$$[C_t] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & k(r) \\ 0 & 0 & 0 \\ k(r) & 0 & 0 \end{bmatrix} (\tau-t) + \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{(\tau-t)^2}{2} \tag{vi}$$

Thus, the Rivlin-Ericksen tensors are [see Eq. (8.9.3)]

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & 0 & k(r) \\ 0 & 0 & 0 \\ k(r) & 0 & 0 \end{bmatrix} \tag{v}$$

$$[\mathbf{A}_2] = \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{vi}$$

$$[\mathbf{A}_n] = 0 \quad \text{for } n \geq 3 \tag{vii}$$

Example 8.9.3

Consider the Couette flow with a velocity field given in cylindrical coordinates as

$$v_r = 0, \quad v_\theta = v(r), \quad v_z = 0 \tag{i}$$

- (a) Obtain the motion using current time  $t$  as reference.
- (b) Compute the relative deformation tensor  $\mathbf{C}_t$ .
- (c) Find the Rivlin-Ericksen tensors.

*Solution.* (a) From the given velocity field, one has

$$\frac{dr'}{d\tau} = 0, \quad r' \frac{d\theta'}{d\tau} = v(r'), \quad \frac{dz'}{d\tau} = 0 \tag{ii}$$

Integration of the above equation gives the pathline equations to be:

$$r' = r, \quad \theta' = \theta + \frac{v(r)}{r} (\tau - t), \quad z' = z \tag{iii}$$

(b) Using Eqs. (8.8.9), one easily obtain the relative right Cauchy-Green deformation tensor to be

$$[\mathbf{C}_t(\tau)] = \begin{bmatrix} 1 + k^2(r)(\tau-t)^2 & k(r)(\tau-t) & 0 \\ k(r)(\tau-t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ = [\mathbf{I}] + (\tau-t) \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{(\tau-t)^2}{2} \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iv}$$

$$k(r) = \left( \frac{dv}{dr} - \frac{v}{r} \right) \tag{v}$$

(c) The nonzero Rivlin Ericksen tensors are

$$[A_1]_{e_r, e_\theta, e_z} = \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{vi}$$

$$[A_2]_{e_r, e_\theta, e_z} = \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{vii}$$

Example 8.9.4

Given the velocity field of a sink flow in spherical coordinates:

$$v_r = -\frac{a}{r^2}, \quad v_\theta = 0, \quad v_\phi = 0 \tag{i}$$

(a) Obtain the motion using current time  $t$  as reference

(b) Compute the relative deformation tensor  $C_t$

(c) Find the Rivlin-Ericksen tensors.

*Solution.* (a) Let the motion be

$$r' = r'(r, \theta, \phi, \tau), \quad \theta' = \theta'(r, \theta, \phi, \tau), \quad \phi' = \phi'(r, \theta, \phi, \tau) \tag{ii}$$

then, from the given velocity field, we have

$$\frac{dr'}{d\tau} = -\frac{a}{r'^2}, \quad \frac{d\theta'}{d\tau} = 0, \quad \frac{d\phi'}{d\tau} = 0 \tag{iii}$$

Integration of these equations with the conditions that at  $\tau = t$ ,  $r' = r$ ,  $\theta' = \theta$  and  $\phi' = \phi$ , we obtain

$$r'^3 = r^3 + 3a(t-\tau), \quad \theta' = \theta, \quad \phi' = \phi \tag{iv}$$

(b) Using Eq. (8.8.16), we obtain, with  $k \equiv dv/dr$

$$[C_t] = \begin{bmatrix} r^4[r^3 + 3a(t-\tau)]^{-4/3} & 0 & 0 \\ 0 & \frac{[r^3 + 3a(t-\tau)]^{2/3}}{r^2} & 0 \\ 0 & 0 & \frac{[r^3 + 3a(t-\tau)]^{2/3}}{r^2} \end{bmatrix} \tag{v}$$

(c) From the result of part (b), we have

$$\begin{aligned} \frac{d(C_t)_{rr}}{d\tau} &= \frac{4a}{r^3} \left[ 1 + \frac{3a(t-\tau)}{r^3} \right]^{-\frac{7}{3}}, \\ \frac{d^2(C_t)_{rr}}{d\tau^2} &= \frac{28a^2}{r^6} \left[ 1 + \frac{3a(t-\tau)}{r^3} \right]^{-\frac{10}{3}}, \\ \frac{d(C_t)_{\theta\theta}}{d\tau} &= \frac{d(C_t)_{\phi\phi}}{d\tau} = -\frac{2a}{r^3} \left[ 1 + \frac{3a(t-\tau)}{r^3} \right]^{-\frac{1}{3}}, \\ \frac{d^2(C_t)_{\theta\theta}}{d\tau^2} &= \frac{d^2(C_t)_{\phi\phi}}{d\tau^2} = -\frac{2a^2}{r^6} \left[ 1 + \frac{3a(t-\tau)}{r^3} \right]^{-\frac{4}{3}}, \\ &\dots\dots\dots \\ &\dots\dots\dots \\ &\text{etc.} \end{aligned}$$

Thus, from Eqs. (8.9.2)

$$[A_1] = \begin{bmatrix} \frac{4a}{r^3} & 0 & 0 \\ 0 & \frac{-2a}{r^3} & 0 \\ 0 & 0 & \frac{-2a}{r^3} \end{bmatrix} \tag{vi}$$

$$[A_2] = \begin{bmatrix} \frac{28a^2}{r^6} & 0 & 0 \\ 0 & \frac{-2a^2}{r^6} & 0 \\ 0 & 0 & \frac{-2a^2}{r^6} \end{bmatrix} \tag{vii}$$

$[A_3], [A_4] \dots$  etc. can be obtained by computing the higher derivatives of the components of  $C_t$  and evaluating them at  $\tau = t$ .

### 8.10 Rivlin-Ericksen Tensor in Terms of Velocity Gradients - The Recursive Formulas

In this section, we show that

$$\mathbf{A}_1 = 2\mathbf{D} = \nabla\mathbf{v} + (\nabla\mathbf{v})^T \tag{8.10.1}$$

$$\mathbf{A}_2 = \frac{D\mathbf{A}_1}{Dt} + \mathbf{A}_1(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{A}_1 \tag{8.10.2}$$

and

$$\mathbf{A}_{N+1} = \frac{D\mathbf{A}_N}{Dt} + \mathbf{A}_N(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{A}_N, \quad N=1,2,3,\dots \tag{8.10.3}$$

where  $\nabla\mathbf{v}$  is the velocity gradient and  $\mathbf{D}$  is the rate of deformation tensor.

We have, at any time  $\tau$

$$ds^{2'} = dx'(\tau) \cdot dx'(\tau) = dx \cdot \mathbf{C}_t dx \tag{i}$$

$$\frac{D}{Dt}(ds^2) \equiv \left[ \frac{\partial}{\partial \tau}(ds^{2'}) \right]_{x_i \text{ - fixed, } \tau=t} = dx \cdot \left( \frac{\partial \mathbf{C}_t}{\partial \tau} \right)_{\tau=t} dx \tag{ii}$$

That is, [see Eqs. (8.9.2)]

$$\frac{D}{Dt}(ds^2) = dx \cdot \mathbf{A}_1 dx \tag{8.10.4}$$

clearly, we also have

$$\frac{D^2}{Dt^2}(ds^2) = dx \cdot \mathbf{A}_2 dx \tag{8.10.5}$$

and

$$\frac{D^N}{Dt^N}(ds^2) = dx \cdot \mathbf{A}_N dx \tag{8.10.6}$$

We now recall from Section 3.13, Eq. (3.13.6a), that

$$\frac{D}{Dt}(ds^2) = 2dx \cdot \mathbf{D} dx$$

where  $\mathbf{D} = \frac{1}{2}[\nabla\mathbf{v} + (\nabla\mathbf{v})^T]$  is the rate of deformation tensor. Thus,

$$\mathbf{A}_1 = 2\mathbf{D}$$

Next, from Eq. (8.10.4),

$$\frac{D^2}{Dt^2}(ds^2) = \left( \frac{D}{Dt} dx \right) \cdot \mathbf{A}_1 dx + dx \cdot \frac{D}{Dt} \mathbf{A}_1 dx + dx \cdot \mathbf{A}_1 \frac{D}{Dt} dx \tag{iii}$$

But

$$\frac{D}{Dt}d\mathbf{x} = (\nabla\mathbf{v})d\mathbf{x} \quad [\text{see Eq. (3.12.4)}]$$

Therefore,

$$\frac{D^2}{Dt^2}(ds^2) = (\nabla\mathbf{v})d\mathbf{x} \cdot \mathbf{A}_1 d\mathbf{x} + d\mathbf{x} \cdot \frac{D\mathbf{A}_1}{Dt} d\mathbf{x} + d\mathbf{x} \cdot \mathbf{A}_1 (\nabla\mathbf{v})d\mathbf{x} \quad (\text{iv})$$

From the definition of transpose

$$(\nabla\mathbf{v})d\mathbf{x} \cdot \mathbf{A}_1 d\mathbf{x} = d\mathbf{x} \cdot (\nabla\mathbf{v})^T \mathbf{A}_1 d\mathbf{x} \quad (\text{v})$$

$$\frac{D^2}{Dt^2}(ds^2) = d\mathbf{x} \cdot \left[ \frac{D\mathbf{A}_1}{Dt} + (\nabla\mathbf{v})^T \mathbf{A}_1 + \mathbf{A}_1 (\nabla\mathbf{v}) \right] d\mathbf{x} \quad (\text{vi})$$

Thus, from Eq. (8.10.5),

$$\mathbf{A}_2 = \frac{D\mathbf{A}_1}{Dt} + \mathbf{A}_1 (\nabla\mathbf{v}) + (\nabla\mathbf{v})^T \mathbf{A}_1$$

Equation (8.10.3) can be similarly proved.

### 8.11 Relation Between Velocity Gradient and Deformation Gradient

From

$$d\mathbf{x}'(\tau) = \mathbf{x}'_t(\mathbf{x} + d\mathbf{x}, \tau) - \mathbf{x}'_t(\mathbf{x}, \tau) \quad (8.11.1)$$

we have

$$d\mathbf{x}' = \mathbf{F}_t(\mathbf{x}, \tau) d\mathbf{x} \quad \text{and} \quad (8.11.2)$$

$$\frac{D}{D\tau}d\mathbf{x}'(\tau) = \mathbf{v}'(\mathbf{x} + d\mathbf{x}, \tau) - \mathbf{v}'(\mathbf{x}, \tau) = \nabla_{\mathbf{x}}\mathbf{v}'(\mathbf{x}, \tau)d\mathbf{x} \quad (8.11.3)$$

From Eq. (8.11.2)

$$\frac{Dd\mathbf{x}'}{D\tau} = \left( \frac{D\mathbf{F}_t}{D\tau} \right) d\mathbf{x} \quad (8.11.4)$$

Comparing Eqs. (8.11.3) and (8.11.4), we have

$$\frac{D\mathbf{F}_t}{D\tau} = \nabla_{\mathbf{x}}\mathbf{v}'(\mathbf{x}, \tau) \quad (8.11.5)$$

and from which

$$\left[ \frac{DF_t}{D\tau} \right]_{\tau=t} = \nabla_{\mathbf{x}^v}(\mathbf{x}, t) \tag{8.11.6}$$

Using this relations, we can obtain the following relations between the rate of deformation tensor  $\mathbf{D}$  and the relative stretch tensor  $\mathbf{U}_t$  as well as the relation between the spin tensor  $\mathbf{W}$  and the relative rotation tensor  $\mathbf{R}_t$ . In fact, from the polar decomposition theorem

$$\mathbf{F}_t(\tau) = \mathbf{R}_t(\tau)\mathbf{U}_t(\tau) \tag{8.11.7}$$

we have

$$\frac{DF_t(\tau)}{D\tau} = \frac{DR_t(\tau)}{D\tau} \mathbf{U}_t(\tau) + \mathbf{R}_t(\tau) \frac{DU_t(\tau)}{D\tau} \tag{8.11.8}$$

Evaluating the above equation at  $\tau = t$ , using Eq. (8.11.6) and noting that  $\mathbf{U}_t(t) = \mathbf{R}_t(t) = \mathbf{I}$ , we obtain

$$\nabla_{\mathbf{x}^v} = \left[ \frac{DR_t(\tau)}{D\tau} \right]_{\tau=t} + \left[ \frac{DU_t(\tau)}{D\tau} \right]_{\tau=t} \tag{8.11.9}$$

Clearly  $\left[ \frac{DU_t(\tau)}{D\tau} \right]_{\tau=t}$  is a symmetric tensor and it can be easily shown that  $\left[ \frac{DR_t(\tau)}{D\tau} \right]_{\tau=t}$  is an antisymmetric tensor. Thus, in view of the fact that the decomposition of a tensor into a symmetric and an antisymmetric tensor is unique, therefore,

$$\left[ \frac{DU_t(\tau)}{D\tau} \right]_{\tau=t} = \mathbf{D}(t) \tag{8.11.10}$$

$$\left[ \frac{DR_t(\tau)}{D\tau} \right]_{\tau=t} = \mathbf{W}(t) \tag{8.11.11}$$

### 8.12 Transformation Laws for the Relative Deformation Tensors under a Change of Frame

The concept of objectivity was discussed in Chapter 5, Section 5.31. We recall that a change of frame, from  $\mathbf{x}$  to  $\mathbf{x}^*$ , is defined by the transformation

$$\mathbf{x}^*(t) = \mathbf{c}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_0) \tag{8.12.1a}$$

$$t^* = t - a \tag{8.12.1b}$$

and if a tensor  $\mathbf{A}$ , in the un-starred frame, transforms to  $\mathbf{A}^*$  in the starred frame in accordance with the relation

$$\mathbf{A}^* = \mathbf{Q}(t)\mathbf{A}\mathbf{Q}^T(t) \tag{8.12.2}$$

then, the tensor  $\mathbf{A}$  is said to be objective, or frame indifferent (i.e., independent of observers). From Eq. (8.12.1), we have

$$d\mathbf{x}^*(t) = \mathbf{Q}(t)d\mathbf{x}(t) \tag{i}$$

and

$$d\mathbf{x}'^*(\tau) = \mathbf{Q}(\tau)d\mathbf{x}'(\tau) \tag{ii}$$

Since  $d\mathbf{x}'(\tau) = \mathbf{F}_t(\tau)d\mathbf{x}(t)$  and  $d\mathbf{x}'^*(\tau) = \mathbf{F}_t^*(\tau)d\mathbf{x}^*(t)$ , therefore from Eq. (ii), we have

$$\mathbf{F}_t^*(\tau)d\mathbf{x}^*(t) = \mathbf{Q}(\tau)d\mathbf{x}'(\tau) = \mathbf{Q}(\tau)\mathbf{F}_t(\tau)d\mathbf{x}(t) \tag{iii}$$

Now, use the inverse of Eq. (i), we get

$$\mathbf{F}_t^*(\tau)d\mathbf{x}^*(t) = \mathbf{Q}(\tau)\mathbf{F}_t(\tau)\mathbf{Q}^T(t)d\mathbf{x}^*(t) \tag{iv}$$

Thus,

$$\mathbf{F}_t^*(\tau) = \mathbf{Q}(\tau)\mathbf{F}_t(\tau)\mathbf{Q}^T(t) \tag{8.12.3}$$

This is the transformation law for  $\mathbf{F}_t(\tau)$  under a change of frame. We see that this tensor is not objective.

Since  $\mathbf{F}_t^* = \mathbf{R}_t^* \mathbf{U}_t^*$  and  $\mathbf{F}_t = \mathbf{R}_t \mathbf{U}_t$ , therefore, from Eq. (8.12.3)

$$\mathbf{R}_t^* \mathbf{U}_t^* = \mathbf{Q}(\tau)\mathbf{R}_t \mathbf{U}_t \mathbf{Q}^T(t) \tag{v}$$

This equation can be written:

$$\mathbf{F}_t^* = \mathbf{R}_t^* \mathbf{U}_t^* = [\mathbf{Q}(\tau)\mathbf{R}_t \mathbf{Q}^T(t)][\mathbf{Q}(t)\mathbf{U}_t \mathbf{Q}^T(t)] \tag{vi}$$

In the above equation, the tensor inside the first bracket is an orthogonal tensor and the tensor inside the second bracket is a symmetric tensor. Since the polar decomposition for  $\mathbf{F}_t^*$  is unique, therefore, we have

$$\mathbf{R}_t^* = \mathbf{Q}(\tau)\mathbf{R}_t \mathbf{Q}^T(t) \tag{8.12.4}$$

$$\mathbf{U}_t^* = \mathbf{Q}(t)\mathbf{U}_t \mathbf{Q}^T(t) \tag{8.12.5}$$

It is a simple matter to show the following transformation laws under a change of frames:

$$\mathbf{C}_t^* = \mathbf{Q}(t)\mathbf{C}_t \mathbf{Q}^T(t) \tag{8.12.6}$$

$$\mathbf{C}_t^{*-1} = \mathbf{Q}(t)\mathbf{C}_t^{-1} \mathbf{Q}^T(t) \tag{8.12.7}$$

$$\mathbf{V}_t^* = \mathbf{Q}(\tau)\mathbf{V}_t \mathbf{Q}^T(\tau) \tag{8.12.8}$$

$$\mathbf{B}_t^* = \mathbf{Q}(\tau)\mathbf{B}_t\mathbf{Q}^T(\tau) \tag{8.12.9}$$

$$\mathbf{B}_t^{*-1} = \mathbf{Q}(\tau)\mathbf{B}_t^{-1}\mathbf{Q}^T(\tau) \tag{8.12.10}$$

Equations (8.12.5),(8.12.6) and (8.12.7) show that the relative right stretch tensor, the relative right Cauchy-Green deformation tensor  $\mathbf{C}_t$  and its inverse  $\mathbf{C}_t^{-1}$  (the relative Finger tensor) are objective. On the other hand,  $\mathbf{V}_t$ ,  $\mathbf{B}_t$  and  $\mathbf{B}_t^{-1}$  are nonobjective. We note, this situation is different from that of the deformation tensor using a fixed reference configuration [See Section 5.31].

From Eq. (8.12.4) and (8.11.11), one can also show that in a change of frame

$$\mathbf{W}^* = \mathbf{Q}(t)\mathbf{W}(t)\mathbf{Q}^T(t) + \frac{d\mathbf{Q}(t)}{dt}\mathbf{Q}^T(t) \tag{8.12.11}$$

which shows, as expected that the spin tensor is non-objective.

Using Eq. (8.12.11), one can derive for any objective tensor  $\mathbf{T}$  (i.e.,  $\mathbf{T}^* = \mathbf{Q}(t)\mathbf{T}\mathbf{Q}^T(t)$ ) that

$$\frac{D\mathbf{T}}{Dt} + \mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T} \tag{8.12.12}$$

is objective, that is, [see Prob. 8.22]

$$\frac{D\mathbf{T}^*}{Dt} + \mathbf{T}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{T}^* = \mathbf{Q}(t)\left[\frac{D\mathbf{T}}{Dt} + \mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T}\right]\mathbf{Q}^T(t) \tag{8.12.13}$$

The expression given in (8.12.12) is known as the **Jaumann derivative** of  $\mathbf{T}$  which will be discussed further in a later section.

### 8.13 Transformation law for the Rivlin-Ericksen Tensors under a Change of Frame

From Eq. (8.12.6)

$$\mathbf{C}_t^*(\tau) = \mathbf{Q}(t)\mathbf{C}_t(\tau)\mathbf{Q}^T(t) \tag{8.13.1}$$

we obtain

$$\frac{D}{Dt}\mathbf{C}_t^*(\tau) = \mathbf{Q}(t)\frac{D}{D\tau}\mathbf{C}_t(\tau)\mathbf{Q}^T(t) \tag{8.13.2}$$

and in fact, for all  $N$ ,

$$\frac{D^N}{Dt^N}\mathbf{C}_t^*(\tau) = \mathbf{Q}(t)\frac{D^N}{D\tau^N}\mathbf{C}_t(\tau)\mathbf{Q}^T(t) \tag{8.13.3}$$

Thus, from Eqs. (8.9.2), we have, for all  $N$

$$\mathbf{A}_N^*(t) = \mathbf{Q}(t)\mathbf{A}_N(\tau)\mathbf{Q}^T(t) \quad (8.13.4)$$

We see therefore that all  $\mathbf{A}_N$  are objective. This is quite to be expected because these tensors characterize the rate and the higher rates of changes of length of material elements at time  $t$  which are independent of the observers.

### 8.14 Incompressible Simple Fluid

An incompressible simple fluid is an isotropic ideal material having the following constitutive equation

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau} \quad (8.14.1)$$

where  $\boldsymbol{\tau}$  depends on the past histories up to the current time  $t$  of the relative deformation tensor  $\mathbf{C}_t$ . In other words, a simple fluid is defined by

$$\mathbf{T} = -p\mathbf{I} + \mathbf{H}[\mathbf{C}_t(\mathbf{x}, \tau); -\infty < \tau \leq t] \quad (8.14.2)$$

where the index  $\tau = -\infty$  to  $t$  indicates that the values of the functional  $\mathbf{H}$  depends on all  $\mathbf{C}_t$  from  $\mathbf{C}_t(\mathbf{x}, -\infty)$  to  $\mathbf{C}_t(\mathbf{x}, t)$ . We note that such a fluid is called "simple" because it depends only on the history of the relative deformation gradient  $\mathbf{F}_t(\tau) = \nabla \mathbf{x}'$  tensor (from which  $\mathbf{C}_t(\tau)$  is obtained), and not on the history of the higher gradient of the relative deformation tensor (e.g.,  $\nabla \nabla \mathbf{x}'$ ).

Obviously, the functional  $\mathbf{H}$  in Eq. (8.14.2) is to be the same for all observers (i.e.,  $\mathbf{H}^* = \mathbf{H}$ ). However, it can not be arbitrary, because it must satisfy the frame indifference requirement. That is, in a change of frame,

$$\mathbf{H}[\mathbf{C}_t^*(\tau)] = \mathbf{Q}(t)\mathbf{H}[\mathbf{C}_t(\tau)]\mathbf{Q}^T(t) \quad (8.14.3)$$

Since  $\mathbf{C}_t(\tau)$  transforms in a change of frame as [see Eq. (8.12.6)]

$$\mathbf{C}_t^* = \mathbf{Q}(t)\mathbf{C}_t\mathbf{Q}^T(t) \quad (8.14.4)$$

therefore, the functional  $\mathbf{H}[\mathbf{C}_t(\mathbf{x}, \tau); -\infty < \tau \leq t]$  must satisfy the condition

$$\mathbf{H}[\mathbf{Q}(t)\mathbf{C}_t\mathbf{Q}^T(t)] = \mathbf{Q}(t)\mathbf{H}[\mathbf{C}_t]\mathbf{Q}^T(t) \quad (8.14.5)$$

We note that Eq. (8.14.5) also states that the fluid defined by Eq. (8.14.2) is an isotropic fluid.

Any function or functional which obeys the condition given by Eq. (8.14.5) is known as an **isotropic function** or **isotropic functional**.

The relationship between stress and deformation given by Eq. (8.14.2) for a simple fluid is completely general. In fact, it includes Newtonian incompressible fluid and Maxwell fluids as

special cases. In this most general form, only very special flow problems can be solved. A class of such flows, called the viscometric flow, will be considered in Part C, using this general form of constitutive equation. However, in the following few sections, we shall first discuss some special constitutive equations. Some of these constitutive equations have been shown to be approximations to the general constitutive equation given in Eq. (8.14.2) under certain conditions (slow flow and/or fading memory). They can also be considered simply as special fluids. For example, a Newtonian incompressible fluid can be considered either as a special fluid by itself or as an approximation to the general simple fluid when it has no memory of its past history of deformation and is under slow flow condition relative to its relaxation time (which is zero).

### 8.15 Special Single Integral Type Nonlinear Constitutive Equations

In Section 8.4, we see that the constitutive equation for the linear Maxwell fluids is defined by

$$\boldsymbol{\tau} = 2 \int_0^\infty f(s) \mathbf{E}(t-s) ds \tag{8.15.1}$$

where  $\mathbf{E}$  is the infinitesimal strain tensor measured with respect to the configuration at time  $t$ . It can be shown that for small deformations, (see Example 8.15.2 below)

$$\mathbf{C}_t - \mathbf{I} = \mathbf{I} - \mathbf{C}_t^{-1} = 2 \mathbf{E} \tag{i}$$

Thus, the following two nonlinear viscoelastic fluids represent natural generalizations of the linear Maxwell fluid in that they reduce to Eq. (8.15.1) under small deformation conditions.

$$\boldsymbol{\tau} = \int_0^\infty f_1(s) [\mathbf{C}_t(t-s) - \mathbf{I}] ds \tag{8.15.2}$$

and

$$\boldsymbol{\tau} = \int_0^\infty f_2(s) [\mathbf{I} - \mathbf{C}_t^{-1}(t-s)] ds \tag{8.15.3}$$

where

$$f_1(s) = f_2(s) = f(s) \tag{8.15.4}$$

and  $f(s)$  may be given by any one of Eqs. (8.4.9), (8.4.10) and (8.4.11).

We note that since  $\mathbf{C}_t(\boldsymbol{\tau})$  is an objective tensor, therefore the constitutive equations defined by Eqs. (8.15.2) and (8.15.3) are frame indifferent. We note also that even though the fluids defined by Eqs. (8.15.2) and (8.15.3), with  $f_1 = f_2$  have the same behaviors at small deformation, they are two different nonlinear viscoelastic fluids, behaving differently at large deformation. Furthermore, if we treat  $f_1(s)$  and  $f_2(s)$  as two different memory functions, then

Eqs. (8.15.2) and (8.15.3) define two nonlinear viscoelastic fluids whose behavior at small deformations are also different.

Example 8.15.1

For the nonlinear viscoelastic fluid defined by Eq. (8.15.2), find the stress components when the fluid is under steady shearing flow defined by the velocity field:

$$v_1 = kx_2, \quad v_2 = v_3 = 0 \tag{i}$$

*Solution.* The relative Cauchy-Green deformation tensor corresponding to this flow was obtained in Example 8.8.1 as:

$$[C_t(\tau)] = \begin{bmatrix} 1 & k(\tau-t) & 0 \\ k(\tau-t) & k^2(\tau-t)^2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{ii}$$

Thus,

$$[C_t(t-s) - \mathbf{I}] = \begin{bmatrix} 0 & -ks & 0 \\ -ks & k^2s^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

Thus, from Eq. (8.15.2)

$$\tau_{11} = \tau_{13} = \tau_{23} = \tau_{33} = 0 \tag{iv}$$

$$\tau_{12} = -k \int_0^\infty s f_1(s) ds \tag{8.15.5}$$

$$\tau_{22} = k^2 \int_0^\infty s^2 f_1(s) ds \tag{8.15.6}$$

We see that for this fluid, the viscosity is given by

$$\mu \equiv \tau_{12}/k = - \int_0^\infty s f_1(s) ds \tag{8.15.7}$$

We also note that the normal stresses are not equal in the simple shearing flow. In fact,

$$T_{11} = -p + \tau_{11} = -p \tag{v}$$

$$T_{22} = -p + \tau_{22} = -p + k^2 \int_0^\infty s^2 f_1(s) ds \tag{vi}$$

$$T_{33} = -p + \tau_{33} = -p \tag{vii}$$

We see from this example that for the nonlinear viscoelastic fluid defined by Eq. (15.2), i.e.,

$$\boldsymbol{\tau} = 2 \int_0^{\infty} f_1(s) [C_t(t-s) - \mathbf{I}] ds \quad (8.15.2)$$

the viscosity function  $\mu(k)$  is given by

$$\mu(k) \equiv \tau_{12} = -k \int_0^{\infty} s f_1(s) ds \quad (8.15.8)$$

and the two normal stress functions are given either by

$$\sigma_1(k) \equiv T_{11} - T_{22} = -k^2 \int_0^{\infty} s^2 f_1(s) ds \quad (8.15.9a)$$

$$\sigma_2(k) \equiv T_{22} - T_{33} = k^2 \int_0^{\infty} s^2 f_1(s) ds \quad (8.15.10a)$$

or

$$\bar{\sigma}_1(k) \equiv T_{22} - T_{33} = k^2 \int_0^{\infty} s^2 f_1(s) ds \quad (8.15.9b)$$

$$\bar{\sigma}_2(k) \equiv T_{11} - T_{33} = 0 \quad (8.15.10b)$$

The shear stress function, and the two normal stress functions (either  $\sigma_1$  and  $\sigma_2$ , or  $\bar{\sigma}_1$  and  $\bar{\sigma}_2$ ) completely describe the material properties of this nonlinear viscoelastic fluid in the simple shearing flow. In part C we will show that these three material functions completely describe the material properties of every simple fluid, of which the present nonlinear fluid is a special case, in viscometric flows, of which the simple shearing flow is a special case.

Similarly, for the nonlinear viscoelastic fluid defined by Eq. (8.15.3),

$$\boldsymbol{\tau} = \int_0^{\infty} f_2(s) [\mathbf{I} - C_t^{-1}(t-s)] ds \quad (8.15.3)$$

the viscosity function and the two normal stress functions can be obtained to be

$$\mu(k) = -k \int_0^{\infty} s f_2(s) ds \quad (8.15.11)$$

$$\sigma_1(k) = -k^2 \int_0^{\infty} s^2 f_2(s) ds \quad (8.15.12)$$

$$\sigma_2(k) = 0 \tag{8.15.13}$$

A special nonlinear viscoelastic fluid defined by Eq. (8.15.3) with a memory function dependent on the second invariant  $I_2$  of the tensor  $C_t$  in the following way

$$\begin{aligned} f_2(s) &= f(s) && \text{when } I_2 \geq B^2 + 3 \\ f_2(s) &= 0 && \text{when } I_2 < B^2 + 3 \end{aligned} \tag{8.15.14}$$

is known as Tanner and Simmons network model fluid. The function  $f(s)$  is given by Eq. (8.4.9). For this fluid, the network “breaks” when a scalar measure of the deformation  $I_2$  reaches a limiting value  $B^2 + 3$ , where  $B$  is called the “strength” of the network.

### Example 8.15.2

Show that for small deformations relative to the configuration at time  $t$

$$C_t - I \approx I - C_t^{-1} \approx 2E \tag{8.15.15}$$

where  $E$  is the infinitesimal strain tensor.

*Solution.* Let  $\mathbf{u}$  denote the displacement vector measured from the configuration at time  $t$ . Then

$$\mathbf{x}'(\tau) = \mathbf{x} + \mathbf{u}(\mathbf{x}, \tau) \tag{i}$$

Thus,

$$\mathbf{F}_t = \nabla \mathbf{x}' = \mathbf{I} + \nabla \mathbf{u} \tag{ii}$$

Now, if  $\mathbf{u}$  is infinitesimal, then

$$C_t = \mathbf{F}_t^T \mathbf{F}_t = [\mathbf{I} + (\nabla \mathbf{u})^T][\mathbf{I} + \nabla \mathbf{u}] \approx \mathbf{I} + 2 \frac{(\nabla \mathbf{u})^T + (\nabla \mathbf{u})}{2} = \mathbf{I} + 2E \tag{iii}$$

Also

$$C_t^{-1} = \mathbf{F}_t^{-1} \mathbf{F}_t^{-1T} \approx [\mathbf{I} - (\nabla \mathbf{u})][\mathbf{I} - (\nabla \mathbf{u})^T] = \mathbf{I} - 2 \frac{(\nabla \mathbf{u})^T + (\nabla \mathbf{u})}{2} = \mathbf{I} - 2E \tag{iv}$$

Thus

$$C_t - I \approx 2E \quad \text{and} \quad I - C_t^{-1} \approx 2E$$

Example 15.3

Show that any polynomial function of a real symmetric tensor  $\mathbf{A}$  can be represented by

$$\mathbf{F}(\mathbf{A}) = f_0 \mathbf{I} + f_1 \mathbf{A} + f_2 \mathbf{A}^{-1} \tag{8.15.16}$$

where  $f_i$  are real valued functions of the scalar invariants of the symmetric tensor  $\mathbf{A}$ .

*Solution.* Let

$$\mathbf{F}(\mathbf{A}) = a_0 \mathbf{I} + a_1 \mathbf{A} + a_2 \mathbf{A}^2 + \dots a_N \mathbf{A}^N \tag{i}$$

Since  $\mathbf{A}$  satisfies its own characteristic equation:

$$\mathbf{A}^3 - I_1 \mathbf{A}^2 + I_2 \mathbf{A} - I_3 \mathbf{I} = 0 \tag{ii}$$

therefore,

$$\mathbf{A}^3 = I_1 \mathbf{A}^2 - I_2 \mathbf{A} + I_3 \mathbf{I} \tag{iii}$$

$$\mathbf{A}^4 = I_1 \mathbf{A}^3 - I_2 \mathbf{A}^2 + I_3 \mathbf{A} = I_1 (I_1 \mathbf{A}^2 - I_2 \mathbf{A} + I_3 \mathbf{I}) - I_2 \mathbf{A}^2 + I_3 \mathbf{A} \tag{iv}$$

etc., Thus, every  $\mathbf{A}^N$  for  $N \geq 3$  can be expressed as a sum of  $\mathbf{A}$ ,  $\mathbf{A}^2$  and  $\mathbf{I}$  with coefficients being functions of the scalar invariants of  $\mathbf{A}$ . Substituting these expressions in Eq. (i), one obtains

$$\mathbf{F}(\mathbf{A}) = b_0(I_i) \mathbf{I} + b_1(I_i) \mathbf{A} + b_2(I_i) \mathbf{A}^2 \tag{v}$$

Now, from Eq. (iii), we can obtain

$$\mathbf{A}^2 = I_1 \mathbf{A} - I_2 \mathbf{I} + I_3 \mathbf{A}^{-1} \tag{vi}$$

therefore, Eq. (v) can also be written as

$$\mathbf{F}(\mathbf{A}) = f_0(I_i) \mathbf{I} + f_1(I_i) \mathbf{A} + f_2(I_i) \mathbf{A}^{-1} \tag{vii}$$

which is Eq. (8.15.16). Actually the representation of  $\mathbf{F}(\mathbf{A})$  given in this example can be shown to be true under the more general condition that the symmetric function  $\mathbf{F}$  of the symmetric tensor  $\mathbf{A}$  is an isotropic function ( of which the polynomial function of  $\mathbf{A}$  is a special case ). An isotropic function  $\mathbf{F}$  is a function which satisfies the condition

$$\mathbf{F}(\mathbf{Q}\mathbf{A}\mathbf{Q}^T) = \mathbf{Q}\mathbf{F}(\mathbf{A})\mathbf{Q}^T \tag{viii}$$

for any orthogonal tensor  $\mathbf{Q}$ . Now, let us identify  $\mathbf{A}$  with  $\mathbf{C}_t$  and  $I_i$  with the scalar invariants of  $\mathbf{C}_t$  ( note however that  $I_3 = 1$  for incompressible fluid ), then the most general representation of  $\mathbf{F}(\mathbf{C}_t)$  is [ we recall that  $\mathbf{F}(\mathbf{C}_t)$  is required to satisfy Eq. (viii) for frame indifference, see Eq. (8.14.5) also],

$$\mathbf{F}(\mathbf{C}_t) = f_1(I_1, I_2) \mathbf{C}_t + f_2(I_1, I_2) \mathbf{C}_t^{-1} \tag{ix}$$

**8.16 General Single Integral Type Nonlinear Constitutive Equations**

From the discussions given in the previous example, we see that the most general single integral type nonlinear constitutive equation for an compressible fluid is defined by

$$\boldsymbol{\tau} = \int_0^\infty [f_1(s, I_1, I_2) \mathbf{C}_t + f_2(s, I_1, I_2) \mathbf{C}_t^{-1}] ds \tag{8.16.1}$$

A special nonlinear viscoelastic fluid, known as the BKZ fluid, is defined by Eq. (8.16.1) with the functions  $f_1(s)$  and  $f_2(s)$  given by

$$f_1(s) = 2 \frac{\partial U}{\partial I_1} \tag{8.16.2a}$$

$$f_2(s) = -2 \frac{\partial U}{\partial I_2} \tag{8.16.2b}$$

where

$$-U = \frac{\alpha}{2} (I_1 - 3)^2 + \frac{9}{2} \beta \ln \frac{I_1 + I_2 + 3}{9} + 24(\beta - c) \ln \frac{I_1 + 15}{I_2 + 15} + c(I_1 - 3) \tag{8.16.2c}$$

where  $\alpha, \beta$  and  $c$  are constants.

**8.17 Differential Type Constitutive Equations for Incompressible Fluids**

We see in Section 8.9 that under the assumption that the Taylor series expansion of the history of the deformation tensor  $\mathbf{C}_t(\mathbf{x}, \tau)$  is justified, the Rivlin-Ericksen tensor  $\mathbf{A}_n (n = 1, 2, \dots, \infty)$  determines the history of  $\mathbf{C}_t(\mathbf{x}, \tau)$ . Thus we may write Eq. (8.14.2) as

$$\mathbf{T} = -p \mathbf{I} + \mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n, \dots) \tag{8.17.1}$$

where  $\mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n, \dots)$  is a function of the Rivlin-Ericksen tensor and  $\text{tr} \mathbf{A}_1 = 0$  which follows from the equation of conservation of mass for an incompressible fluid.

In order to satisfy the frame-indifference condition, the function  $\mathbf{f}$  cannot be arbitrary but must satisfy the relation

$$\mathbf{Q} \mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n) \mathbf{Q}^T = \mathbf{f}(\mathbf{Q} \mathbf{A}_1 \mathbf{Q}^T, \mathbf{Q} \mathbf{A}_2 \mathbf{Q}^T, \dots, \mathbf{Q} \mathbf{A}_n \mathbf{Q}^T) \tag{8.17.2}$$

for any orthogaonal tensor  $\mathbf{Q}$ . We note again, Equation (8.17.2) makes “isotropy of material property” as part of the definition of a simple fluid. Equation.(8.17.2) is obtained in the same way as Eq. (8.14.5) is.

The following are special constitutive equations of this type

(A) *Rivlin-Ericksen incompressible fluid of complexity n*

$$\mathbf{T} = -p \mathbf{I} + \mathbf{f}(\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n) \tag{8.17.3}$$

In particular, a Rivlin-Ericksen liquid of complexity 2 is given by:

$$\begin{aligned} \mathbf{T} = & -p \mathbf{I} + \mu_1 \mathbf{A}_1 + \mu_2 \mathbf{A}_1^2 + \mu_3 \mathbf{A}_2 + \mu_4 \mathbf{A}_2^2 \\ & + \mu_5 (\mathbf{A}_1 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1) + \mu_6 (\mathbf{A}_1 \mathbf{A}_2^2 + \mathbf{A}_2^2 \mathbf{A}_1) \\ & + \mu_7 (\mathbf{A}_1^2 \mathbf{A}_2 + \mathbf{A}_2 \mathbf{A}_1^2) + \mu_8 (\mathbf{A}_1^2 \mathbf{A}_2^2 + \mathbf{A}_2^2 \mathbf{A}_1^2) \end{aligned} \tag{8.17.4}$$

where  $\mu_1, \mu_2, \dots, \mu_8$  are scalar material functions of the following scalar invariants:

$$\begin{aligned} & tr \mathbf{A}_1^2, tr \mathbf{A}_1^3, tr \mathbf{A}_2, tr \mathbf{A}_2^2, tr \mathbf{A}_2^3 \\ & tr \mathbf{A}_1 \mathbf{A}_2, tr \mathbf{A}_1^2 \mathbf{A}_2, tr \mathbf{A}_1 \mathbf{A}_2^2, tr \mathbf{A}_1^2 \mathbf{A}_2^2 \end{aligned} \tag{8.17.5}$$

We note that if  $\mu_2 = \mu_3, \dots = \mu_n = 0$  and  $\mu_1 = \text{a constant}$ , Eq. (8.17.4) reduces to the constitutive equation for a Newtonian liquid with viscosity  $\mu_1$ .

**(B) Second Order Fluid**

$$\mathbf{T} = -p \mathbf{I} + \mu_1 \mathbf{A}_1 + \mu_2 \mathbf{A}_1^2 + \mu_3 \mathbf{A}_2 \tag{8.17.6}$$

where  $\mu_1, \mu_2,$  and  $\mu_3$  are material constants. The second order fluid may be regarded as a special case of the Rivlin-Ericksen fluid. However, it has also been shown that under the assumption of fading memory, small deformation and slow flow, Eq. (8.17.6) provides the second-order approximation whereas the Newtonian fluid provides the first order approximation and the inviscid fluid, the zeroth order approximation.

**Example 8.17.1**

For a second order fluid, compute the stress components in a simple shearing flow given by the velocity field

$$v_1 = kx_2, v_2 = v_3 = 0 \tag{i}$$

*Solution.* From Example 8.9.1, we have for the simple shearing flow,

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

$$[\mathbf{A}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

and

$$\mathbf{A}_3 = \mathbf{A}_4 = \dots = 0 \tag{iv}$$

Now,

$$[\mathbf{A}_1^2] = [\mathbf{A}_1][\mathbf{A}_1] = \begin{bmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{v})$$

therefore, Eq. (8.17.6) gives

$$T_{11} = -p + \mu_2 k^2, \quad T_{22} = -p + \mu_2 k^2 + 2\mu_3 k^2, \quad T_{33} = -p \quad (\text{vi})$$

$$T_{12} = \mu_1 k, \quad T_{13} = T_{23} = 0 \quad (\text{vii})$$

We see that because of the presence of  $\mu_2$  and  $\mu_3$ , normal stresses, in excess of  $p$  on the planes  $x_1 = \text{constant}$  and  $x_2 = \text{constant}$  are necessary to maintain the shearing flow. Furthermore, these normal stress components are not equal. The normal stress difference

$$\sigma_1(k) \equiv T_{11} - T_{22} \quad (\text{viii})$$

and

$$\sigma_2(k) \equiv T_{22} - T_{33} \quad (\text{ix})$$

are given by

$$\sigma_1 = -2\mu_3 k^2, \quad \sigma_2 = \mu_2 k^2 + 2\mu_3 k^2 \quad (\text{x})$$

By measuring the normal stress differences and the shearing stress components  $T_{12}$ , the three material constants can be determined.

### Example 8.17.2

For the simple shearing flow, compute the scalar invariants of Eq. (8.17.5).

*Solution.* Since

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_1^2] = \begin{bmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_1^3] = \begin{bmatrix} 0 & k^3 & 0 \\ k^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{i})$$

$$[\mathbf{A}_2^2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4k^4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_2^3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8k^6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{ii})$$

$$[\mathbf{A}_1][\mathbf{A}_2] = \begin{bmatrix} 0 & 2k^3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_2][\mathbf{A}_1] = \begin{bmatrix} 0 & 0 & 0 \\ 2k^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_1^2][\mathbf{A}_2] = [\mathbf{A}_2][\mathbf{A}_1^2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{iii})$$

$$[\mathbf{A}_1][\mathbf{A}_2^2] = \begin{bmatrix} 0 & 4k^5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_2^2][\mathbf{A}_1] = \begin{bmatrix} 0 & 0 & 0 \\ 4k^5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{A}_1^2][\mathbf{A}_2^2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4k^6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{iv})$$

Therefore,

$$\begin{aligned} \text{tr}\mathbf{A}_1^2 &= 2k^2, \quad \text{tr}\mathbf{A}_1^3 = 0, \quad \text{tr}\mathbf{A}_2 = 2k^2, \\ \text{tr}\mathbf{A}_2^2 &= 4k^4, \quad \text{tr}\mathbf{A}_2^3 = 8k^6, \quad \text{tr}\mathbf{A}_1\mathbf{A}_2 = 0, \\ \text{tr}\mathbf{A}_1^2\mathbf{A}_2 &= 2k^4, \quad \text{tr}\mathbf{A}_1\mathbf{A}_2^2 = 0, \quad \text{tr}\mathbf{A}_1^2\mathbf{A}_2^2 = 4k^6 \end{aligned}$$

### Example 8.17.3

In a simple shearing flow, compute the stress components for the Rivlin-Ericksen liquid.

*Solution.* From Eqs. (8.17.4) and the results of the previous example, we have (note  $\mathbf{A}_3 = \mathbf{A}_4 = \dots = 0$ )

$$T_{11} = -p + k^2[\mu_2(k^2)]$$

$$T_{22} = -p + 2k^2[\mu_3(k^2)] + \frac{1}{2}\mu_2(k^2) + 2(\mu_4(k^2) + \mu_7(k^2))k^2 + 4k^4\mu_8(k^2)$$

$$T_{33} = -p$$

$$T_{12} = k[\mu_1(k^2) + 2k^2\mu_5(k^2) + 4k^4\mu_6(k^2)] = k\eta(k^2) \equiv s(k)$$

$$T_{23} = T_{13} = 0$$

where  $\mu_1(k^2)$  indicates that  $\mu_1$  is a function of  $k^2$ , etc. The normal stress differences  $T_{11} - T_{22}$  and  $T_{22} - T_{33}$  are even functions of  $k$  ( $=$  rate of shear), whereas the shear stress function  $s(k)$  is an odd function of  $k$ .

## 8.18 Objective Rate of Stress

The stress tensor  $\mathbf{T}$  is objective, therefore in a change of frame

$$\mathbf{T}^* = \mathbf{Q}(t)\mathbf{T}\mathbf{Q}^T(t) \quad (8.18.1)$$

Taking material derivative of the above equation, we obtain [note  $D/Dt^* = D/Dt$ ]

$$\frac{DT^*}{Dt} = \frac{dQ}{dt}TQ^T + Q\frac{DT}{Dt}Q^T + QT\left(\frac{dQ}{dt}\right)^T \tag{8.18.2}$$

The above equation shows that the material derivative of stress tensor  $T$  is not objective.

That the stress rate  $D T/Dt$  is not objective is physically quite clear. Consider the case of a time-independent uni-axial state of stress with respect to the first observer. With respect to this observer, the stress rate  $DT/Dt$  is identically zero. Consider a second observer who rotates with respect to the first observer. To the second observer, the given stress state is rotating respect to him and therefore, to him, the stress rate  $DT^*/Dt$  is not zero.

In the following we shall present several stress rates at time  $t$  which are objective

(A) *Jaumann derivative of stress*

Let us consider the tensor

$$J(\tau) = R_\tau^T(\tau)T(\tau)R_\tau(\tau) \tag{8.18.3}$$

We note that since  $R_\tau(t) = R_\tau^T(t) = I$ , therefore, the tensor  $J$  and the tensor  $T$  are the same at time  $t$ . That is

$$J(t) = T(t) \tag{i}$$

However, while  $DT/Dt$  is not an objective stress rate , we will show that

$$\left[\frac{DJ(\tau)}{D\tau}\right]_{\tau=t} \tag{ii}$$

is an objective stress rate. To show this, we note that in Sect.8.12 , we obtained , in a change of frame

$$R_\tau^*(\tau) = Q(\tau)R_\tau(\tau)Q^T(t) \tag{iii}$$

Thus,

$$J^*(\tau) = R_\tau^{*T}(\tau)T^*(\tau)R_\tau^*(\tau) = Q(t)R_\tau^T(\tau)Q^T(\tau)Q(\tau)T(\tau)Q^T(\tau)Q(\tau)R_\tau(\tau)Q^T(t) \tag{iv}$$

$$= Q(t)[R_\tau^T(\tau)T(\tau)R_\tau(\tau)]Q^T(t) \tag{v}$$

Thus,

$$J^*(\tau) = Q(t)J(\tau)Q^T(t) \tag{8.18.4}$$

and

$$\left[ \frac{D^N \mathbf{J}^*(\tau)}{D\tau^N} \right]_{\tau=t} = \mathbf{Q}(t) \left[ \frac{D^N \mathbf{J}(\tau)}{D^N \tau} \right]_{\tau=t} \mathbf{Q}^T(t), \quad N=1,2,3... \quad (8.18.5)$$

That is, the tensor  $\mathbf{J}(\tau)$  as well as its material derivatives evaluated at time  $t$ , is objective. The derivative

$$\frac{D_{cr} \mathbf{T}}{Dt} \equiv \left[ \frac{D\mathbf{J}(\tau)}{D\tau} \right]_{\tau=t} \equiv \left[ \frac{D[\mathbf{R}_t(\tau)\mathbf{T}(\tau)\mathbf{R}_t^T(\tau)]}{D\tau} \right]_{\tau=t} \quad (8.18.6)$$

is called the **first Jaumann derivative** of  $\mathbf{T}$  and the corresponding  $N$ th derivatives are called the  **$N$ th Jaumann derivatives**. They are also called the **co-rotational derivatives**, because they are the derivatives of  $\mathbf{T}$  at time  $t$  as seen by an observer who rotates with the material element (whose rotation tensor is  $\mathbf{R}$ ).

We shall now show that

$$\frac{D_{cr} \mathbf{T}}{Dt} = \frac{D\mathbf{T}}{Dt} + \mathbf{T}(t)\mathbf{W}(t) - \mathbf{W}(t)\mathbf{T}(t) \quad (8.18.7)$$

where  $\mathbf{W}(t)$  is the spin tensor of the element. The right side of Eq. (8.18.6) is

$$\frac{D[(\mathbf{R}_t(\tau)\mathbf{T}(\tau)\mathbf{R}_t^T(\tau))]}{D\tau} = \frac{D\mathbf{R}_t(\tau)}{D\tau} \mathbf{T}(\tau)\mathbf{R}_t^T(\tau) + \mathbf{R}_t(\tau) \frac{D\mathbf{T}(\tau)}{D\tau} \mathbf{R}_t^T(\tau) + \mathbf{R}_t(\tau)\mathbf{T}(\tau) \frac{D\mathbf{R}_t^T(\tau)}{D\tau} \quad (vi)$$

Evaluating the above equation at  $\tau=t$  and noting that

$$\mathbf{R}_t(t) = \mathbf{R}_t^T(t) = I$$

and

$$\left[ \frac{D\mathbf{R}_t(\tau)}{D\tau} \right]_{\tau=t} = \mathbf{W}(t) \quad (vii)$$

$$\left[ \frac{D\mathbf{R}_t^T(\tau)}{D\tau} \right]_{\tau=t} = \mathbf{W}^T(t) = -\mathbf{W}(t) \quad (viii)$$

we obtain immediately

$$\frac{D_{cr} \mathbf{T}}{Dt} = \frac{D\mathbf{T}}{Dt} + \mathbf{T}(t)\mathbf{W}(t) - \mathbf{W}(t)\mathbf{T}(t) \quad (ix)$$

**(B) Oldroyd lower convected derivative**

Let us consider the tensor

$$\mathbf{J}_L(\tau) = \mathbf{F}_t^T(\tau)\mathbf{T}(\tau)\mathbf{F}_t(\tau) \quad (8.18.8)$$

Again, as in (A),

$$\mathbf{J}_L(t) = \mathbf{T}(t) \tag{x}$$

and

$$\left[ \frac{D\mathbf{J}_L(\tau)}{D\tau} \right]_{\tau=t} \tag{xi}$$

is an objective stress rate. To show this, we note that in Sect. 8.12, we obtained, in a change of frame

$$\mathbf{F}_i^*(\tau) = \mathbf{Q}(\tau)\mathbf{F}_i(\tau)\mathbf{Q}^T(t) \tag{xii}$$

Thus,

$$\begin{aligned} \mathbf{J}_L^*(\tau) &= \mathbf{F}_i^{*T}(\tau)\mathbf{T}^*(\tau)\mathbf{F}_i^*(\tau) = \mathbf{Q}(t)\mathbf{F}_i^T(\tau)\mathbf{Q}^T(\tau)\mathbf{Q}(\tau)\mathbf{T}(\tau)\mathbf{Q}^T(\tau)\mathbf{Q}(\tau)\mathbf{F}_i(\tau)\mathbf{Q}^T(t) \\ &= \mathbf{Q}(t)[\mathbf{F}_i^T(\tau)\mathbf{T}(\tau)\mathbf{F}_i(\tau)]\mathbf{Q}^T(t) \end{aligned} \tag{xiii}$$

Thus,

$$\mathbf{J}_L^*(\tau) = \mathbf{Q}(t)\mathbf{J}_L(\tau)\mathbf{Q}^T(t) \tag{8.18.9}$$

and

$$\left[ \frac{D^N \mathbf{J}_L^*(\tau)}{D\tau^N} \right]_{\tau=t} = \mathbf{Q}(t) \left[ \frac{D^N \mathbf{J}_L(\tau)}{D\tau^N} \right]_{\tau=t} \mathbf{Q}^T(t), \quad N=1,2,3... \tag{8.18.10}$$

That is, the tensor  $\mathbf{J}_L(\tau)$  as well as its material derivative evaluated at time  $t$ , is objective. The derivative

$$\frac{D_{Lc}\mathbf{T}}{Dt} \equiv \left[ \frac{D\mathbf{J}_L(\tau)}{D\tau} \right]_{\tau=t} \equiv \left[ \frac{D[\mathbf{F}_i(\tau)\mathbf{T}(\tau)\mathbf{F}_i^T(\tau)]}{D\tau} \right]_{\tau=t} \tag{8.18.11}$$

is called the **first Oldroyd lower convected derivative**. The  $N$ th derivatives of  $\mathbf{J}_L$  are called the  **$N$ th Oldroyd lower derivatives**. Noting that

$$\mathbf{R}_i(t) = \mathbf{R}_i^T(t) = \mathbf{I} \tag{xiv}$$

and

$$\left[ \frac{D\mathbf{F}_i(\tau)}{D\tau} \right]_{\tau=t} = (\nabla\mathbf{v})(t) \tag{xv}$$

it can be shown that

$$\frac{D_{Lc}\mathbf{T}}{Dt} = \frac{D\mathbf{T}}{Dt} + \mathbf{T}(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{T} \tag{8.18.12}$$

and

$$\frac{D_{Lc}^{N+1}\mathbf{T}}{Dt} = \left(\frac{D}{Dt}\right) \left(\frac{D_{Lc}^N\mathbf{T}}{Dt}\right) + \left(\frac{D_{Lc}^N\mathbf{T}}{Dt}\right) (\nabla\mathbf{v}) + (\nabla\mathbf{v})^T \left(\frac{D_{Lc}^N\mathbf{T}}{Dt}\right) \tag{8.18.13}$$

Further, since

$$(\nabla\mathbf{v}) \equiv \mathbf{D} + \mathbf{W} \tag{xvi}$$

Equation (8.18.12) can also be written as

$$\frac{D_{Lc}\mathbf{T}}{Dt} = \frac{D_{cr}\mathbf{T}}{Dt} + \mathbf{T}\mathbf{D} + \mathbf{D}\mathbf{T} \tag{8.18.14}$$

where the first term in the right hand side is the co-rotational derivative of  $\mathbf{T}$  given by Eq. (8.18.7).

(C) *Oldroyd upper convected derivative*

Let us consider the tensor

$$\mathbf{J}_u(\tau) = \mathbf{F}_t^{-1}(\tau)\mathbf{T}(\tau)\mathbf{F}_t^{-1T}(\tau) \tag{8.18.15}$$

Again, as in (A) and (B),

$$\mathbf{J}_u(t) = \mathbf{T}(t) \tag{xvii}$$

and the derivatives

$$\left[ \frac{D^N\mathbf{J}_u(\tau)}{D\tau^N} \right]_{\tau=t}, \quad N=1,2,3\dots \tag{xviii}$$

can be shown to be objective stress rates. [See Prob. 23] These are called the **Oldroyd upper convected derivatives**.

Let

$$\frac{D_{uc}\mathbf{T}}{Dt} \equiv \left[ \frac{D\mathbf{J}_u(\tau)}{D\tau} \right]_{\tau=t} \equiv \left[ \frac{D[\mathbf{F}_t^{-1}(\tau)\mathbf{T}(\tau)\mathbf{F}_t^{-1T}(\tau)]}{D\tau} \right]_{\tau=t} \tag{8.18.16}$$

and note that

$$\frac{D\mathbf{F}_t^{-1}(\tau)}{D\tau} = -\mathbf{F}_t^{-1}(\tau)\frac{D\mathbf{F}(\tau)}{D\tau}\mathbf{F}_t^{-1}(\tau) \tag{8.18.17}$$

one can derive

$$\frac{D_{uc}\mathbf{T}}{Dt} = \frac{D\mathbf{T}}{Dt} - \mathbf{T}(\nabla\mathbf{v})^T - (\nabla\mathbf{v})\mathbf{T} \tag{8.18.18}$$

or more generally

$$\frac{D_{uc}^{N+1}\mathbf{T}}{Dt} = \left(\frac{D}{Dt}\right) \left(\frac{D_{uc}^N\mathbf{T}}{Dt}\right) - \left(\frac{D_{uc}^N\mathbf{T}}{Dt}\right) (\nabla\mathbf{v})^T - (\nabla\mathbf{v}) \left(\frac{D_{uc}^N\mathbf{T}}{Dt}\right) \tag{8.18.19}$$

Again, using Eq. (xvi), Eq. (8.18.18) can also be written

$$\frac{D_{uc}\mathbf{T}}{Dt} = \frac{D_{cr}\mathbf{T}}{Dt} - (\mathbf{T}\mathbf{D} + \mathbf{D}\mathbf{T}) \tag{8.18.20}$$

where the first term in the right hand side is the co-rotational derivative of  $\mathbf{T}$  given by Eq. (8.18.7).

**(D) Other objective stress rates**

The stress rates given in (A)(B)(C) are not the only ones that are objectives. Indeed there are infinitely many. For example, the addition of any term or terms that is (are) objective to any of the above derivatives will give a new objective stress rate. In particular, the derivative

$$\frac{D_{cr}\mathbf{T}}{Dt} + a(\mathbf{T}\mathbf{D} + \mathbf{D}\mathbf{T}) \tag{8.18.21}$$

is objective for any value  $a$ . We note that For  $a = +1$ , it is the Oldroyd lower convected derivative and for  $a = -1$ , the Oldroyd upper convected derivative.

**8.19 The Rate Type Constitutive Equations**

Constitutive equations of the following form are known as the rate type nonlinear constitutive equations:

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau} \tag{8.19.1a}$$

$$\boldsymbol{\tau} + \lambda_1 \frac{D_*\boldsymbol{\tau}}{Dt} + \lambda_2 \frac{D_*^2\boldsymbol{\tau}}{Dt^2} + \dots = 2\mu_1\mathbf{D} + \mu_2 \frac{D_*\mathbf{D}}{Dt} + \dots \tag{8.19.1b}$$

where  $D_*/Dt$ ,  $D_*^2/Dt^2$  etc., denote some objective time derivative and objective higher time derivatives,  $\boldsymbol{\tau}$  is the extra stress and  $\mathbf{D}$  is rate of deformation tensor. Equation (8.19.1) may be regarded as a generalization of the generalized linear Maxwell fluid defined in Sect. 8.2.

The following are some examples:

(a) *The convected Maxwell fluid*

The convected Maxwell fluid is defined by the constitutive equation

$$\boldsymbol{\tau} + \lambda \frac{D_{cr} \boldsymbol{\tau}}{Dt} = 2\mu \mathbf{D} \quad (8.19.2)$$

where  $\frac{D_{cr}}{Dt}$  is the corotational derivative. That is

$$\frac{D_{cr} \boldsymbol{\tau}}{Dt} = \frac{D\boldsymbol{\tau}}{Dt} + \boldsymbol{\tau} \mathbf{W} - \mathbf{W} \boldsymbol{\tau} \quad (8.19.3)$$

Example 8.19.1

Obtain the stress components for the convected Maxwell fluid in a simple shearing flow.

*Solution.* With the velocity field for a simple shearing flow given by

$$v_1 = kx_2, \quad v_2 = v_3 = 0 \quad (i)$$

the rate of deformation tensor and the spin tensor are given by

$$[\mathbf{D}] = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ \frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [\mathbf{W}] = \begin{bmatrix} 0 & \frac{k}{2} & 0 \\ -\frac{k}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (ii)$$

Thus,

$$[\boldsymbol{\tau} \mathbf{W}] = \begin{bmatrix} -\frac{k}{2} \tau_{12} & \frac{k}{2} \tau_{11} & 0 \\ -\frac{k}{2} \tau_{22} & \frac{k}{2} \tau_{12} & 0 \\ -\frac{k}{2} \tau_{23} & \frac{k}{2} \tau_{13} & 0 \end{bmatrix} \quad (iii)$$

$$[\mathbf{W} \boldsymbol{\tau}] = \begin{bmatrix} \frac{k}{2} \tau_{12} & \frac{k}{2} \tau_{22} & \frac{k}{2} \tau_{23} \\ -\frac{k}{2} \tau_{11} & -\frac{k}{2} \tau_{12} & -\frac{k}{2} \tau_{13} \\ 0 & 0 & 0 \end{bmatrix} \quad (iv)$$

$$[\boldsymbol{\tau}\mathbf{W}-\mathbf{W}\boldsymbol{\tau}] = \begin{bmatrix} -k\tau_{12} & \frac{k}{2}(\tau_{11}-\tau_{22}) & -\frac{k}{2}\tau_{23} \\ \frac{k}{2}(\tau_{11}-\tau_{22}) & k\tau_{12} & \frac{k}{2}\tau_{13} \\ -\frac{k}{2}\tau_{23} & \frac{k}{2}\tau_{13} & 0 \end{bmatrix} \quad (\text{v})$$

Since the flow is steady and the rate of deformation is a constant independent of position, therefore, the stress field is also independent of time and position. Thus, the material derivative  $D\boldsymbol{\tau}/Dt$  is zero so that Eq. (v) is the corotational derivative of  $\boldsymbol{\tau}$  (see Eq. (8.19.3)). Substituting this equation into the constitutive equation, we obtain

$$\tau_{11} - k\lambda\tau_{12} = 0 \quad (\text{vi})$$

$$\tau_{12} + \frac{k\lambda}{2}(\tau_{11} - \tau_{22}) = \mu k \quad (\text{vii})$$

$$\tau_{13} - \frac{k\lambda}{2}\tau_{23} = 0 \quad (\text{viii})$$

$$\tau_{22} + k\lambda\tau_{12} = 0 \quad (\text{ix})$$

$$\tau_{23} + \frac{k\lambda}{2}\tau_{13} = 0 \quad (\text{x})$$

$$\tau_{33} = 0 \quad (\text{xi})$$

From Eqs. (viii) and (x), we obtain,

$$\tau_{13} = \tau_{23} = 0 \quad (\text{xii})$$

From Eqs. (vi) and (ix), respectively

$$\tau_{11} = k\lambda\tau_{12} \quad (\text{xiii})$$

and

$$\tau_{22} = -k\lambda\tau_{12} \quad (\text{xiv})$$

Using the above two equations, we obtain from Eq. (vii) the shear stress function  $\tau(k)$

$$\tau(k) = \tau_{12} = \frac{\mu k}{1 + k^2\lambda^2} \quad (\text{xv})$$

The apparent viscosity  $\eta$  is

$$\eta(k) \equiv \frac{\tau(k)}{k} = \frac{\mu}{1 + k^2\lambda^2} \quad (\text{xi})$$

The normal stress functions are

$$\sigma_1 \equiv T_{11} - T_{22} = \frac{2\mu k^2 \lambda}{1 + k^2 \lambda^2} \quad (\text{xii})$$

$$\sigma_2 \equiv T_{22} - T_{33} = \frac{-\mu k^2 \lambda}{1 + k^2 \lambda^2} \quad (\text{xiii})$$

(b) *The Corotational Jeffrey Fluid*

The corotational Jeffrey Fluid is defined by the constitutive equation

$$\mathbf{T} = -p \mathbf{I} + \boldsymbol{\tau} \quad (8.19.4a)$$

$$\boldsymbol{\tau} + \lambda_1 \frac{D_{cr} \boldsymbol{\tau}}{Dt} = 2\mu \left( \mathbf{D} + \lambda_2 \frac{D_{cr} \mathbf{D}}{Dt} \right) \quad (8.19.4b)$$

Example 8.19.2

Obtain the stress components for the corotational Jeffrey fluid in simple shearing flow.

*Solution.* The corotational derivative of the extra stress is the same as the previous example, thus,

$$\left[ \frac{D_{cr} \boldsymbol{\tau}}{DT} \right] = [\mathbf{0}] + [\boldsymbol{\tau} \mathbf{W} - \mathbf{W} \boldsymbol{\tau}] = \begin{bmatrix} -k\tau_{12} & \frac{k}{2}(\tau_{11} - \tau_{22}) & -\frac{k}{2}\tau_{23} \\ \frac{k}{2}(\tau_{11} - \tau_{22}) & k\tau_{12} & \frac{k}{2}\tau_{13} \\ -\frac{k}{2}\tau_{23} & \frac{k}{2}\tau_{13} & 0 \end{bmatrix} \quad (\text{i})$$

$$\left[ \frac{D_{cr} \mathbf{D}}{DT} \right] = [\mathbf{0}] + [\mathbf{D} \mathbf{W} - \mathbf{W} \mathbf{D}] = \begin{bmatrix} -\frac{k^2}{2} & 0 & 0 \\ 0 & \frac{k^2}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{ii})$$

Substituting the above two equations and  $\mathbf{D}$  from the previous example into Eq. (8.19.4), we obtain

$$\tau_{11} - k \lambda_1 \tau_{12} = -\mu \lambda_2 k^2 \quad (\text{iii})$$

$$\tau_{12} + \frac{k\lambda_1}{2}(\tau_{11} - \tau_{22}) = \mu k \quad (\text{iv})$$

$$\tau_{13} - \frac{k\lambda_1}{2}\tau_{23} = 0 \quad (\text{v})$$

$$\tau_{22} + k\lambda_1\tau_{12} = \mu\lambda_2 k^2 \quad (\text{vi})$$

$$\tau_{23} + \frac{k\lambda_1}{2}\tau_{13} = 0 \quad (\text{vii})$$

$$\tau_{33} = 0 \quad (\text{viii})$$

Proceeding as in Example 8.19.1, we obtain the apparent viscosity  $\eta$  and the normal stress functions as:

$$\eta(k) \equiv \frac{\tau_{12}}{k} = \mu \frac{1 + \lambda_1\lambda_2 k^2}{1 + \lambda_1^2 k^2} \quad (\text{ix})$$

$$\sigma_1 \equiv T_{11} - T_{22} = \frac{2\mu(\lambda_1 - \lambda_2)k^2}{1 + \lambda_1^2 k^2} \quad (\text{x})$$

$$\sigma_2 \equiv T_{22} - T_{33} = -\frac{\mu(\lambda_1 - \lambda_2)k^2}{1 + \lambda_1^2 k^2} \quad (\text{xi})$$

(c) *The Oldroyd 3-constant fluid*

The Oldroyd 3-constant model (also known as the Oldroyd fluid A) is defined by the following constitutive equation:

$$\mathbf{T} = -p\mathbf{I} + \boldsymbol{\tau} \quad (8.19.5a)$$

$$\boldsymbol{\tau} + \lambda_1 \frac{D_{up}\boldsymbol{\tau}}{Dt} = 2\mu \left( \mathbf{D} + \lambda_2 \frac{D_{up}\mathbf{D}}{Dt} \right) \quad (8.19.5b)$$

where  $D_{up}/Dt$  denote the Oldroyd upper convected derivative defined in Section 8.18. That is

$$\frac{D_{up}\boldsymbol{\tau}}{Dt} = \frac{D_{cr}\boldsymbol{\tau}}{Dt} - (\boldsymbol{\tau}\mathbf{D} + \mathbf{D}\boldsymbol{\tau}) \quad (\text{i})$$

and

$$\frac{D_{up}\mathbf{D}}{Dt} = \frac{D_{cr}\mathbf{D}}{Dt} - (\mathbf{D}^2 + \mathbf{D}^2) \quad (\text{ii})$$

where again  $D_{cr}/Dt$  denote the corotational derivative. By considering the simple shearing flow as was done in the previous two models, we can obtain that the viscosity of this fluid is a constant independent of the shear rate  $k$ , i.e.,

$$\eta(k) = \mu \tag{iii}$$

The normal stress functions are:

$$\sigma_1 \equiv T_{11} - T_{22} = 2\mu (\lambda_1 - \lambda_2) k^2 \tag{iv}$$

$$\sigma_2 \equiv T_{22} - T_{33} = -2\mu (\lambda_1 - \lambda_2) k^2 \tag{v}$$

(d) *The Oldroyd 4-constant fluid*

the Oldroyd 4-constant fluid is defined by the following constitutive equation:

$$\mathbf{T} = -p \mathbf{I} + \boldsymbol{\tau} \tag{8.19.6a}$$

$$\boldsymbol{\tau} + \lambda_1 \frac{D_{up}\boldsymbol{\tau}}{Dt} + \mu_o (\text{tr } \boldsymbol{\tau}) \mathbf{D} = 2\mu \left( \mathbf{D} + \lambda_2 \frac{D_{up}\mathbf{D}}{Dt} \right) \tag{8.19.6b}$$

We note that in this model an additional term  $\mu_o(\text{tr } \boldsymbol{\tau}) \mathbf{D}$  is added to the left hand side. This term is obviously an objective term since both  $\boldsymbol{\tau}$  and  $\mathbf{D}$  are objective. The inclusion of this term will make the viscosity of the fluid dependent on the rate of deformation.

By considering the simple shearing flow as was done in the previous models, we can obtain the apparent viscosity to be

$$\eta(k) \equiv \frac{\tau_{12}}{k} = \mu \frac{1 + \lambda_2 \mu_o k^2}{1 + \lambda_1 \mu_o k^2} \tag{i}$$

The normal stress functions are:

$$\sigma_1 = T_{11} - T_{22} = 2\mu \lambda_1 \left( \frac{\eta(k)}{\mu} - \frac{\lambda_2}{\lambda_1} \right) \tag{ii}$$

$$\sigma_2 \equiv T_{22} - T_{33} = 0 \tag{iii}$$

**Part C Viscometric Flows of an Incompressible Simple Fluid**

**8.20 Viscometric Flow**

Viscometric flows may be defined to be the class of flows which satisfies the following conditions:

(i) At all time and at every material point, the history of the relative right Cauchy-Green deformation tensor can be expressed as

$$C_t(\tau) = \mathbf{I} + (\tau-t)\mathbf{A}_1 + \frac{(\tau-t)^2}{2}\mathbf{A}_2. \tag{8.20.1}$$

(ii) There exists an orthogonal basis  $(\mathbf{n}_i)$ , with respect to which, the only nonzero Rivlin-Erickson tensors are given by

$$[\mathbf{A}_1] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{n}_i} \quad [\mathbf{A}_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{n}_i} \tag{8.20.2}$$

The orthogonal basis  $\{\mathbf{n}_i\}$  in general depends on the position of the material element.

The statement given in (ii) is equivalent to the following: There exists an orthogonal basis  $(\mathbf{n}_i)$  with respect to which

$$\mathbf{A}_1 = k(\mathbf{N} + \mathbf{N}^T) \tag{8.20.3}$$

$$\mathbf{A}_2 = 2k^2\mathbf{N}^T\mathbf{N} \tag{8.20.4}$$

where the matrix of  $\mathbf{N}$  with respect to  $(\mathbf{n}_i)$  is given by

$$[\mathbf{N}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{n}_i} \tag{8.20.5}$$

In the following examples, we demonstrate that simple shearing flow, plane Poiseuille flow, Poiseuille flow and Couette flow are all viscometric flows.

Example 8.20.1

Consider the uni-directional flow with a velocity field given in Cartesian coordinates as:

$$v_1 = v(x_2), \quad v_2 = v_3 = 0 \tag{i}$$

Show that it is a viscometric flow. We note that the uni-directional flow includes the simple shearing flow (where  $v(x_2)=kx_2$ ) and the plane Poiseuille flow.

*Solution.* In Example 8.9.1, we obtained that for this flow, the history of  $C_t(\tau)$  is given by Eq. (8.20.1) and the matrix of the two non-zero Rivlin-Ericksen tensors  $\mathbf{A}_1$  and  $\mathbf{A}_2$ , with respect to the rectangular Cartesian basis, are given by Eqs. (8.20.2) where  $k = k(x_2)$ . Thus, the given uni-directional flows are viscometric flows and the basis  $\{\mathbf{n}_i\}$  with respect to which,  $\mathbf{A}_1$  and  $\mathbf{A}_2$  have the forms given in Eq. (8.20.2), is clearly given by

$$\mathbf{n}_1 = \mathbf{e}_1, \quad \mathbf{n}_2 = \mathbf{e}_2, \quad \mathbf{n}_3 = \mathbf{e}_3 \quad (\text{ii})$$

### Example 8.20.2

Consider the axisymmetric flow with a velocity field given in cylindrical coordinates as:

$$v_r = 0, \quad v_\theta = 0, \quad v_z = v(r) \quad (\text{i})$$

Show that this is a viscometric flow. Find the basis  $\{\mathbf{n}_i\}$  with respect to which,  $\mathbf{A}_1$  and  $\mathbf{A}_2$  have the forms given in Eq. (8.20.2).

*Solution.* In Example 8.9.2, we obtained that for this flow, the history of the right Cauchy-Green deformation tensor  $\mathbf{C}_t(\tau)$  is given by an equation of the same form as Eq. (8.20.1)

where the two non-zero Rivlin-Ericksen tensors are given by

$$[\mathbf{A}_1]_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z} = \begin{bmatrix} 0 & 0 & k(r) \\ 0 & 0 & 0 \\ k(r) & 0 & 0 \end{bmatrix}_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z} \quad (\text{ii})$$

$$[\mathbf{A}_2]_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z} = \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z} \quad (\text{iii})$$

Let

$$\mathbf{n}_1 = \mathbf{e}_z, \quad \mathbf{n}_2 = \mathbf{e}_r, \quad \mathbf{n}_3 = \mathbf{e}_\theta \quad (\text{iv})$$

then

$$(\mathbf{A}_1)_{11} = (\mathbf{A}_1)_{zz}, \quad (\mathbf{A}_1)_{12} = (\mathbf{A}_1)_{zt}, \quad (\mathbf{A}_1)_{13} = (\mathbf{A}_1)_{z\theta} \quad \text{etc.}$$

Then with respect to the basis  $\{\mathbf{n}_i\}$ ,

$$[\mathbf{A}_1]_{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3} = \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{A}_2]_{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2(r) & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{n}_i} \quad (\text{v})$$

Thus, this is a viscometric flow for which the basis  $\{\mathbf{n}_i\}$  is related to the cylindrical basis  $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z)$  by Eq. (iv) [see figure 8.4].

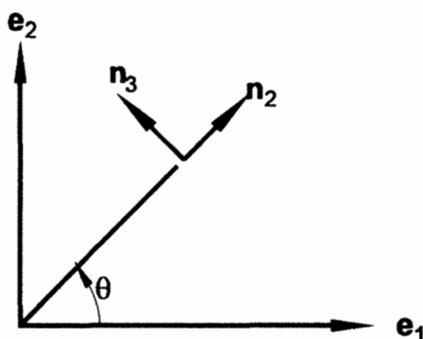


Fig. 8.4

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 Example 8.20.3

Consider the Couette flow with a velocity field given in cylindrical coordinates as

$$v_r = 0, \quad v_\theta = v(r) = r\omega(r), \quad v_z = 0 \quad (\text{i})$$

Show that this is a viscometric flow and find the basis  $\{\mathbf{n}_i\}$  with respect to which,  $\mathbf{A}_1$  and  $\mathbf{A}_2$  have the forms given in Eq. (8.20.2).

*Solution.* For the given velocity field, we obtained in Example 8.9.3

$$[\mathbf{C}_t(\tau)] = [\mathbf{I}] + (\tau - t) \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{(\tau - t)^2}{2} \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{ii})$$

where

$$k(r) = \left( \frac{dv}{dr} - \frac{v}{r} \right) = r \frac{d\omega}{dr} \quad (\text{iii})$$

The nonzero Rivlin-Ericksen tensors are

$$[\mathbf{A}_1]_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z} = \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{A}_2]_{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z} = \begin{bmatrix} 2k^2(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{iv})$$

$$[A_n]_{e_r, e_\theta, e_z} = 0 \quad \text{for } n = 3 \tag{v}$$

Comparing Eqs. (iv)(v) and (vi) with Eqs. (8.20.2), we see that the Couette flow is a viscometric flow. However, the basis  $\{n_1, n_2, n_3\}$  with respect to which,  $A_1$  and  $A_2$  have the forms given in Eq. (8.20.2), is

$$n_1 = e_\theta, \quad n_2 = e_r, \quad n_3 = e_z \tag{vi}$$

**8.21 Stresses in Viscometric Flow of an Incompressible Simple Fluid**

When a simple fluid is in viscometric flow, its history of deformation tensor  $C_t(t-\tau)$  is completely characterized by the two non-zero Rivlin-Ericksen tensor  $A_1$  and  $A_2$ . Thus, the functional in Eq. (8.14.2) becomes simply a function of  $A_1$  and  $A_2$ . That is

$$T = -p I + f(A_1, A_2) \tag{8.21.1}$$

where the Rivlin -Ericksen tensors  $A_1$  and  $A_2$  are expressible as

$$A_1 = k(N + N^T) \tag{8.21.2}$$

$$A_2 = 2k^2 N^T N \tag{8.21.3}$$

where the matrix of  $N$  relative to some choice of basis  $n_i$  is

$$[N] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.21.4}$$

Furthermore, the objectivity condition, Eq. (8.14.5) demands that for all orthogonal tensors  $Q$

$$Qf(A_1, A_2)Q^T = f(QA_1Q^T, QA_2Q^T) \tag{8.21.5}$$

In the following, we shall show that for a simple fluid in viscometric flow, with respect the basis  $n_i$ ,

$$T_{13} = T_{23} = 0$$

and that the normal stresses are all different from one another.

Let us choose a orthogonal tensor  $Q$  such that

$$[Q]_{n_i} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{i}$$

Then,

$$[\mathbf{Q}][\mathbf{N}][\mathbf{Q}^T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\mathbf{N}] \quad (\text{ii})$$

Also

$$[\mathbf{Q}][\mathbf{N}^T\mathbf{N}][\mathbf{Q}^T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\mathbf{N}^T\mathbf{N}] \quad (\text{iii})$$

i.e., for this choice of  $\mathbf{Q}$ ,

$$\mathbf{Q}\mathbf{N}\mathbf{Q}^T = \mathbf{N} \quad (\text{iv})$$

and

$$\mathbf{Q}(\mathbf{N}^T\mathbf{N})\mathbf{Q}^T = \mathbf{N}^T\mathbf{N} \quad (\text{v})$$

Thus,

$$\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T = k\mathbf{Q}(\mathbf{N} + \mathbf{N}^T)\mathbf{Q}^T = k(\mathbf{N} + \mathbf{N}^T) = \mathbf{A}_1 \quad (\text{vi})$$

and

$$\mathbf{Q}\mathbf{A}_2\mathbf{Q}^T = 2k^2\mathbf{Q}(\mathbf{N}^T\mathbf{N})\mathbf{Q}^T = 2k^2\mathbf{N}^T\mathbf{N} = \mathbf{A}_2 \quad (\text{vii})$$

Now, from Eq. (8.21.1), we have, for this particular choice of  $\mathbf{Q}$ ,

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = -p\mathbf{I} + \mathfrak{f}(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T, \mathbf{Q}\mathbf{A}_2\mathbf{Q}^T) = -p\mathbf{I} + \mathfrak{f}(\mathbf{A}_1, \mathbf{A}_2) \quad (\text{viii})$$

i.e., for this  $\mathbf{Q}$

$$\mathbf{Q}\mathbf{T}\mathbf{Q}^T = \mathbf{T} \quad (\text{ix})$$

Thus,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \quad (\text{x})$$

Carrying out the matrix multiplications, one obtains

$$\begin{bmatrix} T_{11} & T_{12} & -T_{13} \\ T_{21} & T_{22} & -T_{23} \\ -T_{31} & -T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \quad (\text{xi})$$

The above equation states that

$$T_{13} = T_{23} = 0 \quad (8.21.6)$$

Since  $A_1$  and  $A_2$  depend only on  $k$ , therefore, the nonzero stress components with respect to the basis  $\mathbf{n}_i$  are:

$$T_{12} = \tau(k) \quad (\text{xii})$$

$$T_{11} = -p + \alpha(k) \quad (\text{xiii})$$

$$T_{22} = -p + \beta(k) \quad (\text{xiv})$$

$$T_{33} = -p + \gamma(k) \quad (\text{xv})$$

where  $\alpha$ ,  $\beta$ , and  $\gamma$  are functions of  $k$ . Defining the normal stress functions by the equations

$$\sigma_1 \equiv T_{11} - T_{22} \quad (8.21.7)$$

$$\sigma_2 \equiv T_{22} - T_{33} \quad (8.21.8)$$

We can write the stress components of a simple fluid in viscometric flows as follows

$$T_{12} = \tau(k) \quad (8.21.9)$$

$$T_{11} = T_{22} + \sigma_1(k) \quad (8.21.9b)$$

$$T_{22} = T_{33} + \sigma_2(k) \quad (8.21.9c)$$

and

$$T_{13} = T_{23} = 0. \quad (8.21.9d)$$

As mentioned earlier in Section B, the function  $\tau(k)$  is called the **shear stress function** and the function  $\sigma_1(k)$ , and  $\sigma_2(k)$  are called the **normal stress functions**. [we recall that other definitions of the normal stress functions such as those given in Eq. (8.15.9) have also been used]. These three functions are known as the **viscometric functions**. These functions, when determined from the experiment on *one* viscometric flow of a fluid, determine completely the properties of the fluid in any other viscometric flow.

It can be shown that

$$\tau(-k) = -\tau(k) \quad (8.21.10)$$

$$\sigma_1(-k) = \sigma_1(k) \quad (8.21.11)$$

$$\sigma_2(-k) = \sigma_2(k) \quad (8.21.12)$$

That is,  $\tau$  is an odd function of  $k$ , while  $\sigma_1$  and  $\sigma_2$  are even functions of  $k$ .

For the fluid in simple shearing flow,  $k$  is a constant so that all stress components are independent of spatial positions. Being accelerationless, it is clear that all momentum equations are satisfied so long as  $k$  remains constant. For a Newtonian fluid, such as water, the simple shearing flow gives

$$\tau(k) = \mu k, \quad \sigma_1(k) = 0, \quad \sigma_2 = 0 \quad (8.21.13)$$

For a non-Newtonian fluid, such as a polymeric solution, for small  $k$ , the viscometric functions can be approximated by a few terms of their Taylor series expansion. Noting the  $\tau$  is an odd function of  $k$ , we have

$$\tau(k) = \mu k + \mu_1 k^3 + \dots \quad (8.21.14)$$

and noting that  $\sigma_1$  and  $\sigma_2$  are even function of  $k$ , we have

$$\sigma_1 = s_1 k^2 + \dots \quad (8.21.15)$$

$$\sigma_2 = s_2 k^2 + \dots \quad (8.21.16)$$

Since the deviation from Newtonian behavior is of the order of  $k^2$  for  $\sigma_1$  and  $\sigma_2$  and of  $k^3$  for  $\tau$ , therefore, it is expected that the deviation of the normal stresses will manifest themselves within the range of  $k$  in which the response of the shear stress remains essentially the same as that of a Newtonian fluid.

## 8.22 Channel Flow

We now consider the steady shearing flow between two infinite parallel fixed plates. That is,

$$v_1 = v(x_2), \quad v_2 = v_3 = 0 \quad (8.22.1)$$

with

$$v\left(\pm \frac{h}{2}\right) = 0 \quad (8.22.2)$$

We saw in Section 8.20 that the basis  $\mathbf{n}_i$  for which the stress components are given by Eqs. (8.21.9), is the Cartesian basis.  $\mathbf{e}_i$ . That is, with  $k(x_2) \equiv dv/dx_2$

$$\tau_{12} = \tau(k), \quad \tau_{13} = \tau_{23} = 0, \quad \tau_{11} = \tau_{22} + \sigma_1(k), \quad \tau_{22} = \tau_{33} + \sigma_2(k) \quad (8.22.3)$$

Substituting the above equation in the equations of motion, we get, in the absence of body forces, [noting that  $k$  depends only on  $x_2$ ]

$$-\frac{\partial p}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2} = 0, \quad \frac{\partial \tau_{22}}{\partial x_2} - \frac{\partial p}{\partial x_2}, \quad \frac{\partial p}{\partial x_3} = 0 \quad (i)$$

Differentiating the above three equations with respect to  $x_1$ , we get

$$\frac{\partial}{\partial x_1} \frac{\partial p}{\partial x_1} = \frac{\partial}{\partial x_2} \frac{\partial p}{\partial x_1} = \frac{\partial}{\partial x_3} \frac{\partial p}{\partial x_1} = 0 \quad (ii)$$

Thus,  $-\frac{\partial p}{\partial x_1} = \text{a constant}$ . Let this constant be denoted by  $f$ , which is the driving force for the flow, we have,

$$-\frac{\partial p}{\partial x_1} = -f \quad (8.22.4)$$

Now, the first equation in Eq. (i) gives

$$\frac{\partial \tau(k)}{\partial x_2} - \frac{\partial p}{\partial x_1} = 0 \quad (iii)$$

so that

$$\tau(k) = -f x_2 \quad (8.22.5)$$

where the integration constant is taken to be zero because the flow is symmetric with respect to the plane  $x_2 = 0$ . Inverting Eq. (8.22.5), we have,

$$k = \tau^{-1}(-f x_2) \equiv \gamma(-f x_2) = -\gamma(f x_2) \quad (8.22.6)$$

where  $\gamma(S)$ , the inverse function of  $\tau(k)$ , is an odd function because  $\tau(k)$  is an odd function. Since  $k(x_2) \equiv dv/dx_2$ , therefore, the above equation gives

$$\frac{dv}{dx_2} = -\gamma(f x_2) \quad (8.22.7)$$

Integrating, we get

$$v(x_2) = -\int_{-\frac{h}{2}}^{x_2} \gamma(f x_2) dx_2 \quad (8.22.8)$$

For a given simple fluid with a known shear stress function  $\tau(k)$ ,  $\gamma(S)$  is also known, the above equation can be integrated to give the velocity distribution in the channel. The volume flux per unit width  $Q$  is given by

$$Q = \int_{-\frac{h}{2}}^{\frac{h}{2}} v(x_2) dx_2 \quad (8.22.9)$$

Equation (8.22.9) can be written in a form suitable for determining the function  $\gamma(S)$  from an experimentally measured relationship between  $Q$  and  $f$ . Indeed, integration by part gives

$$Q = x_2 v(x_2) \Big|_{-h/2}^{h/2} - \int_{-h/2}^{h/2} x_2 dv = - \int_{-h/2}^{h/2} x_2 \frac{dv}{dx_2} dx_2 \quad (iv)$$

Using Eq. (8.22.7), we obtain

$$Q = \int_{-h/2}^{h/2} x_2 \gamma(f x_2) dx_2 = \frac{1}{f^2} \int_{-fh/2}^{fh/2} S \gamma(S) dS \quad (v)$$

or,

$$Q = \frac{2}{f^2} \int_0^{fh/2} S \gamma(S) dS \quad (8.22.10)$$

Thus,

$$\frac{\partial(f^2 Q)}{\partial f} = 2 \left( \frac{fh}{2} \right) \gamma \left( \frac{fh}{2} \right) \left( \frac{h}{2} \right) \quad (vi)$$

so that

$$\gamma \left( \frac{fh}{2} \right) = \frac{2}{fh^2} \frac{\partial}{\partial f} (f^2 Q) \quad (8.22.11)$$

Now, if the variation of  $Q$  with the driving force  $f$  (the pressure gradient), is measured experimentally, then the right hand side of the above equation is known so that the inverse shear stress function  $\gamma(S)$  is obtained from the above equation.

#### Example 8.22.1

Use Eq. (8.22.8) to calculate the volume discharge per unit width across a cross section of the channel for a Newtonian fluid.

*Solution.* For a Newtonian fluid,

$$S = \tau(k) = \mu k, \text{ so that } k = \frac{S}{\mu} \equiv \gamma(S) \quad (i)$$

Thus,

$$\gamma(-fx) = \frac{-fx}{\mu} \quad (ii)$$

$$v(x_2) = \int_{-h/2}^{x_2} \frac{-fx}{\mu} dx = - \left[ \frac{f x^2}{\mu 2} \right]_{-h/2}^{x_2} = - \frac{f}{\mu} \left( \frac{x_2^2}{2} - \frac{h^2}{8} \right) \quad (iii)$$

and

$$Q = - \int_{-d/2}^{d/2} x \left( - \frac{fx}{\mu} \right) dx = \left[ \frac{fx^3}{\mu 3} \right]_{-d/2}^{d/2} = \frac{fd^3}{12\mu} \quad (iv)$$

### 8.23 Couette Flow

Couette flow is defined to be the two dimensional steady laminar flow between two concentric infinitely long cylinders which rotate with angular velocities  $\Omega_1$  and  $\Omega_2$ . The velocity field is given by

$$v_r = 0, \quad v_\theta = v(r) = r \omega(r), \quad v_z = 0 \tag{8.23.1}$$

and in the absence of body forces, there is no pressure gradient in the  $\theta$  and  $z$  directions.

In example 8.20.3, we see that the Couette flow is a viscometric flow and with

$$\mathbf{n}_1 = \mathbf{e}_\theta, \quad \mathbf{n}_2 = \mathbf{e}_r, \quad \mathbf{n}_3 = \mathbf{e}_z \tag{i}$$

the nonzero Rivlin-Ericksen tensors are given by

$$[\mathbf{A}_1]_{\mathbf{e}_\theta, \mathbf{e}_r, \mathbf{e}_z} = \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{ii}$$

$$[\mathbf{A}_2]_{\mathbf{e}_\theta, \mathbf{e}_r, \mathbf{e}_z} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2k^2(r) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{iii}$$

where

$$k(r) = r \frac{d\omega}{dr} \tag{iv}$$

Thus, the stress components with respect to the basis  $\{\mathbf{n}_i\}$  are given by (see Section 8.21)

$$\tau_{\theta r} = \tau(k) \tag{8.23.2}$$

$$\tau_{\theta\theta} - \tau_{rr} = \sigma_1(k) \tag{8.23.3}$$

$$\tau_{rr} - \tau_{zz} = \sigma_2(k) \tag{8.23.4}$$

$$\tau_{rz} = \tau_{\theta z} = 0 \tag{8.23.5}$$

where  $\tau(k)$ ,  $\sigma_1(k)$ , and  $\sigma_2(k)$  are the shear stress function, the first normal stress function and the second normal stress function respectively. These three functions completely characterize the fluid in any viscometric flow, of which the present Couette flow is one. For a given simple fluid, these three functions are assumed to be known. On the other hand, we may use any one of the viscometric flows to measure these functions for use with the same fluid in other viscometric flows.

Let us first assume that we know these functions, then our objective is to find the velocity distribution,  $v(r)$  and the stress distributions  $\tau_{ij}(r)$  in this flow when the externally applied torque  $M$  per unit height in the axial direction is given.

In cylindrical coordinates, the equations of motion are

$$\frac{d\tau_{rr}}{dr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) - \frac{dp}{dr} = -\rho r \omega^2 \quad (8.23.6)$$

$$\frac{1}{r} \frac{d}{dr}(r^2 \tau_{r\theta}) = 0 \quad (8.23.7)$$

The z-equation of motion is identically satisfied, in view of Eq. (8.23.5) and the fact that  $\tau_{zz}$  does not depend on z.

Equation (8.23.7) gives

$$\tau_{r\theta} = \frac{C}{r^2} \quad (v)$$

where  $C$  is the integration constant. The torque per unit height of the cylinders needed to maintain the flow is clearly given by

$$M = (2\pi r \tau_{r\theta})r \quad (8.23.8)$$

thus,

$$C = \frac{M}{2\pi} \quad (vi)$$

Now, to find the velocity distribution  $v(r)$ , from the known shear stress function  $\tau(k)$ , we first note that  $\tau_{r\theta} = \tau(k)$  so that by Eq. (8.23.8), we have

$$\tau(k) = \frac{M}{2\pi r^2} \quad (vii)$$

Now, we wish to determine the function  $k(r) \equiv r \frac{d\omega(r)}{dr}$  in Couette flow. To do this, we let

$$S(r) \equiv \tau(k(r)) \quad (8.23.9a)$$

and

$$k(r) = \gamma(S(r)) \quad (8.23.9b)$$

where  $\gamma(S)$  is the inverse of the function  $\tau(k)$ , and is therefore a known function when the function  $\tau(k)$  is known.

Since  $k(r) = r \frac{d\omega}{dr}$ , therefore, from Eq. (8.23.9b), we have

$$r \frac{d\omega}{dr} = \gamma(S) \quad (viii)$$

where from Eqs. (vii) and (8.23.9a),

$$S = \frac{M}{2\pi r^2} \tag{ix}$$

Now,

$$\frac{d\omega}{dr} = \frac{d\omega}{dS} \frac{dS}{dr} = \frac{d\omega}{dS} \left( \frac{M}{2\pi} \right) \left( \frac{-2}{r^3} \right) = \frac{M}{2\pi r^2} \frac{2}{r} \frac{d\omega}{dS} = -2S r \frac{d\omega}{dS} \tag{x}$$

Thus,

$$\gamma(S) = r \frac{d\omega}{dr} = -2S \frac{d\omega}{dS} \tag{xi}$$

from which we have

$$d\omega = -\frac{\gamma(S)}{2S} dS \tag{xii}$$

Integration of the above equation gives

$$\omega(r) = \Omega_1 - \frac{1}{2} \int \frac{\frac{M}{2\pi r}}{\frac{M}{2\pi R_1^2}} \frac{\gamma(S)}{S} dS \tag{8.23.10}$$

where  $\Omega_1$  is the angular velocity of the inner cylinder. For given  $\gamma(S)$ , the above equation gives the desired function  $\omega(r)$  from which  $v_\theta = r\omega(r)$  can be obtained.

Next, we wish to obtain the normal stresses  $T_{rr}$  at the two cylindrical surfaces  $r = R_1$  and  $r = R_2$ .

From the  $r$ -equation of motion, Eq. (8.23.6), we have

$$\frac{d\tau_{rr}}{dr} + \frac{1}{r}(\tau_{rr} - \tau_{\theta\theta}) = -r\rho\omega^2 \tag{xiii}$$

That is, using Eq. (8.23.3)

$$\frac{d\tau_{rr}}{dr} - \frac{\sigma_1}{r} = -r\rho\omega^2 \tag{xiv}$$

Integrating, we get

$$\tau_{rr}(r) - \tau_{rr}(R_1) = \int_{r=R_1}^r \frac{\sigma_1}{r} dr - \rho \int_{r=R_1}^r r \omega^2 dr \tag{xv}$$

In particular,

$$[-T_{rr}(R_2)] - [-T_{rr}(R_1)] = \rho \int_{R_1}^{R_2} r \omega^2(r) dr - \int_{R_1}^{R_2} \frac{\sigma_1}{r} dr \tag{8.23.11}$$

In the right hand side of the above equation, the first term is always positive, stating that the centrifugal force effects always make the pressure on the outer cylinder larger than that on the inner cylinder. On the other hand, for a fluid which has a positive normal stress function  $\bar{\sigma}_1$ , the second term in the above equation is negative, stating that the contribution to the pressure difference due to the normal stress effect is in the opposite direction to that due to the centrifugal force effect. Indeed, all known polymeric solutions have a positive  $\bar{\sigma}_1$  and in many instances, this normal stress effects actually causes the pressure on the inner cylinder to be larger than that on the outer cylinder.

We now consider the reverse problem of determining the material function  $\gamma(S)$  and therefore  $\tau(S)$  from a measured relationship between the torque  $M$  needed to maintain the Couette flow and the angular velocity difference  $\Omega_2 - \Omega_1$ .

Since

$$\tau_1 = \frac{M}{2\pi R_1^2} \tag{xvi}$$

therefore,

$$\tau_2 = \frac{M}{2\pi R_2^2} = \frac{M}{2\pi R_1^2} \left(\frac{R_1}{R_2}\right)^2 \tag{xvii}$$

That is

$$\tau_2 = \beta \tau_1 \tag{xviii}$$

where

$$\beta = \left(\frac{R_1}{R_2}\right)^2 < 1 \tag{xix}$$

Now, from Eq. (8.23.10), we have

$$\Delta\Omega = \int_{\tau_2}^{\tau_1} \frac{\gamma(S)}{2S} dS \tag{8.23.12}$$

where

$$\Delta\Omega = \Omega_2 - \Omega_1 \tag{xx}$$

and

$$\tau_1 = \frac{M}{2\pi R_1^2} \text{ and } \tau_2 = \frac{M}{2\pi R_2^2} \quad (\text{xxi})$$

Differentiating the above equation with respect to  $M$  gives

$$2M \frac{\partial \Delta \Omega}{\partial M} = \gamma(\tau_1) - \gamma(\tau_2) \quad (8.23.13)$$

Using Eq. (xix), we have

$$2M \frac{\partial \Delta \Omega}{\partial M} = \gamma(\tau_1) - \gamma(\beta \tau_1) \quad (8.23.14)$$

Defining

$$\Gamma(\tau_1) \equiv \gamma(\tau_1) - \gamma(\beta \tau_1) \quad (8.23.15)$$

we have,

$$2M \frac{\partial \Delta \Omega}{\partial M} = \Gamma(\tau_1) \quad (8.23.16)$$

i.e.,

$$2M \frac{\partial \Delta \Omega}{\partial M} = \Gamma \left( \frac{M}{2\pi R_1^2} \right) \quad (8.23.17)$$

Equation (8.23.17) allows the determination of  $\Gamma(\tau_1)$  from experimental results relating  $\Delta \Omega$  with  $M$ . To obtain  $\gamma(S)$  and therefore  $\tau(k)$ , from  $\Gamma(\tau_1)$ , we note the following

$$\Gamma(\tau_1) = \gamma(\tau_1) - \gamma(\beta \tau_1) \quad (\text{xxii})$$

$$\Gamma(\beta \tau_1) = \gamma(\beta \tau_1) - \gamma(\beta^2 \tau_1) \quad (\text{xxiii})$$

$$\Gamma(\beta^2 \tau_1) = \gamma(\beta^2 \tau_1) - \gamma(\beta^3 \tau_1) \quad (\text{xxiv})$$

Thus,

$$\sum_n^N \Gamma(\beta^n \tau_1) = \gamma(\tau_1) - \gamma(\beta^{N+1} \tau_1) \quad (\text{xxv})$$

Since  $\beta < 1$ ,  $\lim_{N \rightarrow \infty} \beta^{N+1} \rightarrow 0$ ,  $\gamma(\beta^{N+1} \tau_1) \rightarrow \gamma(0) = 0$ , therefore,

$$\gamma(\tau) = \sum_{n=0}^{\infty} \Gamma(\beta^n \tau) \quad (8.23.18)$$

Thus, from the known function  $\Gamma(\tau)$ , the above equation allows one to obtain the inverse shear function  $\gamma(S)$  from which the shear function  $\tau(S)$  can be obtained.

If the gap  $R_2 - R_1$  is very small, the rate of shear  $k$  will be essentially a constant independent of  $r$  given by

$$k = \frac{R_1 \Delta \Omega}{R_2 - R_1}$$

Thus,  $k = \gamma(\tau_1)$  leads to

$$\gamma\left(\frac{M}{2\pi R_1}\right) = \frac{R_1 \Delta \Omega}{R_2 - R_1} \quad (8.23.19)$$

By measuring the relationship between  $M$  and  $\Delta \Omega$ , the above equation determines the inverse shear stress function  $\gamma(S)$ .

## PROBLEMS

8.1. Show that for an incompressible Newtonian fluid in Couette flow, the pressure at the outer cylinder ( $r = R_o$ ) is always larger than that at the inner cylinder. That is, obtain

$$[-T_{rr}(R_o)] - [-T_{rr}(R_i)] = \rho \int_{R_i}^{R_o} r \omega^2(r) dr$$

8.2. Obtain the force-displacement relationship for N-Maxwell elements connected in parallel. Neglect inertia effects.

8.3. Obtain the force-displacement relationship for the Kelvin-Voigt solid which consists of a dashpot (with damping coefficient  $\eta$ ) and a spring (with spring constant  $G$ ) connected in parallel. Also, obtain its relaxation function. Neglect inertia effects.

8.4. Obtain the force-displacement relationship for a dashpot (damping coefficient  $\eta_o$ ) and a Kelvin-Voigt solid (damping coefficient  $\eta$  and spring constant  $G$ , see the previous problem) connected in series. Also, obtain the relaxation function.

8.5. A linear Maxwell fluid, defined by Eq.(8.1.1), is between two parallel plates which are one unit apart. Starting from rest, at time  $t = 0$ , the top plate is given a displacement  $u = v_o t$  while the bottom plate remains fixed. Neglect inertia effects, obtain the shear stress history.

8.6. Obtain Eq.(8.3.1) by solving the linear, nonhomogeneous ordinary differential equation Eq.(8.1.1b).

8.7. Show that for the linear Maxwell fluid defined by Eqs. (8.1.1)

$$\int_{-\infty}^t \phi(t-t') J(t') dt' = t$$

where  $\phi(t)$  is the relaxation function and  $J(t)$  is the creep compliance function.

8.8. Obtain the storage modulus and loss modulus for the linear Maxwell fluid with a continuous relaxation spectrum defined by Eq.(8.4.1).

8.9. Show that the viscosity  $\mu$  of a linear Maxwell fluid defined by Eqs. (8.1.1) is related to the relaxation function  $\phi(t)$  and the memory function  $f(s)$  by the relation

$$\mu = \int_0^{\infty} \phi(s) ds = - \int_0^{\infty} s f(s) ds$$

8.10. Show that the relaxation function for the Jeffrey's model, Eq.(8.2.5) with  $a_2 = 0$  is given by

$$\phi(t) = \frac{b_o}{a_1} \left[ \left( 1 - \frac{b_1}{b_o a_1} \right) e^{-t/a_1} + \frac{2 b_1}{b_o} \delta(t) \right]$$

where  $\delta(t)$  is the Dirac function.

**8.11.** For the following velocity field, obtain (a) the particle pathline equation using the current time as the reference time, (b) the relative right Cauchy-Green deformation tensor and (c) the Rivlin-Ericksen tensor.

$$v_1 = 0, \quad v_2 = v(x_1), \quad v_3 = 0$$

**8.12.** Given the velocity field

$$v_1 = -k x_1, \quad v_2 = k x_2, \quad v_3 = 0$$

- (a) Obtain the relative right Cauchy-Green deformation tensor.  
 (b) Using Eq.(8.9.2), obtain the Rivlin-Ericksen tensors.  
 (c) Obtain the Rivlin-Ericksen tensors from the recursive equation, Eq.(8.10.3).  
 (d) Is this velocity field a viscometric flow?

**8.13.** Do the previous problem for the velocity field

$$v_1 = k x_1, \quad v_2 = k x_2, \quad v_3 = -2 k x_3$$

**8.14.** Given the velocity field

$$v_1 = k x_2, \quad v_2 = k x_1, \quad v_3 = 0$$

- (a) Obtain the pathline equations using the current time as the reference time.  
 (b) Obtain the relative right Cauchy-Green deformation tensor.  
 (c) Using Eq.(8.9.2) to obtain the Rivlin-Ericksen tensor.  
 (d) Using Eq.(8.10.3) to obtain the Rivlin-Ericksen tensor.

**8.15.** Given the velocity field

$$v_1 = -k x_1, \quad v_2 = k x_2, \quad v_3 = 0$$

- (a) Obtain the stress field  $\mathbf{T}$  for a Newtonian fluid.  
 (b) Obtain the co-rotational derivative of the stress tensor  $\mathbf{T}$ .  
 (c) Obtain the upper convected derivative of the stress tensor  $\mathbf{T}$ .  
 (d) Obtain the lower convected derivative of the stress tensor  $\mathbf{T}$ .

**8.16.** Do the previous problem for the following velocity field

$$v_1 = k x_1, \quad v_2 = k x_2, \quad v_3 = -2 k x_3$$

**8.17.** Given the velocity field

$$v_1 = -k x_1, \quad v_2 = k x_2, \quad v_3 = 0$$

- (a) Obtain the stress field  $\mathbf{T}$  for a second-order fluid.  
 (b) Obtain the co-rotational derivative of the stress tensor  $\mathbf{T}$ .  
 (c) Obtain the upper convected derivative of the stress tensor  $\mathbf{T}$ .  
 (d) Obtain the lower convected derivative of the stress tensor  $\mathbf{T}$ .

8.18. Do the previous problem for the following velocity field

$$v_1 = k x_1, \quad v_2 = k x_2, \quad v_3 = -2 k x_3$$

8.19. Derive Eqs.(8.8.4).

8.20. Derive Eqs. (8.8.9b) and (8.8.9e).

8.21. Derive Eq. (8.10.3).

8.22. Derive Eq. (8.18.5)

8.23. Show from Eq.(8.18.15), that Oldroyd's upper convected derivative is objective.

8.24. The Reiner-Rivlin fluid is defined by the constitutive equation

$$\mathbf{T} = -p \mathbf{I} + \boldsymbol{\tau}$$

$$\boldsymbol{\tau} = \phi_1(I_2, I_3)\mathbf{D} + \phi_2(I_2, I_3)\mathbf{D}^2$$

where  $I_i$  are the scalar invariants of  $\mathbf{D}$ . Obtain the stress components for this fluid in a simple shearing flow.

8.25. The exponential of a tensor  $\mathbf{A}$  is defined as

$$\exp \mathbf{A} = \mathbf{I} + \sum_1^{\infty} \frac{1}{n!} \mathbf{A}^n$$

If  $\mathbf{A}$  is an objective tensor, is  $\exp \mathbf{A}$  also objective?

8.26. Why is it that the following constitutive equation is not acceptable?

$$\mathbf{T} = -p \mathbf{I} + \boldsymbol{\tau}$$

$$\boldsymbol{\tau} = \alpha (\nabla \mathbf{v})$$

where  $\mathbf{v}$  is the velocity vector and  $\alpha$  is a constant.

8.27. Let  $da$  and  $d\mathbf{A}$  denote the differential area vector at time  $\tau$  and at time  $t$  respectively. For an incompressible fluid, show that

$$\left[ \frac{D^N da^2}{D \tau^N} \right]_{\tau=t} = d\mathbf{A} \cdot \left[ \frac{D^N \mathbf{C}_t^{-1}}{D \tau^N} \right]_{\tau=t} d\mathbf{A} \equiv -d\mathbf{A} \cdot \mathbf{M}_N d\mathbf{A}$$

where  $da$  is the magnitude of  $d\mathbf{A}$ , and the tensor  $\mathbf{M}_N$  are known as the White-Metzner tensors

8.28. (a) Verify that Oldroyd's lower convected derivatives of the identity tensor are the Rivlin-Ericksen tensors  $\mathbf{A}_N$ .

(b) Verify that Oldroyd's upper convected derivatives of the identity tensor are the negative White-Metzner tensors (see Prob. 8.27).

8.29. Show that the derivative given in Eq.(xviii) of Section 8.18 is objective.

8.30. Obtain Eq. (8.18.12) for Oldroyd's lower convected derivative.

8.31. Obtain Eq.(8.18.18) for Oldroyd's upper convected derivative.

8.32. Show that the lower convected derivative of the first Rivlin-Ericksen tensor  $A_1$  is the second Rivlin-Ericksen tensor  $A_2$ .

8.33. Consider the following constitutive equation

$$\boldsymbol{\tau} + \lambda \frac{D_* \boldsymbol{\tau}}{Dt} = 2\mu \mathbf{D}$$

where  $\frac{D_* \boldsymbol{\tau}}{Dt} = \frac{D_{cr} \boldsymbol{\tau}}{Dt} + a(\mathbf{D} \boldsymbol{\tau} + \boldsymbol{\tau} \mathbf{D})$  and  $a$  is a constant. Obtain the shear stress function and the two normal stress functions for this fluid. We note that  $a = 1$  corresponds to Eq.(8.18.4) and  $a = -1$  corresponds to Eq.(8.18.20).

8.34. Let  $\mathbf{Q}$  be a tensor whose matrix with respect to the basis  $\mathbf{n}_i$  is

$$[\mathbf{Q}]_{\mathbf{n}_i} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) Verify the following relations for the tensor  $\mathbf{N}$  whose matrix with respect to  $\mathbf{n}_i$  is given by Eq.(8.20.5):  $\mathbf{Q}\mathbf{N}\mathbf{Q}^T = -\mathbf{N}$  and  $\mathbf{Q}\mathbf{N}^T\mathbf{N}\mathbf{Q} = \mathbf{N}^T\mathbf{N}$

(b) For  $\mathbf{A}_1$  and  $\mathbf{A}_2$  given by Eq.(8.20.3) and Eq.(8.20.4), verify the relations

$$\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T = -\mathbf{A}_1 \quad \text{and} \quad \mathbf{Q}\mathbf{A}_2\mathbf{Q}^T = \mathbf{A}_2$$

(c) From  $\mathbf{T} = -p\mathbf{I} + \mathfrak{f}(\mathbf{A}_1, \mathbf{A}_2)$  where  $\mathbf{A}_1$ , and  $\mathbf{A}_2$  are given by Eq.(8.20.3) and (8.20.4) and

$$\mathbf{Q} \mathfrak{f}(\mathbf{A}_1, \mathbf{A}_2) \mathbf{Q}^T = \mathfrak{f}(\mathbf{Q}\mathbf{A}_1\mathbf{Q}^T, \mathbf{Q}\mathbf{A}_2\mathbf{Q}^T)$$

show that

$$\mathbf{Q}\mathbf{T}(k)\mathbf{Q}^T = \mathbf{T}(-k)$$

(d) From the results of part (c), show that the viscometric functions have the properties

$$\tau(k) = -\tau(-k), \quad \sigma_1(k) = \sigma_1(-k), \quad \sigma_2(k) = \sigma_2(-k)$$

8.35. For the velocity field given in Example 8.20.2, i.e.,

$$v_r = 0, \quad v_\theta = 0, \quad v_z = v(r),$$

(a) Obtain the stress components in terms of the shear stress function  $\tau(k)$  and the normal stress functions  $\sigma_1(k)$  and  $\sigma_2(k)$ , where  $k = dv/dr$ . (b) Obtain the following velocity distribution for the Poiseuille flow under a pressure gradient of  $-f$ :

$$v(r) = \int_r^R \gamma \left( \frac{fr}{2} \right) dr$$

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where  $\gamma$  is the inverse shear stress function

(c) Obtain the relation

$$\gamma\left(\frac{Rf}{2}\right) = \frac{1}{\pi R^3 f^2} \frac{\partial(f^3 Q)}{\partial f}$$

where  $Q$  is the volume flux.

## APPENDIX: MATRICES

### Matrix

A matrix is an aggregate of elements arranged in rectangular array.

The following is a matrix of  $m$  rows and  $n$  columns

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ T_{21} & T_{22} & \dots & T_{2n} \\ \dots & \dots & \dots & \dots \\ T_{m1} & T_{m2} & \dots & T_{mn} \end{bmatrix} \quad (1)$$

There are  $m \times n$  elements in the above matrix. The first subscript of each element refers to the row in which the element is located and the second subscript refers to the column. Thus  $T_{32}$  is located in the third row and second column of the matrix. In general,  $T_{ij}$  is the element in the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column. The matrix in Eq.(1) may also be denoted by

$$[\mathbf{T}] = [T_{ij}] \quad (2)$$

where  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$

Whenever necessary, we shall use the notation  $[\mathbf{T}]^{m \times n}$  to indicate there are  $m$  rows and  $n$  columns in the matrix  $[\mathbf{T}]$ .

### Transpose of a Matrix

Let  $[\mathbf{S}]^{m \times n} = [S_{ij}]$  be a matrix of  $m$  rows and  $n$  columns. If  $S_{ij} = T_{ji}$ , then  $[\mathbf{S}]$  is called the transpose of  $[\mathbf{T}]$ , and will be denoted by  $[\mathbf{S}]^T$ .

For example, if

$$[\mathbf{T}]^{2 \times 3} = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix}$$

then

$$[\mathbf{S}]^{3 \times 2} = \begin{bmatrix} 1 & 4 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$$

### Square Matrix

A matrix with equal number of rows and columns is called a square matrix. For example,

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

is a square matrix of the third order. The elements  $T_{11}, T_{22}, T_{33}$  are called the diagonal elements. Example:

$$[\mathbf{T}] = \begin{bmatrix} T_{11} & 4 & -1 \\ 4 & T_{22} & 2 \\ -1 & 2 & T_{33} \end{bmatrix}$$

Note that for a **symmetric matrix**

$$[\mathbf{T}] = [\mathbf{T}]^T$$

### Diagonal Matrix

A square matrix with the property that all nondiagonal elements are zero is called a diagonal matrix. Example:

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Note that a diagonal matrix is a symmetric matrix.

### Scalar Matrix

A diagonal matrix with the property that  $T_{11} = T_{22} = T_{33} = \alpha$  is called a scalar matrix. Example:

$$[\mathbf{T}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

### Identity Matrix

A scalar matrix with the property that the diagonal elements equal unity is called the identity matrix. We shall denote the identity matrix by  $[\mathbf{I}]$ . Thus

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

### Row Matrix

A matrix with only one row is called a row matrix. Only one index is necessary to locate the position of an element. Thus

$$[\mathbf{a}] = [a_1, a_2, a_3] = [a_i]_r$$

where the subscript  $r$  indicates that  $[\mathbf{a}]$  is a row matrix.

### Column Matrix

A matrix with only one column is called a column matrix. Thus

$$[\mathbf{a}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = [a_i]_c$$

is a column matrix. Note that

$$[\mathbf{a}]_c^T \text{ is the row matrix given by } [a_1, a_2, a_3]$$

### Matrix Operation

1. If  $[\mathbf{T}]^{m \times n} = [T_{ij}]$  and  $[\mathbf{S}]^{m \times n} = [S_{ij}]$ , then  $[\mathbf{T}] = [\mathbf{S}]$ , if and only if  $T_{ij} = S_{ij}$

2. If  $\alpha$  is a scalar, then

$$\alpha[\mathbf{T}] = [\mathbf{T}]\alpha = [\alpha T_{ij}]$$

Example:

$$3 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

3.  $[\mathbf{T}]^{m \times n} + [\mathbf{S}]^{m \times n} \equiv [T_{ij} + S_{ij}]$

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 6 \end{bmatrix}$$

The following rules follows from the operation rules of scalars

$$[\mathbf{T}] + [\mathbf{S}] = [\mathbf{S}] + [\mathbf{T}]$$

$$[\mathbf{T}] + ([\mathbf{S}] + [\mathbf{R}]) = ([\mathbf{T}] + [\mathbf{S}]) + [\mathbf{R}]$$

$$(\alpha + \beta)[\mathbf{T}] = \alpha[\mathbf{T}] + \beta[\mathbf{T}]$$

$$\alpha[\mathbf{T} + \mathbf{S}] = \alpha[\mathbf{T}] + \alpha[\mathbf{S}]$$

4. If  $[T_{ij}]^{m \times n}$  is a matrix of  $m$  rows and  $n$  columns and  $[S_{ij}]^{n \times p}$  is a matrix of  $n$  rows and  $p$  columns, then

$$[\mathbf{T}]^{m \times n} [\mathbf{S}]^{n \times p} \equiv [\mathbf{R}]^{m \times p}$$

where the elements of  $[\mathbf{R}]$  are given by

$$R_{ij} = \sum_{k=1}^n T_{ik} S_{kj} \quad i=1,2,\dots,m, \quad j=1,2,\dots,p$$

Example:

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \\ S_{31} & S_{32} \end{bmatrix}$$

$$= \begin{bmatrix} T_{11} S_{11} + T_{12} S_{21} + T_{13} S_{31} & T_{11} S_{12} + T_{12} S_{22} + T_{13} S_{32} \\ T_{21} S_{11} + T_{22} S_{21} + T_{23} S_{31} & T_{21} S_{12} + T_{22} S_{22} + T_{23} S_{32} \end{bmatrix} \\ [a_1, a_2, a_3] \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

It is important to note that only when the number of columns of the first matrix is the same as the number of rows of the second matrix is multiplication defined. Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

But

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

is not defined.

The following examples show even if both  $[T][S]$  and  $[S][T]$  are defined, they are in general not equal. Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 7 \\ 11 & 15 \end{bmatrix}$$

But

$$\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 11 & 16 \end{bmatrix}$$

However, if  $[T]$  is a scalar matrix, i.e.,

$$[T] = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

then,

$$[T][S] = [S][T]$$

provided both  $[T][S]$  and  $[S][T]$  are defined (that is, provided  $[S]$  is also a square matrix and of the same order as that of the  $[T]$ ). In particular, if  $[S]$  is a square matrix of the same order as that of the unit matrix  $[I]$ , then

$$[I][S] = [S][I] = [S]$$

It can be shown that the matrix product has the following properties:

$$[T]([S][R]) = ([T][S])[R]$$

$$\alpha[TS] = (\alpha[T])[S] = [T](\alpha[S])$$

$$[T + S][R] = [TR] + [SR]$$

$$[R][T + S] = [R][T] + [R][S]$$

### The Reversed Rule for a Transposed Product

In the following, we shall show that the transpose of a product of matrices is equal to the product of their transpose in reverse order, i.e.,

$$[\mathbf{TS}]^T = [\mathbf{S}]^T [\mathbf{T}]^T$$

*Proof* : Let

$$\begin{aligned} [\mathbf{T}] &= [T_{ij}]^{m \times n}, & [\mathbf{S}] &= [S_{ij}]^{n \times p} \\ [\mathbf{T}]^T \equiv [\mathbf{A}] &= [A_{ij}]^{n \times m} & [\mathbf{S}]^T \equiv [\mathbf{B}] &= [B_{ij}]^{p \times n} \\ [\mathbf{TS}] \equiv [\mathbf{C}] &= [C_{ij}]^{m \times p} & [\mathbf{TS}]^T \equiv [\mathbf{D}] &= [D_{ij}]^{p \times m} \end{aligned}$$

and

$$[\mathbf{S}]^T [\mathbf{T}]^T = [\mathbf{B}] [\mathbf{A}] \equiv [\mathbf{E}] = [E_{ij}]^{p \times m}$$

then,

$$C_{ij} = \sum_{k=1}^n T_{ik} S_{kj} \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, p$$

so that

$$D_{ij} = C_{ji} = \sum_{k=1}^n T_{jk} S_{ki} \quad j = 1, 2, \dots, m, \quad i = 1, 2, \dots, p$$

On the other hand

$$\begin{aligned} E_{ij} &= \sum_{k=1}^n B_{ik} A_{kj} \quad i = 1, 2, \dots, p, \quad j = 1, 2, \dots, m \\ &= \sum_{k=1}^n S_{ki} T_{jk} \quad i = 1, 2, \dots, p, \quad j = 1, 2, \dots, m \end{aligned}$$

Thus,

$$E_{ij} = D_{ij}$$

That is

$$[\mathbf{TS}]^T = [\mathbf{S}]^T [\mathbf{T}]^T$$

### Inverse of a Matrix

A square matrix  $[\mathbf{S}]$  is called the inverse of the square matrix  $[\mathbf{T}]$  if

$$[\mathbf{T}] [\mathbf{S}] = [\mathbf{I}]$$

where  $[\mathbf{I}]$  is the unit matrix. We shall denote  $[\mathbf{S}]$  by  $[\mathbf{T}]^{-1}$ . Thus,

$$[\mathbf{T}] [\mathbf{T}]^{-1} = [\mathbf{I}]$$

It can be proved that if the determinant of  $[\mathbf{T}]$ , ie.,  $|T_{ij}|$  is not equal to zero, then the inverse of  $[\mathbf{T}]$  exists and that

$$[\mathbf{T}] [\mathbf{T}]^{-1} = [\mathbf{T}]^{-1} [\mathbf{T}] = [\mathbf{I}]$$

where  $[\mathbf{T}]^{-1}$  is unique. The proof will not be given here.

### The Reverse Rule of the Inverse of a Product of Matrices.

In the following we shall show that the inverse of a product of matrices is equal to the product of their inverses in reversed order, i.e.,

$$[\mathbf{TS}]^{-1} = [\mathbf{S}]^{-1} [\mathbf{T}]^{-1}$$

*Proof.*

$$[\mathbf{TS}] [\mathbf{S}]^{-1} [\mathbf{T}]^{-1} = [\mathbf{T}] [\mathbf{S}] [\mathbf{S}]^{-1} [\mathbf{T}]^{-1} = [\mathbf{T}] [\mathbf{T}]^{-1} = [\mathbf{I}].$$

Thus,  $[\mathbf{S}]^{-1} [\mathbf{T}]^{-1}$  is the unique inverse of  $[\mathbf{TS}]$ .

### Differentiation of a Matrix

If  $T_{ij}$  are functions of  $x, y, z, t$ , then

$$\frac{\partial}{\partial x} [\mathbf{T}] = \left[ \frac{\partial T_{ij}}{\partial x} \right]$$

$$\frac{\partial}{\partial y} [\mathbf{T}] = \left[ \frac{\partial T_{ij}}{\partial y} \right]$$

etc.

## Answers to Problems

### CHAPTER 2

2A1. (a) 5 (b) 28 (c) 23

2A2. (a) and (c)

2A5. (a)  $T_{11} = T_{22} = T_{33} = 0$ ,  $T_{12} = T_{21} = 0$

$$T_{23} = -T_{32} = 1, T_{31} = -T_{13} = 2$$

(b)  $c_1 = 3$ ,  $c_2 = 2$ ,  $c_3 = 0$

2B5.

$$[\mathbf{mn}]_{\mathbf{e}_i} = \frac{1}{2} \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}_{\mathbf{e}_i}$$

2B10. (b)  $[\mathbf{S}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}_{\mathbf{e}_i}$  (c)  $[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}_{\mathbf{e}_i}$

2B17. (a)  $[\mathbf{Q}]_{\mathbf{e}_i} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}_{\mathbf{e}_i}$  (b)  $[\mathbf{a}] = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}_{\mathbf{e}_i'}$

2B19.  $T_{11}' = \frac{4}{5}$ ;  $T_{12}' = -3\sqrt{5}$ ;  $T_{31}' = \frac{2}{5}$

2B20. (a)  $[T_{ij}'] = \begin{bmatrix} 0 & -5 & 0 \\ -5 & 1 & 5 \\ 0 & 5 & 1 \end{bmatrix}$

$$(b) T_{ii} = T_{ii}' = 2; \det[T_{ij}] = \det[T_{ij}'] = -25$$

$$2B28. \quad [T^S] = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}, \quad [T^A] = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}, \quad (b) [t^A] = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

2B31. Eigenvector of  $\mathbf{T}$  is  $\mathbf{n}$ , Eigenvector of  $\mathbf{T}$  is  $r_1 s_1$

$$2B34. \quad (a) \mathbf{n} = \frac{\mathbf{a}}{|\mathbf{a}|} \quad (b) \mathbf{a} \cdot \mathbf{b} = a_i b_i$$

$$2B35. \quad (a) [\mathbf{R}]_{\mathbf{e}_i} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad (b) \mathbf{n} = \pm \frac{1}{\sqrt{3}} (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$$

$$2B36. \quad (b) [\mathbf{R}^S]_{\mathbf{e}_i'} = \begin{bmatrix} \cos\theta & 0 & 0 \\ 0 & \cos\theta & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad [\mathbf{R}^A]_{\mathbf{e}_i'} = \begin{bmatrix} 0 & -\sin\theta & 0 \\ \sin\theta & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$(c) \lambda_1 = \cos\theta; \quad \mathbf{n}_1 = \alpha \mathbf{e}_1' + \beta \mathbf{e}_2', \quad \lambda_2 = \cos\theta; \quad \mathbf{n}_2 = -\beta \mathbf{e}_1' + \alpha \mathbf{e}_2'$$

$$\lambda_3 = 1; \quad \mathbf{n}_3 = \mathbf{e}_3', \quad \alpha^2 + \beta^2 = 1$$

$$(d) R_{ii}' = 2\cos\theta + 1; \quad (e) t^A = \sin\theta \mathbf{e}_3', \quad (f) \theta = \arccos(-1/2) = 120^\circ$$

$$\mathbf{e}_3' = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$$

$$2B41. \quad (a) I_1 = 3; \quad I_2 = -16; \quad I_3 = -48$$

$$\lambda_1 = 3; \quad \mathbf{n}_1 = \mathbf{e}_1, \quad \lambda_2 = 4; \quad \mathbf{n}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_2 + \mathbf{e}_3), \quad \lambda_3 = -4; \quad \mathbf{n}_3 = \frac{1}{\sqrt{2}}(-\mathbf{e}_2 + \mathbf{e}_3)$$

$$(b) [\mathbf{T}]_{\mathbf{n}_i} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}_{\mathbf{n}_i}$$

(c) No, the first invariants are not equal

$$2C2. \quad (a) \mathbf{n} = \mathbf{e}_3 \quad (b) |\nabla\varphi| = 2 \quad (c) \frac{d\varphi}{dr} = \sqrt{2}$$

$$2C3. \quad \mathbf{n} = \left( \frac{x}{a^2} \mathbf{e}_1 + \frac{y}{b^2} \mathbf{e}_2 + \frac{z}{c^2} \mathbf{e}_3 \right) \left( \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{-1/2}$$

$$2C4. \quad (a) \mathbf{q} = -3k(\mathbf{e}_1 + \mathbf{e}_2) \quad (b) \mathbf{q} = -3k(\mathbf{e}_1 + 2\mathbf{e}_2)$$

$$2C5. \quad (a) \mathbf{E} = -\nabla\varphi = -\alpha(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2) \quad (b) \mathbf{D} = -\varepsilon_1 \alpha \cos\theta \mathbf{e}_1 - \varepsilon_2 \alpha \sin\theta \mathbf{e}_2$$

$$(c) |\mathbf{D}|^2 = \alpha^2(\varepsilon_1^2 \cos^2\theta + \varepsilon_2^2 \sin^2\theta); \quad |\mathbf{D}|_{\max} \text{ for } \theta = 0; \frac{\pi}{2}; \frac{3\pi}{2} \dots$$

$$2C7. \quad (a) [\nabla \mathbf{v}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \quad (b) [(\nabla \mathbf{v})\mathbf{v}] = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

(c)  $\operatorname{div} \mathbf{v} = 2$    (d)  $\operatorname{curl} \mathbf{v} = 2\mathbf{e}_1$    (e)  $d\mathbf{v} = 2ds(\mathbf{e}_1 + \mathbf{e}_3)$

## CHAPTER 3

$$3.1. \quad (a) \mathbf{v} = k\mathbf{e}_1, \quad \mathbf{a} = \mathbf{0} \quad (b) \frac{D\theta}{Dt} = Ak \quad (c) \frac{D\theta}{Dt} = 0$$

$$3.2. \quad (b) \mathbf{v} = 2kX_1^2 t \mathbf{e}_2, \quad \mathbf{a} = 2kX_1^2 \mathbf{e}_2, \quad (c) \text{Use } X_1 = x_1$$

$$3.3. \quad (b) v_1 = \frac{2ktx_2^2}{(1+kt)^2}, \quad v_2 = \frac{kx_2}{1+kt}, \quad v_3 = 0 \quad (c) a_1 = \frac{2kx_2^2}{(1+kt)^2}, \quad a_2 = 0, \quad a_3 = 0$$

$$3.4. \quad (b) \mathbf{v} = k\mathbf{e}_1, \quad \mathbf{a} = \mathbf{0} \quad (c) \mathbf{v} = \left( \frac{k}{1+t} \right) \mathbf{e}_1, \quad \mathbf{a} = \mathbf{0}$$

$$3.7. \quad (b) \mathbf{v} = \left( \frac{x_1}{1+t} \right) \mathbf{e}_1$$

$$3.15. \quad (b) \mathbf{a} = -\frac{1}{4}(\mathbf{e}_1 + \mathbf{e}_2), \quad \frac{D\theta}{Dt} = 2k$$

$$3.19. \quad (b) E'_{11} = k/2$$

$$3.20. \quad (a) \text{Unit elongation: } 5k, 2k. \quad \text{Decrease in angle: } k$$

$$3.21. \quad (a) k/3$$

$$3.22. \quad (a) \frac{58}{9} \times 10^{-4} \quad (b) \frac{32}{3\sqrt{5}} \times 10^{-4}$$

$$3.25. \quad \left( \frac{\Delta l}{l} \right)_{\max} = 3 \times 10^{-6}$$

$$3.31. \quad E_{11} = a, \quad E_{22} = c, \quad E_{12} = b - \frac{1}{2}(a + c)$$

$$3.34. \quad E_{11} = a, \quad E_{22} = \frac{1}{3}(2b + 2c - a), \quad E_{12} = \frac{b - c}{\sqrt{3}}$$

$$3.37. \quad (b) 3k$$

$$3.38. \quad \frac{1}{ds_1} \frac{Dds_1}{Dt} = -(k+1), \quad \frac{1}{ds_2} \frac{Dds_2}{Dt} = -\frac{1}{2}(k+1)$$

$$3.48. \quad k = 1$$

$$3.52. \quad v_1 = v_1(x_2, x_3), \quad v_2 = v_3 = 0$$

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3.53. (a)  $\rho = \frac{\rho_0}{(1+t)}$  (b)  $\rho = \frac{\alpha}{x_1}$

3.54.  $\rho = \rho_0 e^{-t^2}$

CHAPTER 4

4.1. (a) On  $\mathbf{e}_2$ ,  $T_n = 4$  MPa (b) On  $\mathbf{e}_3$ ,  $T_s = 5.83$  MPa

4.2. (a)  $\mathbf{t} = \frac{1}{3}(5\mathbf{e}_1 + 6\mathbf{e}_2 + 5\mathbf{e}_3)$  MPa (b)  $T_n = 3$  MPa  $T_s = \frac{\sqrt{5}}{3}$  MPa

4.4.  $\mathbf{t} = \frac{100}{4}(\sqrt{3}\mathbf{e}_1 + \mathbf{e}_2 - \sqrt{3}\mathbf{e}_3)$

4.6.  $T'_{11} = -6.43$  MPa,  $T'_{13} = 18.6$  MPa

4.7.  $\mathbf{F} = 4\beta \mathbf{e}_2$ ,  $\mathbf{M} = -\frac{4\alpha}{3} \mathbf{e}_3$

4.9.  $\mathbf{F} = 4\alpha \mathbf{e}_1$ ,  $\mathbf{M} = -\frac{4}{3}\alpha \mathbf{e}_1$

4.10 (a)  $\mathbf{t} = \mathbf{0}$ ,  $\mathbf{t} = \alpha x_3 \mathbf{e}_2 - \alpha x_2 \mathbf{e}_3$ ,  $\mathbf{t} = -\alpha x_3 \mathbf{e}_2 + \alpha x_2 \mathbf{e}_3$

(b)  $\mathbf{F} = \mathbf{0}$ ,  $\mathbf{M} = 8\pi a \mathbf{e}_1$

4.22.  $T_{11} = 1$  MPa  $T_{33} = 1$  MPa

4.25. (a)  $T_1 = \tau$ ,  $T_2 = 0$ ,  $T_3 = -\tau$ ,  $\mathbf{n}_1 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$ ,  $\mathbf{n}_2 = \mathbf{e}_3$ ,  $\mathbf{n}_3 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_2)$

(b)  $T_s]_{\max} = \tau$ ,  $\mathbf{n} = \mathbf{e}_1$  and  $\mathbf{e}_2$

4.26.  $T_s]_{\max} = \frac{1}{2}(\sigma_1 - \sigma_3)$ ,  $\mathbf{n} = \frac{\sqrt{2}}{2}(\mathbf{e}_1 \pm \mathbf{e}_3)$

4.28.  $T_{12} = 2x_1 - x_2 + 3$

4.29.  $T_{33} = (1 + \rho g/\alpha)x_3 + f(x_1, x_2)$

4.31. Yes

CHAPTER 5

5.4.  $E_Y = 207$  GPa ( $30 \times 10^6$  psi),  $\nu = 0.30$ ,  $k = 172$  GPa ( $25 \times 10^6$  psi)

5.6.  $\nu = 0.27$ ,  $\lambda = 91$  GPa ( $13.2 \times 10^6$  psi),  $k = 141$  GPa ( $20.5 \times 10^6$  psi)

5.8. (a)  $[\mathbf{T}] = \begin{bmatrix} 6.6 & -2.3 & 0 \\ -2.3 & 3.2 & 0 \\ 0 & 0 & 8.9 \end{bmatrix} \times 10^3$  psi =  $\begin{bmatrix} 45.5 & -15.9 & 0 \\ -15.9 & 22.1 & 0 \\ 0 & 0 & 61.4 \end{bmatrix}$  MPa

5.10. (a)  $[\mathbf{E}] = \begin{bmatrix} 3.33 & 1.26 & 0 \\ 1.26 & -2.32 & 0 \\ 0 & 0 & -0.43 \end{bmatrix} \times 10^{-5}$  (b)  $\Delta V = 3.04 \times 10^{-3} \text{ cm}^3$

5.14. (a)  $[\mathbf{T}] = k\mu \begin{bmatrix} 0 & 2X_3 & 2X_1 + X_2 \\ 2X_3 & 0 & X_1 - 2X_2 \\ 2X_1 + X_2 & X_1 - 2X_2 & 0 \end{bmatrix}$  (b) Yes

5.17.  $\frac{c_L}{c_T} = 2; 7.14; 22.4$

5.19. (a) Transverse wave in  $e_3$  direction. (b)  $c = c_T$  (c)  $\alpha = -1$  (d)  $\beta = \frac{n\pi}{l}$ ,  $n = 1, 2, 3, \dots$

5.20. (c)  $\alpha = +1$  (d)  $\beta = \frac{n\pi}{2l}$ ,  $n = 1, 3, 5, \dots$

5.21. (c)  $\alpha = +1$  (d)  $\beta = \frac{n\pi}{l}$ ,  $n = 1, 2, 3, \dots$

5.22. (a) Longitudinal wave in  $e_3$  direction (b)  $c = c_L$  (c)  $\alpha = -1$  (d)  $\beta = \frac{n\pi}{l}$ ,  $n = 1, 2, 3, \dots$

5.27. (a)  $\alpha_1 = \alpha_2 = \alpha_3 = 0$ ,  $\epsilon_2 = \epsilon_1$ ,  $\epsilon_3 = 0$  (b)  $\alpha_3 = 31.2^\circ$ ,  $\frac{\epsilon_2}{\epsilon_1} = 0.74$ ,  $\frac{\epsilon_3}{\epsilon_1} = 0.50$

5.28.  $\alpha_1 = 30^\circ$

5.34. (a)  $u_1(x_1, t) = \alpha \cos \omega t \left[ \cos \frac{\omega x_1}{c_L} + \left( \tan \frac{\omega l}{c_L} \right) \sin \frac{\omega x_1}{c_L} \right]$ , (b)  $\omega = \frac{n\pi c_L}{2l}$ ,  $n = 1, 3, 5, \dots$

5.39. (a)  $T_n]_{\max} = 11,300 \text{ psi (78 MPa)}$  (b)  $\Delta l = 3.62 \times 10^{-2} \text{ inch (} 9.2 \times 10^{-2} \text{ cm)}$   
 $T_s]_{\max} = 5650 \text{ psi (39 MPa)}$

5.40.  $\delta = 0.72 \text{ cm}$

5.45.  $P_1 = \frac{P}{12}$ ,  $P_2 = \frac{P}{3}$ ,  $P_3 = \frac{7P}{12}$

5.48.  $d = 5.8 \text{ cm}$

5.49.  $d = 3.38 \text{ inch (8.59 cm)}$

5.51.  $M_1 = \frac{M_t}{\frac{l_1 \left( \frac{a_2}{a_1} \right)^4}{l_2 \left( \frac{a_1}{a_1} \right)^4} + 1}$ ,  $M_2 = \frac{M_t}{1 + \frac{l_2 \left( \frac{a_1}{a_2} \right)^4}{l_1 \left( \frac{a_1}{a_2} \right)^4}}$

5.53.  $M_L = M \frac{b + 2c}{(a + b + c)}$

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5.54. (b)  $T_n]_{\max} = 21.4 \text{ksi} (147 \text{MPa})$ ,  $T_s]_{\max} = 16.6 \text{ksi} (114 \text{MPa})$

5.62.  $d = 10.5 \text{cm}$

5.63.  $b = 10 \text{inch} (25.4 \text{cm})$

5.69. (a) Yes (b)  $T_{11} = 0$ ,  $T_{12} = -2\alpha x_1$ ,  $T_{22} = 2\alpha$ ,  $T_{33} = 2\nu\alpha$

(c) On  $x_1 = a$ ,  $\mathbf{t} = -2\alpha a \mathbf{e}_2$ , on  $x_1 = b$ ,  $\mathbf{t} = -2\alpha x_1 \mathbf{e}_1 + 2\alpha \mathbf{e}_2$

5.70. (a) If  $\alpha = -\beta$

5.71. (a) Yes (b)  $T_{11} = 2\alpha x_1$ ,  $T_{22} = 0$ ,  $T_{12} = -2\alpha x_2 - 3\beta x_2^2$

CHAPTER 6

6.1.  $B = 5.10 \times 10^4 \text{N}$

6.5.  $p_A = p_o + \rho(g + a)h$

6.8.  $z = \frac{\omega^2 r^2}{2g}$

6.10.  $p = p_o e^{-\rho_o g(z - z_o)/p_o}$  for  $n = 1$

6.11. (b)  $T_n = -p + \mu$ ,  $T_s = 0$

6.12. (a)  $-4\mu$  (b)  $8\mu$

6.23.  $v_1 = \frac{\rho g \sin \theta d^2}{\mu} \left[ \left( \frac{x_2}{d} \right) - \frac{1}{2} \left( \frac{x_2}{d} \right)^2 \right]$

6.28. (a)  $v = A \ln \frac{r}{a} + v_a$ ,  $A = \frac{v_b - v_a}{\ln(b/a)}$

(b)  $v = \frac{\beta}{4}(r^2 - a^2) + A \ln \frac{r}{a}$ ,  $\beta = \frac{1}{\mu} \frac{dp}{dx}$ ,  $A = \frac{\beta(a^2 - b^2)}{4 \ln(b/a)}$

6.30.  $A = \frac{\beta a^2 b^2}{2(a^2 + b^2)}$ ,  $B = -A$

6.31.  $A = -\frac{\beta \sqrt{3}}{6b}$   $B = 0$

6.37.  $\theta = \left( \frac{\partial p}{\partial x} \right)^2 \frac{(b^4 - x_2^4)}{12 \kappa \mu} + \frac{(\theta_2 - \theta_1)x_2}{2b} + \frac{\theta_1 + \theta_2}{2}$

6.40. (b)  $T_{11} = -p + 2\mu k$ ,  $T_{22} = -p - 2\mu k$ ,  $T_{33} = -p$ , remaining  $T_{ij} = 0$

(c)  $\mathbf{a} = k^2(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2)$  (d) and (e)  $p = -\frac{\rho k^2}{2}(x_1^2 + x_2^2) + p_o$

(f)  $\Phi_{inc} = 4k^2\mu$

6.41. (b)  $T_{11} = -p + 4\mu kx_1$ ,  $T_{22} = -p - 4\mu kx_1$ ,  $T_{12} = -4\mu kx_2$

$T_{33} = -p$ ,  $T_{13} = T_{23} = 0$

(c)  $a_1 = 2k^2(x_1^3 + x_1x_2^2)$ ,  $a_2 = 2k^2(x_2^3 + x_1^2x_2)$

6.46.  $Q = 0.0762 \text{ m}^3/\text{sec}$

6.28.  $Q = 0.113\text{m}^3/\text{sec}$

**CHAPTER 7**

7.7. (b)  $3A\rho_0 e^{-t}$ ,  $-3A\rho$  (c)  $-3A\rho$

7.8. (b)  $m = A\rho_0$  (c)  $\mathbf{p} = \frac{9}{2}\rho_0 A e^{t-t_0}\mathbf{e}_1$

7.11. (a)  $\frac{4A\rho_0}{(1+t)^2}$ ,  $-\frac{8A\rho_0}{(1+t)^3}$  (b)  $\frac{8A\rho_0}{(1+t)^3}$  (c) 0

(d)  $\frac{13}{3}\frac{\rho_0}{(1+t)^3}$ ,  $-\frac{13\rho_0}{(1+t)^4}$  (e)  $\frac{13\rho_0 A}{(1+t)^4}$

7.12. (a)  $\frac{9}{2}\rho_0 A e^{-t}\mathbf{e}_1$ ,  $-\frac{9}{2}\rho_0 A e^{-t}\mathbf{e}_1$  (b)  $9\rho_0 A e^{-t}\mathbf{e}_1$  (c)  $\frac{9}{2}\rho_0 A e^{-t}\mathbf{e}_1$

7.14.  $\mathbf{F} = 8\rho(\mathbf{e}_1 + \mathbf{e}_2)$   $\mathbf{M} = 8\rho(-\mathbf{e}_1 + \mathbf{e}_2)$

7.18.  $283\left(\frac{1}{2}\mathbf{e}_1 - \frac{\sqrt{3}}{2}\mathbf{e}_2\right)\text{N}$

7.20.  $p v_0^2 A$

7.21.  $y_2 = -\frac{y_1}{2} + \left[\left(\frac{y_1}{2}\right)^2 + \frac{2Q^2}{gy_1}\right]^{1/2}$

7.23. 612 rad/sec

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